

Lecture 22: Disjoint Sets

Agenda:

- Introduction
- Application — graph (connected) components
- 1st implementation — an array of representatives

Reading:

- Textbook pages 498 – 505

Overview:

- An abstract data type (ADT)
- Analysis: via sequences of operations
- Implementation: array of representatives
- Application: graph components
- Implementation: forest of rooted trees (next lecture)
 - basic implementation
 - improvement: union by rank
 - improvement: compressed find

Disjoint sets:

- An abstract data type
- It maintains: pairwise disjoint sets
- One element of each set is the representative
- Operations:
 - **MakeSet**(x) — make x itself into a set and use x to be the representative: $S_x \leftarrow \{x\}$
 - **Find**(x) — return the representative of the set $S_{f(x)}$ containing x , which is $f(x)$
 - **Union**(x, y) — find the sets containing x and y , respectively, and union them into a new set with representative z : $S_z \leftarrow S_{f(x)} \cup S_{f(y)}$
- Analysis over a sequence of operations on n elements
 - **MakeSet**(1), **MakeSet**(2), ..., **MakeSet**(n)
 - all the **Union** and **Find** operations
 - m — number of operations
 - running time for all $m = n + |U| + |F|$ operations?

Simplest DS implementation

— array R of representatives:

- $R(x)$ — the representative of the set containing x
 - $\Theta(n)$ space
 - how to describe a set now:
elements with the same representative are in a same set
 - report a set in $\Theta(n)$ time

- procedure MakeSet(x) ****initialize representative for x**

$R(x) \leftarrow x$

- procedure Find(x) ****representative of x**

return $R(x)$

- procedure Union(x, y)

$rx \leftarrow R(x)$

$ry \leftarrow R(y)$

for $j \leftarrow 1$ to n do

if $R(j) = ry$ then

$R(j) \leftarrow rx$

if $R(j) = rx$ then

$R(j) \leftarrow ry$

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An example:

sets at start	{1}	{2}	{3}	{4}	{5}	{6}	{7}	{8}
index	1	2	3	4	5	6	7	8
representative	1	2	3	4	5	6	7	8
Union(1,4)	1	2	3	1	5	6	7	8
Find(1)	1							
Find(4)				1				
Union(2,3)	1	2	2	1	5	6	7	8
Union(5,1)	5	2	2	5	5	6	7	8
Union(1,8)	5	2	2	5	5	6	7	5
Union(6,5)	6	2	2	6	6	6	7	6
Find(6)						6		
Find(3)			2					
sets at finish	{1, 4, 5, 6, 8}, {2, 3}, {7}							

A DSUF application

— finding connected components of a graph:

- procedure `ConnectedComponents( $G$ )`

```

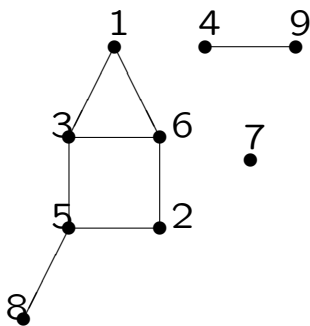
for each vertex  $v \in V(G)$  do
  MakeSet( $v$ )
for each edge  $(x, y) \in E(G)$  do
  if not SameComponent( $x, y, G$ ) then
    Union( $x, y$ )

```
- procedure `SameComponent( $x, y, G$ )`

```

return Find( $x$ ) = Find( $y$ )

```
- An example:



$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

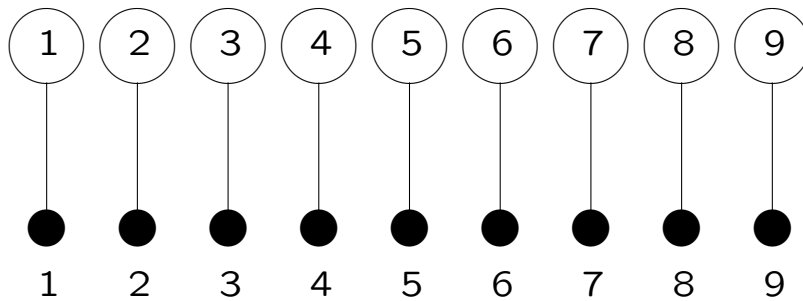
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After MakeSets:



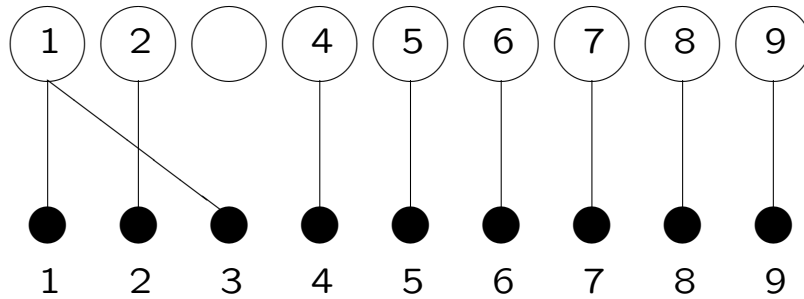
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(1, 3)$:



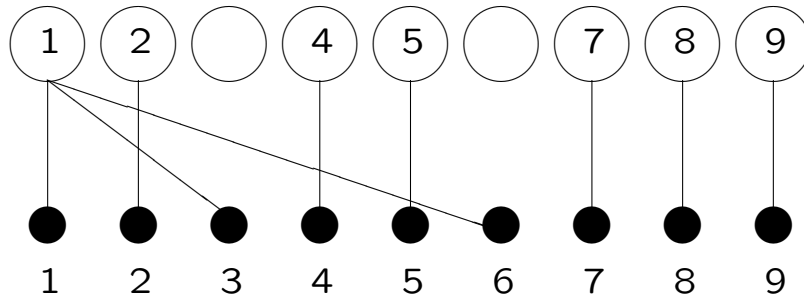
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(1, 6)$:



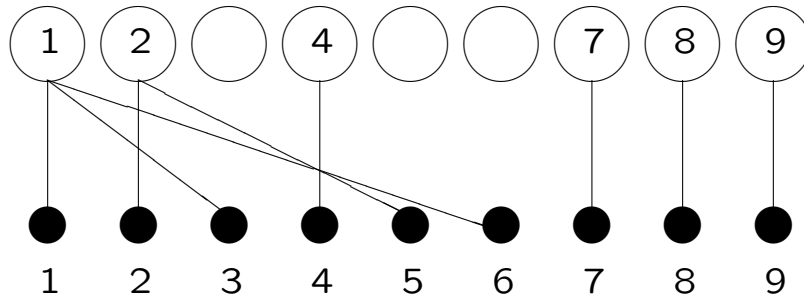
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(2, 5)$:



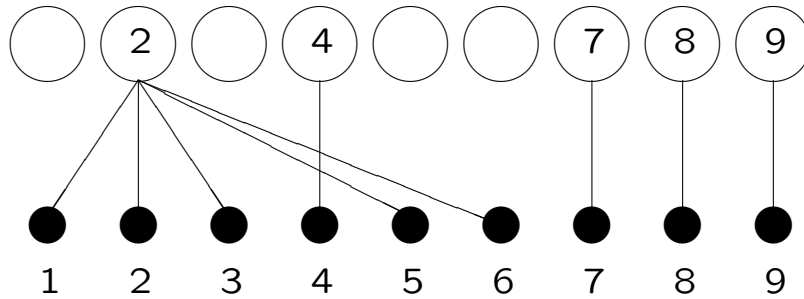
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(2, 6)$:



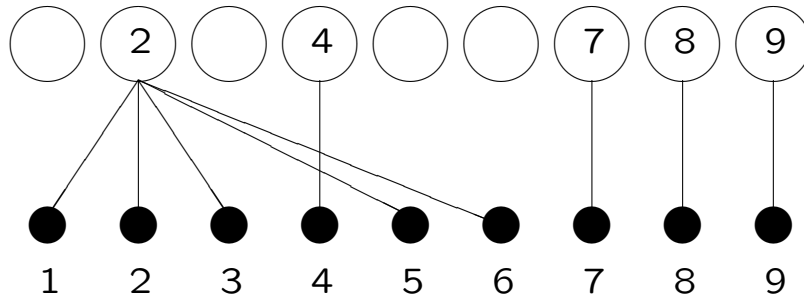
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edges $(3, 5)$ and $(3, 6)$



(no change since 3, 5, 6 are already in a same component):

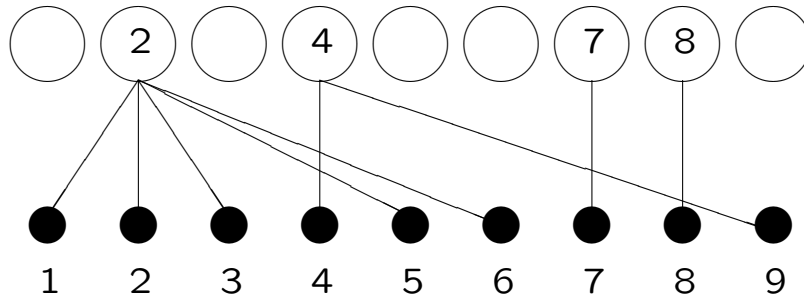
Finding connected components of a graph:

- Graph $G = (V, E)$:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

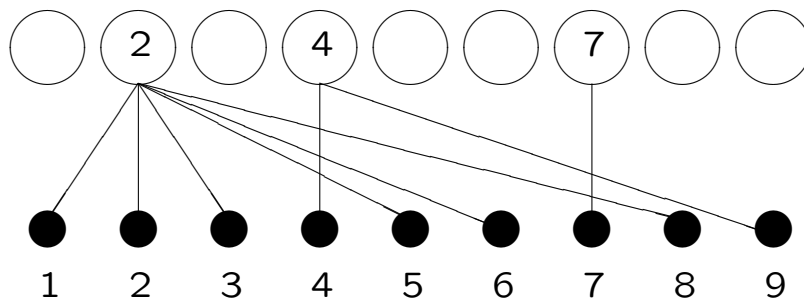
$$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$$

- After considering edge $(4, 9)$:

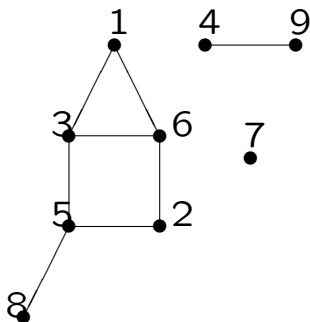


Finding connected components of a graph:

- Graph $G = (V, E)$:
 $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
 $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$
- After considering edge $(5, 8)$:



- Sets at finish:
 $\{1, 2, 3, 5, 6, 8\}$, $\{4, 9\}$, and $\{7\}$
- Therefore, there are 3 connected components



Analysis of the simplest implementation:

- n MakeSet and $(m - n)$ Union/Find
- Each MakeSet — $\Theta(1)$ time
- Each Find — $\Theta(1)$ time

since we record for every element x its representative as $R(x)$

- Each Union — $\Theta(n)$ time

since we need to check for every element in order to update its representative and for every element it takes $\Theta(1)$ time

- Worst case:
 $\Theta(n + (m - n)n) = \Theta(mn)$ time (assuming $m \gg n$)
- On average, $\Theta(n)$ per operation

amortized running time analysis

Lecture 22: Disjoint Sets

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	disjoint sets?
☐	☐	☐	3 operations
☐	☐	☐	array of representatives
☐	☐	☐	finding connected components
☐	☐	☐	running time analysis