

Lecture 23: Disjoint Sets

Agenda:

- 1st implementation — an array of representatives (review)
- 2nd implementation — forest of rooted trees

Reading:

- Textbook pages 501 – 509

An array of representatives (recall)

- $R(x)$ — the representative of the set containing x
 - $\Theta(n)$ space
 - how to describe a set now:
elements with the same representative are in a same set
 - report a set in $\Theta(n)$ time

- procedure `MakeSet( $x$ )` ****initialize representative for x**

$R(x) \leftarrow x$

- procedure `Find( $x$ )` ****representative of x**

return $R(x)$

- procedure `Union( $x, y$ )`

$rx \leftarrow R(x)$
 $ry \leftarrow R(y)$
 for $j \leftarrow 1$ to n do
 if $R(j) = ry$ then
 $R(j) \leftarrow rx$

- Running time per operation $\Theta(n)$

2nd implementation — forest of rooted trees

- Forest of rooted trees:
 - elements of a set $\longleftrightarrow$ nodes in the rooted trees
 - representative of a set $\longleftrightarrow$ root of the tree
 - each node needs only ‘parent’ $\longrightarrow$ implement via an array
 - $P(x)$ — parent of x , for $x = 1, 2, \dots, n$

- procedure MakeSet(x) ****initialize parent for x**

$$P(x) \leftarrow x$$

- procedure Find(x) ****return root of the tree containing x**

```

while  $P(x) \neq x$  do
   $x \leftarrow P(x)$ 
return  $x$ 

```

- procedure Union(x, y) ****make root of x 's tree
a child of root of y 's tree

```

 $rx \leftarrow \text{Find}(x)$ 
 $ry \leftarrow \text{Find}(y)$ 
 $P(rx) \leftarrow ry$             $P(ry) \leftarrow rx$ 

```

- Running time per operation ???

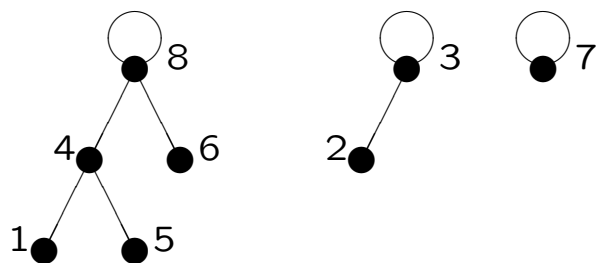
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An example:

sets at start	{1}	{2}	{3}	{4}	{5}	{6}	{7}	{8}
index	1	2	3	4	5	6	7	8
parent	1	2	3	4	5	6	7	8
Union(1,4)	4	2	3	4	5	6	7	8
Find(1)	4							
Find(4)				4				
Union(2,3)	4	3	3	4	5	6	7	8
Union(5,1)	4	3	3	4	4	6	7	8
Union(1,8)	4	3	3	8	4	6	7	8
Union(6,5)	4	3	3	8	4	8	7	8
Find(6)						8		
Find(3)			3					

sets at finish {1, 4, 5, 6, 8}, {2, 3}, {7}

forest at finish:



Finding connected components of a graph:

- procedure `ConnectedComponents( $G$ )`

```

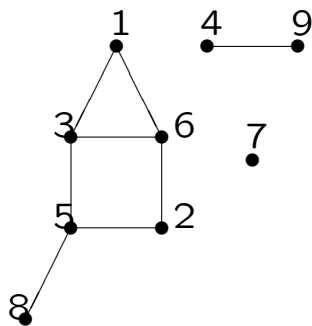
    for each vertex  $v \in V(G)$  do
        MakeSet( $v$ )
    for each edge  $(x, y) \in E(G)$  do
        if not SameComponent( $x, y, G$ ) then
            Union( $x, y$ )

```
- procedure `SameComponent( $x, y, G$ )`

```

    return Find( $x$ ) = Find( $y$ )

```
- An example:



$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

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Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After MakeSets:



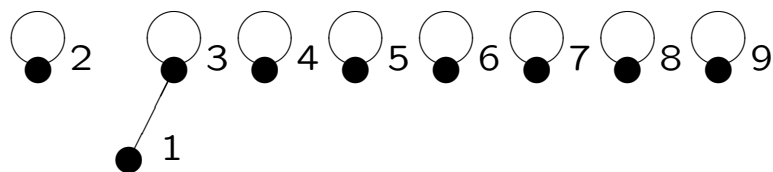
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(1, 3)$:



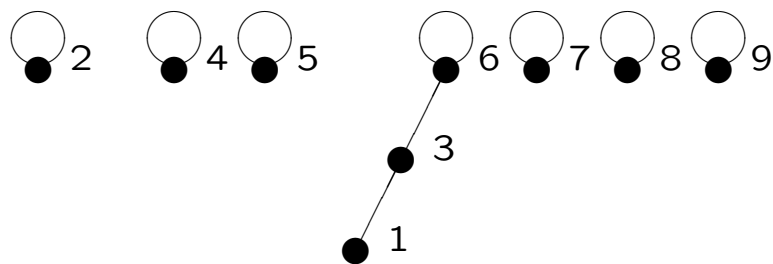
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(1, 6)$:



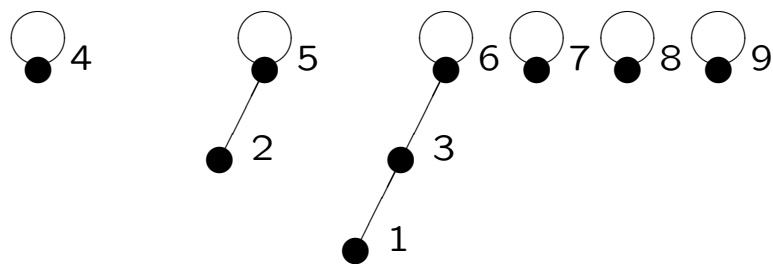
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

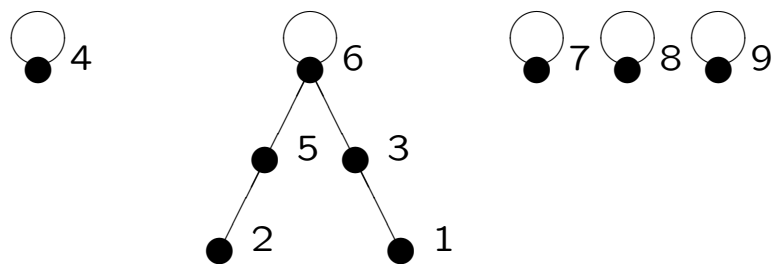
$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(2, 5)$:



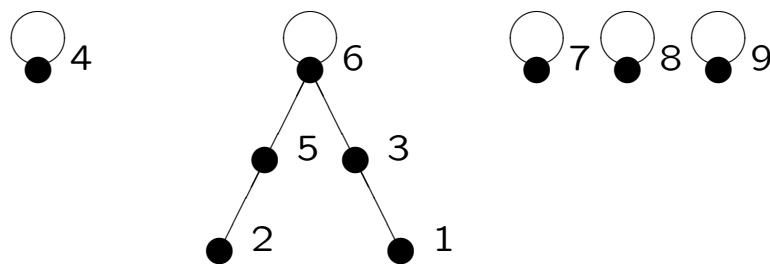
Finding connected components of a graph:

- Graph $G = (V, E)$:
 $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
 $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$
- After considering edge $(2, 6)$:



Finding connected components of a graph:

- Graph $G = (V, E)$:
 $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
 $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$
- After considering edges $(3, 5)$ and $(3, 6)$



(no change since 3, 5, 6 are already in a same component):

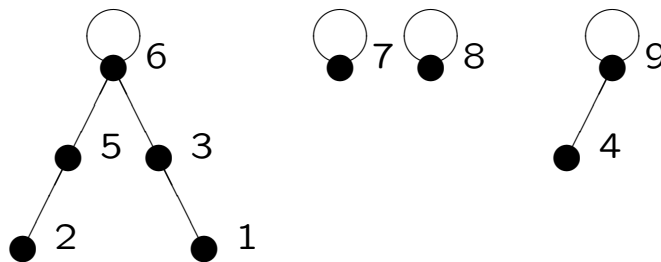
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

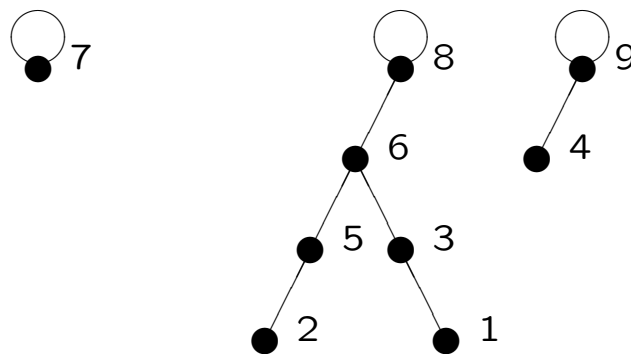
$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(4, 9)$:

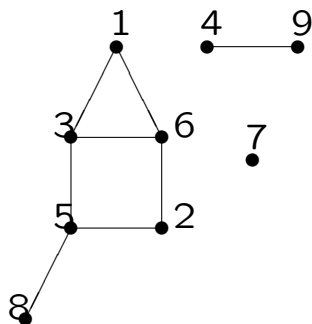


Finding connected components of a graph:

- Graph $G = (V, E)$:
 $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
 $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$
- After considering edge (5, 8):



- Sets at finish:
 $\{1, 2, 3, 5, 6, 8\}$ with representative 8
 $\{4, 9\}$ with representative 9
 $\{7\}$ with representative 7
- Therefore, there are 3 connected components



Analysis of the implementation:

- n MakeSet and $(m - n)$ Union/Find
- Each MakeSet — $\Theta(1)$ time
- Each Find — $\Theta(\text{depth}(x))$ time

since we need to get to the root of the tree containing element x

- Each Union — $\Theta(\text{depth}(x) + \text{depth}(y))$ time

since we need to find the representatives for x and y and then use constant time to update

- Worst case:
 $\Theta(n + (m - n)n) = \Theta(mn)$ time (assuming $m \gg n$)
- On average, $\Theta(n)$ per operation

amortized running time analysis

Conclusion: Forest of rooted trees is NOT better than Array of representatives

Yet it allows speedup, just adding some tricks ...
(next lecture)

Lecture 23: Disjoint Sets

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	disjoint sets?
☐	☐	☐	3 operations
☐	☐	☐	forest of rooted trees
☐	☐	☐	finding connected components
☐	☐	☐	running time analysis