

Lecture 8: Recurrences

Agenda:

- Iterated substitution — leading to
- Master theorem (simple version)

Reading:

- Textbook pages 63 – 75

Solving recurrence relations:

- Iterated substitution (done)
- Recursion tree (done)
- Master theorem (now)
- Divide-and-conquer: what form of recurrence relation does it have?
- Typical procedure:

```

Proc dnq( $n$ )
    .....
    dnq( $\frac{n}{b}$ ) ... dnq( $\frac{n}{b}$ )
    .....
    return
end dnq
    
```

- For the call $\text{dnq}(n)$ assume:
 - running time is n^c — excluding recursive calls
 - there are a total of a calls to $\text{dnq}(\frac{n}{b})$
- Recurrence relation for total time $T(n)$

$$T(n) = \begin{cases} \text{bounded,} & \text{if } n < b \\ a \times T(\frac{n}{b}) + n^c, & \text{if } n \geq b \end{cases}$$
- Closed form solution?
 - Iterated substitution !!!
 - Simplifying assumption to $n = b^k$

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Iterated substitution for $T(n)$:

$$\begin{aligned}
 & T(b^k) \\
 = & a \times T\left(\frac{b^k}{b}\right) + b^{kc} && \text{definition} \\
 = & a \times T(b^{k-1}) + b^{kc} && \text{arithmetic} \\
 \\
 = & a \times \left(a \times T\left(\frac{b^{k-1}}{b}\right) + b^{(k-1)c}\right) + b^{kc} && \text{definition} \\
 = & a^2 \times T(b^{k-2}) + a \times b^{(k-1)c} + b^{kc} && \text{arithmetic} \\
 \\
 = & a^2 \times \left(T\left(\frac{b^{k-2}}{b}\right) + b^{(k-2)c}\right) + a \times b^{(k-1)c} + b^{kc} && \text{definition} \\
 = & a^3 \times T(b^{k-3}) + a^2 \times b^{(k-2)c} + a \times b^{(k-1)c} + b^{kc} && \text{arithmetic} \\
 \\
 = & a^3 \times \left(T\left(\frac{b^{k-3}}{b}\right) + b^{(k-3)c}\right) + a^2 \times b^{(k-2)c} + a \times b^{(k-1)c} + b^{kc} && \text{definition} \\
 = & a^4 \times T(b^{k-4}) + a^3 \times b^{(k-3)c} + a^2 \times b^{(k-2)c} + a \times b^{(k-1)c} + b^{kc} && \text{arithmetic} \\
 \\
 = & a^5 \times T(b^{k-5}) + a^4 \times b^{(k-4)c} + a^3 \times b^{(k-3)c} \\
 & \quad + a^2 \times b^{(k-2)c} + a \times b^{(k-1)c} + b^{kc} && \text{guessing} \\
 \\
 = & \dots\dots\dots \\
 \\
 = & a^k \times T(b^{k-k}) + \sum_{i=0}^{k-1} \left(a^i \times b^{(k-i)c}\right) && \text{guessing} \\
 \\
 = & a^k \times T(1) + \left(\sum_{i=0}^{k-1} \left(\frac{a}{b^c}\right)^i\right) b^{kc} && \text{arithmetic} \\
 = & a^k \times T(1) + \begin{cases} b^{kc} \frac{1 - \left(\frac{a}{b^c}\right)^k}{1 - \frac{a}{b^c}} & \text{if } a \neq b^c \\ b^{kc} \times k & \text{if } a = b^c \end{cases} && \text{arithmetic}
 \end{aligned}$$

Replace k by $\log_b n$, we have

$$T(n) = a^{\log_b n} \times T(1) + \begin{cases} n^c \frac{1 - \left(\frac{a}{b^c}\right)^{\log_b n}}{1 - \frac{a}{b^c}} & \text{if } a \neq b^c \\ n^c \times \log_b n & \text{if } a = b^c \end{cases}$$

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Growth of function $T(n)$:

$$T(n) = n^{\log_b a} \times T(1) + \begin{cases} n^c \frac{1 - (\frac{a}{b^c})^{\log_b n}}{1 - \frac{a}{b^c}} & \text{if } a \neq b^c \\ n^c \times \log_b n & \text{if } a = b^c \end{cases}$$

- If $a = b^c$ then $\log_b a = c$ and

$$T(n) = T(1)n^{\log_b a} + n^c \log_b n \in \Theta(n^c \log n).$$

- If $a < b^c$ then $\log_b a < c$ and

$$T(n) = T(1)n^{\log_b a} + n^c \frac{1 - (\frac{a}{b^c})^{\log_b n}}{1 - \frac{a}{b^c}} = T(1)n^{\log_b a} + \frac{n^c - n^{\log_b a}}{1 - \frac{a}{b^c}} \in \Theta(n^c).$$

- If $a > b^c$ then $\log_b a > c$ and

$$T(n) = T(1)n^{\log_b a} + n^c \frac{(\frac{a}{b^c})^{\log_b n} - 1}{\frac{a}{b^c} - 1} = T(1)n^{\log_b a} + \frac{n^{\log_b a} - n^c}{\frac{a}{b^c} - 1} \in \Theta(n^{\log_b a}).$$

Master Theorem (simple version):

Let $a \geq 1$ and $b > 1$ be constants, let $f(n) \in \Theta(n^c)$ for some positive c , and let $T(n)$ be defined by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

Then $T(n)$ can be bounded asymptotically as follows:

1. if $c = \log_b a$, then $T(n) \in \Theta(n^c \log n)$,
2. if $c > \log_b a$, then $T(n) \in \Theta(n^c)$,
3. if $c < \log_b a$, then $T(n) \in \Theta(n^{\log_b a})$,

Algorithm design consequences:

- $c \neq \log_b a \rightarrow T(n) \uparrow$ if $\log_b a \uparrow$ or $c \uparrow$
Redesign: no
- $c > \log_b a \rightarrow T(n) \downarrow$ if $\log_b a \uparrow$ or $c \downarrow$
Redesign: more/bigger calls, less body
- $c < \log_b a \rightarrow T(n) \downarrow$ if $\log_b a \downarrow$ or $c \uparrow$
Redesign: fewer/less calls, more body

Some examples:

$$1. \quad T(n) = \begin{cases} 2T(\frac{n}{2}) + n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$2. \quad T(n) = \begin{cases} 4T(\frac{n}{2}) + n^3 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$3. \quad T(n) = \begin{cases} 7T(\frac{n}{2}) + n^2 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$4. \quad T(n) = \begin{cases} 7T(\frac{n}{2}) + \frac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

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Some examples:

$$1. \quad T(n) = \begin{cases} 2T(\frac{n}{2}) + n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 2, b = 2, c = 1 \implies c = \log_b a, \text{ so } T(n) \in \Theta(n \log n)$$

$$2. \quad T(n) = \begin{cases} 4T(\frac{n}{2}) + n^3 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 4, b = 2, c = 3 \implies c > \log_b a = 2, \text{ so } T(n) \in \Theta(n^3)$$

$$3. \quad T(n) = \begin{cases} 7T(\frac{n}{2}) + n^2 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$$a = 7, b = 2, c = 2 \implies c < \log_b a, \text{ so } T(n) \in \Theta(n^{\log_2 7})$$

$$4. \quad T(n) = \begin{cases} 7T(\frac{n}{2}) + \frac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 7, b = 2, c = ??? \implies$, so simple Master Theorem does NOT apply

Exercise:

In the last example, show that $T(n) \in O(n^{\log_2 7})$.

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Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	iterated substitution method
☐	☐	☐	closed form guessing
☐	☐	☐	prove by math induction
☐	☐	☐	logarithm (base substitution)
☐	☐	☐	Master Theorem: simple version
☐	☐	☐	applying Master Theorem