CMPUT204
Department of Computing Science
University of Alberta

Homework Assignment #1
Due: Sept 22, 2004

Question 1. (10 marks) Prove or disprove each of the following statements:

a. (2 marks) $1^k + 2^k + \ldots + n^k \in O(n^{k+1})$ for every $k \in \mathbb{N}$.

b. (2 marks) For any functions $f, g : \mathbb{N} \longrightarrow R^+$, either $f \in O(g)$ or $g \in O(f)$.

c. (2 marks) For any functions $f, g : \mathbb{N} \longrightarrow R^+$, $f \in O(g)$ if and only if $g \in \Omega(f)$.

d. (2 marks) For any functions $f, f', g, g' : \mathbb{N} \longrightarrow R^+$, if $f \in O(f')$ and $g \in O(g')$ then $(f + g) \in O(\max(f', g'))$ where $(f + g)(n)$ is defined to be $f(n) + g(n)$ for any value of $n \in \mathbb{N}$.

e. (2 marks) For any functions $f, g : \mathbb{N} \longrightarrow R^{\geq 1}$, $\log(f(n)) \in \Theta(\log(g(n))$ if and only if $f(n) \in \Theta(g(n))$.

Question 2. (10 marks) Prove by induction that using an unlimited supply of 4-cent and 7-cent postage stamps we can make (exactly) any amount of postage that is 18 cents or more.

Question 3. (10 marks) Consider the following recursive function:

$$f(n) = \begin{cases} 3 & \text{if } n = 0 \\ 6 & \text{if } n = 1 \\ 5f(n-1) - 6f(n-2) + 2 & \text{if } n > 1 \end{cases}$$

Prove that $f(n) = 1 + 2^n + 3^n$ for all values of $n \geq 0$.

Question 4. (10 marks) Consider the following program:

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Smallest (A, f, l)
**Precondition: A[f ... l] is an array of integers, f ≤ l and f, l ∈ ℕ.
**Postcondition: Returns the smallest element of A[f ... l].
if f = l then
    return A[f]
else
    m = ⌊(f+l)/2⌋
    return min(Smallest(A, f, m), Smallest(A, m + 1, l))
end if
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a. (3 marks) Write a recurrence relation which describes the time complexity of this program in terms of the size of array A.

b. (2 marks) Using the master theorem, give a Θ bound on the complexity of this program.

c. (3 marks) Write a non-recursive version of Smallest(A, f, l) (in pseudo-code).

d. (2 marks) Find the worst case running time of this new program and express it in Θ notation.

Question 5. (12 marks) True or False? for each case give a justification of no more than two lines.

a. (2 marks) $n \log^5 n \in O(n^{\frac{5}{4}})$

b. (2 marks) $2^{3n} \in \Theta(6^n)$.

c. (2 marks) $(\log\log n)^{\log n} \in \Omega(1.1^n)$

d. (2 marks) $5^{\sqrt{\log n}} \in \Omega(n)$

e. (2 marks) $n^{\frac{\log\log n}{\log n}} \in O(n^{0.01})$.

f. (2 marks) $n! \in O(2^{n\log n})$

Question 6. (12 marks) P2-2 on page 38.