

Lecture 9: Recurrences

Agenda:

- Master theorem (full version from textbook)
- The limit of master theorems
- Recurrence conclusions

Reading:

- Textbook pages 62 – 75

Master Theorem (simple version):

Let $a \geq 1$ and $b > 1$ be constants, let $f(n) \in \Theta(n^c)$ for some positive c , and let $T(n)$ be defined by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

Then $T(n)$ can be bounded asymptotically as follows:

1. if $c = \log_b a$, then $T(n) \in \Theta(n^c \log n)$,
2. if $c < \log_b a$, then $T(n) \in \Theta(n^{\log_b a})$,
3. if $c > \log_b a$, then $T(n) \in \Theta(n^c)$.

Some examples:

$$1. \quad T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$2. \quad T(n) = \begin{cases} 4T\left(\frac{n}{2}\right) + n^3 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$3. \quad T(n) = \begin{cases} 7T\left(\frac{n}{2}\right) + n^2 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$4. \quad T(n) = \begin{cases} 7T\left(\frac{n}{2}\right) + \frac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

Lecture 9: Recurrences

Some examples:

$$1. \quad T(n) = \begin{cases} 2T(\frac{n}{2}) + n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 2, b = 2, c = 1 \implies c = \log_b a, \text{ so } T(n) \in \Theta(n \log n)$$

$$2. \quad T(n) = \begin{cases} 4T(\frac{n}{2}) + n^3 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 4, b = 2, c = 3 \implies c > \log_b a = 2, \text{ so } T(n) \in \Theta(n^3)$$

$$3. \quad T(n) = \begin{cases} 7T(\frac{n}{2}) + n^2 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 7, b = 2, c = 2 \implies c < \log_b a, \text{ so } T(n) \in \Theta(n^{\log_2 7})$$

$$4. \quad T(n) = \begin{cases} 7T(\frac{n}{2}) + \frac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 7, b = 2, c = ??? \implies$, so simple Master Theorem does NOT apply

Exercise:

In the last example, show that $T(n) \in O(n^{\log_2 7})$.

Master Theorem (full version):

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a} \log n)$,
2. if $f(n) \in O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$,
3. if $f(n) \in \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) \in \Theta(f(n))$.

The last example:

$$T(n) = \begin{cases} 7T(\frac{n}{2}) + \frac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 7, b = 2, \log_b a > 2$$

$$f(n) = \frac{n^2}{\lg n} \in O(n^{\log_b a - \epsilon}) \text{ for any constant } 0 < \epsilon < \log_b a - 2.$$

Therefore, $T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\lg 7})$.

Lecture 9: Recurrences

Limit of Master Theorem:

$$T(n) = \begin{cases} 4T(\frac{n}{2}) + \frac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 4, b = 2, \log_b a = 2$$

$$f(n) = \frac{n^2}{\lg n} \notin \Theta(n^2);$$

$f(n) = \frac{n^2}{\lg n} \in O(n^2)$ but $f(n) = \frac{n^2}{\lg n} \notin O(n^{2-\epsilon})$ for any positive constant ϵ .

What we can do to get the closed form?

— iterated substitution!!!

$$\begin{aligned} & T(2^k) \\ &= 4 \times T(2^{k-1}) + \frac{2^{2k}}{k} \\ &= 4^2 \times T(2^{k-2}) + 4 \times \frac{2^{2(k-1)}}{k-1} + \frac{2^{2k}}{k} \\ &= 4^2 \times T(2^{k-2}) + \frac{2^{2k}}{k-1} + \frac{2^{2k}}{k} \\ &= 4^3 \times T(2^{k-3}) + 4^2 \times \frac{2^{2(k-2)}}{k-2} + \frac{2^{2k}}{k-1} + \frac{2^{2k}}{k} \\ &= 4^3 \times T(2^{k-3}) + \frac{2^{2k}}{k-2} + \frac{2^{2k}}{k-1} + \frac{2^{2k}}{k} \\ &= 4^k \times T(1) + \frac{2^{2k}}{k-(k-1)} + \dots + \frac{2^{2k}}{k-1} + \frac{2^{2k}}{k} \\ &= 4^k \times T(1) + 2^{2k} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} \right) \\ &= 4^k \times T(1) + 4^k \times H(k) \end{aligned}$$

Therefore, $T(n) = n^2 \times T(1) + n^2 \times H(\lg n) \in \Theta(n^2 H(\lg n))$.

Further we have $H(k) \in \Theta(\log k)$ — you should be able to prove it, thus $T(n) \in \Theta(n^2 H(\lg n)) = \Theta(n^2 \log \log n)$.

Lecture 9: Recurrences

Limit of Master Theorem:

$$T(n) = \begin{cases} 4T(\frac{n}{2}) + n^2 \lg n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 4, b = 2, \log_b a = 2$$

$$f(n) = n^2 \lg n \notin \Theta(n^2);$$

$f(n) = n^2 \lg n \in \Omega(n^2)$ but $f(n) = n^2 \lg n \notin \Omega(n^{2+\epsilon})$ for any positive constant ϵ .

What we can do to get the closed form?

— iterated substitution!!!

$$\begin{aligned} & T(2^k) \\ = & 4 \times T(2^{k-1}) + 2^{2k}k \\ = & 4^2 \times T(2^{k-2}) + 4 \times 2^{2(k-1)}(k-1) + 2^{2k}k \\ = & 4^2 \times T(2^{k-2}) + 2^{2k}(k-1) + 2^{2k}k \\ \\ = & 4^3 \times T(2^{k-3}) + 4^2 \times 2^{2(k-2)}(k-2) + 2^{2k}(k-1) + 2^{2k}k \\ = & 4^3 \times T(2^{k-3}) + 2^{2k}(k-2) + 2^{2k}(k-1) + 2^{2k}k \\ \\ = & 4^k \times T(1) + 2^{2k}(k - (k-1)) + \dots + 2^{2k}(k-1) + 2^{2k}k \\ = & 4^k \times T(1) + 2^{2k}(1 + 2 + 3 + \dots + k) \\ = & 4^k \times T(1) + 4^k \times \frac{k(k+1)}{2} \end{aligned}$$

Therefore, $T(n) = n^2 \times T(1) + n^2 \times \frac{\lg n (\lg n + 1)}{2} \in \Theta(n^2 \lg^2 n)$.

Lecture 9: Recurrences

An exercise — dealing with floor & ceiling:

Prove that $T(n)$ defined by the following recurrence is in $O(\log n)$:

$$T(n) = \begin{cases} 1, & \text{if } n = 1, \\ T(\lceil \frac{n}{2} \rceil) + 1, & \text{if } n \geq 2 \end{cases}$$

- Examine some small cases:

$$T(1) = 1$$

$$T(2) = 2$$

$$T(3) = T(4) = 3$$

$$T(5) = T(6) = T(7) = T(8) = 4$$

...

Guess: $T(n) = k + 1$, for any $2^{k-1} < n \leq 2^k$

- **Prove** the above guessed (by induction).
- Now you only need to get the closed form for n being a power of 2 ...
- By iterated substitution, $T(2^k) = k + 1$ (again, **prove** by induction)
- So, $T(n = 2^k) = \lg n + 1$
- So, guess for any n , $T(n) \leq \lceil \lg n \rceil + 1$
Prove it by induction
- Conclusion: since $T(n) \leq \lceil \lg n \rceil + 1 \leq \lg n + 2 \leq 2 \lg n$, for $n \geq 4$,

$$T(n) \in O(\lg n) = O(\log n)$$

Lecture 9: Recurrences

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	Master Theorem: simple version
☐	☐	☐	Master Theorem: full version
☐	☐	☐	applying Master Theorem
☐	☐	☐	limit of Master Theorem
☐	☐	☐	iterated substitution method
☐	☐	☐	closed form guessing
☐	☐	☐	prove by math induction