Sampling functions and sparse reconstruction methods

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Outlines:

- Motivations
- Sparse Reconstruction
  - Sparseness imposing norms
  - Minimum Weighted Norm Interpolation
- Sampling operators (irregular vs. regular)
- Synthetic examples
  - 1D, 2D and 3D examples
- Real data example (3D)
- Conclusion
Motivations I:

Main Goal:
Analyzing the effects of various sampling operators on sparse reconstruction methods
Sparse Reconstruction

(Minimum Weighted Norm Interpolation)
Band-Limited Minimum Weighted Norm Interpolation (BLMWNI)

Analogy Between the time-space and the frequency-wavenumber domain
$$\hat{x} = F^H \gamma F G^T (G F^H \gamma F G^T + \mu I)^{-1} y$$

MWNI

Spatial Domain

Wavenumber Domain

Data in FX domain

Data in FK domain

Final reconstructed data in FX domain

$F$: Fourier Operator

$\gamma$: Weighting Operator

$G$: Sampling Operator
Sampling Operators
(Irregular Vs. Regular)
Non-uniform sampling and its spectra (Continuous signal)

\[ s(t) = \frac{T}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} \delta(t - kT - u[l] \frac{T}{N}) \quad 0 \leq t \leq T \]

\[ S(\omega) = \frac{T}{2\pi} \int_{t=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} \delta(t - kT - u[l] \frac{T}{N}) e^{-j\omega t} dt \]

Doing some math tricks

\[ F_s(\omega) = N \sum_{n=-\infty}^{\infty} F(\omega - \frac{n2\pi}{T}) Q(\frac{n2\pi}{T}) \]

\[ Q(\omega) = \frac{1}{N} \sum_{l=0}^{N-1} e^{-j\omega u[l]T/N} \]

\[ F_s(\frac{m2\pi}{T}) = N \sum_{n=-\infty}^{\infty} F(\frac{m2\pi}{T} - \frac{n2\pi}{T}) Q(\frac{n2\pi}{T}) \]
Non-uniform sampling operator for discrete signals

Original signal
\[ X = X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi nk/N} \quad k = 0, 1, 2, \ldots, N - 1 \]

\[ \mathcal{H} = \{ h(0), h(1), h(2), \ldots, h(M - 1) \} \]

Sampled signal
\[ x_s = x_s(n) = \begin{cases} x(n) & n \in \mathcal{H} \\ 0 & n \notin \mathcal{H} \end{cases} \]

\[ X_s = X_s(k) = \sum_{n=0}^{M-1} x(h(n)) e^{-i2\pi h(n)k/N} \quad k = 0, 1, 2, \ldots, N - 1 \]

Sampling operator
\[ Q = Q(k) = \frac{1}{N} \sum_{m=0}^{M-1} e^{-i2\pi h(m)k/N} \quad k = 0, 1, 2, \ldots, N - 1 \]

Convolution
\[ X_s = X \ast Q \]
Spectrum of regular sampling operators with various decimations
(from top to bottom: decimation factors equal to 1, 2, 3, 4 and 5)
Spectrum of random sampling operators with 50% missing samples
Spectrum of random sampling operators with 90% missing samples
Spectrum of sampling operators with Gaps
(from top to bottom: big gaps to small gaps)
Synthetic Examples
1D example: random sampling

Spatial domain

Original

Missing

Reconstructed

Fourier domain

b)

c)

d)

e)

f)
1D example: regularly missing samples
(every other traces)
1D example: random sampling from regularly missing signal (every other trace)
1D example: random sampling from regularly missing signal (decimation factor equal to 3)
1D example: random sampling (with the distance between two consecutive samples is always bigger than 4).
1D example: Error Vs. Percentage of available samples (for random sampling operators)
2D example: Irregular sampling

T-X Domain

(a) Original

(b) Missing

(c) MWNI-reconstructed
2D example: Irregular sampling

F-K Domain

(a) Original
(b) Missing
(c) MWNI-reconstructed
2D example: Regular sampling

(a) Regularly missing traces

(b) MWNI-Reconstructed
2D example: Regular sampling

F-K Domain

Normalized wave-number

Normalized frequency

(c) Regularly missing traces

(d) MWNI-Reconstructed
2D example: Gap example

Original

Missing

MWNLI-reconstructed
2D example: Gap example

F-K Domain

Original

Missing

MWNl-reconstructed
Original 3D data with linear events
Randomly missing 90% of 3D data
MWNI result for randomly missing 90% of 3D data
Regularly missing 50% of 3D data
MWNI result for regularly missing 50% of 3D data
Original 3D data with mild hyperbolic events
Randomly missing 75% of original data
3D Gulunay’s f-k interpolation
19 shot gathers from the Gulf of Mexico

91 traces per shot
4 shots of the original shots
Regularly decimated shots with factor of 4
Non-zero traces of the regularly decimated data
MWNI reconstruction of the regularly decimated data
Randomly eliminating 75% of the traces
Non-zero traces of the randomly eliminated data
Sparse reconstruction of the randomly eliminated data
Conclusions:

- Random sampling causes low amplitude noises in the spectrum domain and therefore sparse reconstruction methods perform successfully (if signal is Sparse).
- With the regular sampling, the resulted artifacts are as high amplitude as the original spectrum of signal. Hence imposing sparseness constraints can not recover missing samples.
- Sparse reconstruction can reconstruct big gaps to an acceptable level of accuracy.
- MSAR reconstruction can overcome the failure of sparse reconstruction methods for regularly or almost-regularly sampled data (Our next talk).
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Regularized Inversion (quadratic cost function):

\[ d = Gm \quad \text{Ill-posed linear system} \quad G : \text{Inverse DFT} \]

\[ J = \| C_n^{-\frac{1}{2}} (d - Gm) \|^2_2 + \| C_m^{-\frac{1}{2}} (m - m_0) \|^2_2 \quad \text{Cost function} \]

\[ \hat{m} = \left( G^H C_n^{-1} G + C_m^{-1} \right)^{-1} \left( G^H C_n^{-1} d - C_m^{-1} m_0 \right) \quad \text{Solution} \]

\[ C_n^{-1} = \frac{1}{c^2} W \quad C_m^{-1} = \sigma_m^2 I \quad m_0 = 0 \]

\[ \hat{m} = (G^H W G + \frac{c^2}{\sigma_m^2} I)^{-1} G^H W d \quad \text{Minimum Norm Solution} \]

\[ \lambda = \frac{c^2}{\sigma_m^2} \quad W = I \]

\[ \hat{m} = (G^H G + \lambda I)^{-1} G^H d \quad \text{This solution is more familiar} \]
Regularized Sparse Inversion:

Cost function

\[ J = \frac{1}{c^2} \| W^{\frac{1}{2}} (d - Gm) \|_2^2 + \rho(m) \]

Solution

\[ \hat{m} = (G^H W G + c^2 C_m^{-1})^{-1} G^H W d \]

\[ C_{m,ii}^{-1} = \frac{1}{m_i} \frac{\partial \rho(m_i)}{\partial m_i} \]

Cauchy norm

\[ \rho(x) = \frac{1}{2} \ln(1 + x^2) \]

\[ c^2 C_m^{-1} = c^2 C_{m,ii}^{-1} = \frac{c^2}{m_i^* m_i + \sigma_m^2} = \lambda \frac{1}{1 + \frac{m_i^* m_i}{\sigma_m^2}} \]

\[ \sigma_m^2 \leq m_i^* m_i : \text{Less damping (Faithful to available data)} \]

\[ \sigma_m^2 \geq m_i^* m_i : \text{More damping (Constrained to model data (zero))} \]