Seismic data interpolation and de-noising in the frequency-wavenumber domain

Mostafa Naghizadeh†

ABSTRACT
I introduce a unified approach for de-noising and interpolation of seismic data in the frequency-wavenumber ($f-k$) domain. First an angular search in the $f-k$ domain is carried out to identify a sparse number of dominant dips, not only using low frequencies but over the whole frequency range. Then, an angular mask function is designed based on the identified dominant dips. The mask function is utilized with the least-squares fitting principle for optimal de-noising or interpolation of data. The least-squares fit is directly applied in the $t-x$ domain. The proposed method can be used to interpolate regularly sampled data as well as randomly sampled data on a regular grid. Synthetic and real data examples are provided to examine the performance of the proposed method.

INTRODUCTION
Interpolation and noise attenuation of seismic records are considered key steps in seismic data processing methods, as interpolated and de-noised seismic records can increase the resolution of the migrated seismic images. This can be achieved by obtaining seismic records which have better event coherency, as well as reduced random and coherent seismic noise. Despite the obvious differences in the outputs of interpolation and noise attenuation methods, they are nearly identical operations with the sole purpose of creating a cleaner final stacked image. The similarities between de-noising and interpolation methods are evident when using the Fourier transform. The Fourier transform provides the possibility of looking for sparse dominant energy harmonics of data, which often contain the desired seismic signals (Naghizadeh and Sacchi, 2010). Retrieving a sparse Fourier representation of data is a common goal for both de-noising and interpolation methods.

The frequency-space ($f$-$x$) domain methods comprise a large group of seismic data interpolation and de-noising methods. The seismic data are transformed from time-space ($t$-$x$) domain to the $f$-$x$ domain by applying Fourier transform along the time axis. Prediction filters are used by Canales (1984) in the $f$-$x$ domain for seismic data de-noising. Other methods, such as projection filters (Soubaras, 1994); Singular Value Decomposition (Trickett, 2003); Cadzow de-noising (Cadzow, 1988; Trickett and Burroughs, 2009); and Singular Spectrum Analysis (Oropeza and Sacchi, 2011) have also been used for random

*This paper was presented in “SPNA 2: Seismic Interpolation and Regularization” session at the 80th Annual Meeting of the SEG in Denver, Colorado.
noise attenuation in the $f$-$x$ domain. Also, the current industrial multidimensional interpolation methods such as minimum weighted norm interpolation (MWNI) (Liu and Sacchi, 2004; Trad, 2009), anti-leakage Fourier transform (ALFT) (Xu et al., 2005), projection on convex sets (POCS) (Abma and Kabir, 2005), and multicomponent matching pursuit algorithms (Ozbek et al., 2010; Ozdemir et al., 2010; Vassallo et al., 2010) are applied in the $f$-$x$ domain. However, one can also consider the latter methods as frequency-wavenumber ($f$-$k$) methods, since the final stage of interpolation is carried out for the spatial samples of each frequency slice by transforming them into the wavenumber domain. All of the $f$-$x$ de-noising and interpolation methods are based on the assumption that the spatial signals at each single frequency are composed of a sum of a limited number of complex harmonics. Another class of $f$-$x$ domain seismic-trace interpolation methods have been proposed by Spitz (1991), Porsani (1999), and Naghizadeh and Sacchi (2007, 2009). These methods utilize the low frequency portion of data for a robust and alias-free interpolation of high frequencies.

Another important transform that can be used to de-alias seismic records is the Radon transform. The data in the slant stack (or $\tau - p$) domain can be characterized based on their dip information. Nichols (1992) used the continuity of unaliased energy along a single slowness trace in the $\tau - p$ domain to de-alias band-limited seismic records. Herrmann et al. (2000) and Moore and Kostov (2002) have also explored the concept of high-resolution Radon transform (Sacchi and Ulrych, 1995) to obtain an alias-free and stable domain to represent seismic data. Recently, Wang et al. (2009) also introduced a greedy least-squares method which utilizes local high resolution Radon transform for interpolation of aliased seismic records. In addition, Guitton and Claerbout (2010) have utilized the pyramid transform for beyond alias interpolation of seismic records by searching for a small collection of plane waves across all frequencies.

The $f$-$k$ domain is another domain which has been frequently used for interpolation and de-noising purposes. The $f$-$k$ domain is obtained by applying Fourier transforms along both time and space axes of the seismic data. One of the prominent attributes of the $f$-$k$ domain is the separation of signals according to their dip. This property makes the $f$-$k$ domain a suitable option for ground-roll elimination due to the distinctive dip information of ground-roll and reflection seismic data. The separation of signals according to dip is also a prominent feature of the $\tau - p$ domain, however the $f$-$k$ domain is a simpler alternative with lower computational costs. The most commonly used $f$-$k$ domain de-noising method is the dip filtering method. Dip filters in the $f$-$k$ domain are used to diminish energies in specific dip ranges.

The spatial interpolation of seismic records can also be viewed as noise attenuation (or wavenumber deconvolution) problem in the $f$-$k$ domain. Depending on the sampling function, noise introduced by decimation of data in the $f$-$k$ domain can be considered as incoherent (for random spatial sampling) or coherent (for regular spatial sampling). Gulunay (2003) has introduced the $f$-$k$ equivalent of $f$-$x$ interpolation methods by creating a mask function from low frequencies. Zwartjes and Sacchi (2007) combined Gulunay’s $f$-$k$ interpolation method with sparse Fourier inversion (Sacchi et al., 1998) to interpolate irregularly sampled aliased seismic records. Schonewille et al. (2009) utilized dip information of the low frequencies in the $f$-$k$ domain to introduce an anti-alias ALFT interpolation method. Recently, Curry (2010) has introduced an $f$-$k$ interpolation method, using a Fourier-radial adaptive thresholding strategy, in an attempt to utilize the continuity of events along the
frequency axis. Also, Naghizadeh (2010) introduced a f-k domain method which utilizes information from all desired frequencies for robust interpolation or de-noising of seismic data. This paper is an expansion of the latter with more synthetic and real data examples.

The proposed f-k method in this paper consists of three steps. The first step is an angular search for a range of dips, over all frequencies or any arbitrary band of frequencies (not just low frequencies), to identify the dominant energy dips. The origin of angular rays is located on the origin of the f-k domain. Next, a mask function is designed on the f-k domain over the dominant energy dips. Finally, the components of the f-k domain data which fall under the mask function, are fitted to the available samples in the time-space (t-x) domain by a least-squares algorithm. Synthetic and real data examples are provided to illustrate the performance of the proposed method.

THEORY

Identifying dominant dips in the f-k domain

Suppose $d(t, x)$ is the data in the t-x domain and $D(\omega, k)$ is the f-k spectra of data obtained by applying 2D Fourier transform. The first step of the proposed method is an angular search over a range of dips in the f-k domain to identify dominant dips in the data. The origin of the angular rays is located on the origin of the f-k domain, $(\omega, k) = (0, 0)$. Due to the symmetrical property of the frequency axis in the f-k domain, we only use the positive frequencies to explain the methodology. Also, for theoretical simplicity, we use the concept of normalized frequencies and wavenumbers. Normalized frequency and wavenumber axes are obtained by considering $\Delta t = 1$ and $\Delta x = 1$, respectively. This leads to the ranges $0 < \omega < 0.5$ for normalized frequencies and $-0.5 < k < 0.5$ for normalized wavenumbers. A map of dominant dips is produced by summation along the angular rays

$$M(p) = \sum_{n=1}^{N_\omega} D(\omega_n, k = p\omega_n - \lfloor p\omega_n + 0.5 \rfloor),$$

where, $p$ is the slope of the summation path in the f-k domain. The parameter $n$ represents the index of normalized frequency and can include any interval of frequencies. The operator $\lfloor . \rfloor$ denotes the nearest smaller integer value or in other words “rounding towards $-\infty$”. Notice that equation 1 allows the ray path to wrap around the frequency axis in order to account for aliased data in the f-k domain. These aliased dips are defined by $p > 1$ and $p < -1$. Figure 1 shows a schematic representation of angular rays for dips equal to -1, 0, 1, 2, and 4. The range of $p$ values is determined based on the alias severity and slope limits of data and can be different from one case to another. This range can be divided into equal segments to obtain the $p$ values required for the angular search. It is important to chose a fine enough distance between consecutive $p$ values in order to capture the correct variation of the dominant slopes.

The peak values in the function $M(p)$ are indicators of the dips or slopes with dominant energy. A slope in the function $M(p)$ was considered a peak value if it was bigger than its two neighboring values. Several peak values can be identified in $M(p)$ but we need only keep those with the largest values. This can be achieved in two ways. The first approach is to keep the $L$ highest peak values. The alternative is to set a threshold value and keep...
all of the peak values which are larger than the chosen threshold value. Regardless of the
adopted approach, let’s assume that we have been able to identify $L$ dominant slopes with
the values equal to $p_1, p_2, \ldots, p_L$.

**Building a mask function after identifying dominant dips**

After identifying the dominant dips of function $M(p)$, we deploy straight lines along the
correspondent angles of the dominant dips in the $f$-$k$ domain. The goal here is to transfer
the dominant dips $p_1, p_2, \ldots, p_L$ into a 2D mask function with the size of the original data
in the $f$-$k$ domain. Initiating $H$ matrix with zeros, the expression

$$H(\omega_n, k = p_j \omega_n - \lfloor p_j \omega_n + 0.5 \rfloor) = 1, \quad \begin{cases} n = 1, 2, \ldots, N_\omega, \\ j = 1, 2, \ldots, L, \end{cases}$$

deployes values of 1 along the desired angles in the $f$-$k$ domain. Convolving $H$ with a 1D
box car function, $B(1, L_b)$, gives

$$W(\omega, k) = H(\omega, k) \circledast B,$$

where $L_b$ is the length of box function along the wavenumber axis, $W$ is the final mask
function, and $\circledast$ represents 2D circulant convolution. Notice that $B$ is a vector along the
wavenumber axis and does not have elements along the frequency axis. Therefore, the
convolution in expression 3 only widens the mask function along the wavenumber axis. In
other words, expression 3 is identical to a 1D convolving of each frequency slice with a box
function of length $L_b$. This helps to take into account the uncertainties involved in the
prediction of dominant dips. Also, any value larger than one in the final mask function $W$ must be set equal to 1. The designed mask function can serve to eliminate the unwanted artifacts (which fall under the zero values of the mask function) and preserve the original signal. The dip search strategy explained here can be easily incorporated into other Fourier reconstruction methods like MWNI, ALFT, and POCS to create anti-alias versions of the aforementioned interpolation methods.

**Least-squares de-noising and interpolation using the mask function**

The mask function, $W(\omega, k)$, is 1 for the signal portion of noisy data and 0 for elsewhere in the $f$-$k$ domain. The mask function can be deployed inside a least-squares fitting algorithm for optimal interpolation or de-noising of the data. Suppose $d_v$ and $D_v$ are the column-wise-ordered long vector of the 2D signal $d$ and its Fourier spectrum $D$, respectively. Then, a stable and unique solution can be found by minimizing the following cost function (Tikhonov and Goncharsky, 1987)

$$J = ||d_v - T F^H \Upsilon_w D_v||^2_2 + \mu^2 ||D_v||^2_2. \quad (4)$$

where $F^H$ is the inverse 2D Fourier transform, $\Upsilon_w$ is a diagonal matrix built from the column-wise-ordered long vector of mask function $W$, and $T$ is the sampling matrix which maps the fully sampled desired seismic data to the available samples. For de-noising problems, assuming that all data samples are available, the sampling matrix is equal to the identity matrix $T = I$. In expression 4, $\mu$ is the trade-off parameter between the misfit and model norm and is used to balance noise suppression and data fitting. The minimum of the cost function $J$ can be computed using the method of conjugate gradients (Hestenes and Stiefel, 1952). For de-noising purposes 1 or 2 iterations of conjugate gradients suffices.

**EXAMPLES**

**Synthetic de-noising example**

In order to examine the performance of the proposed method, a synthetic seismic section with 3 linear events was created. Figures 2a and 2b show the original synthetic data in the $t$-$x$ and $f$-$k$ domains, respectively. The original data was contaminated with random noise, with the signal to noise ratio (SNR) equal to 1, to obtain the noisy data in Figure 3a. Figure 3b shows the de-noised data using the proposed method. Figures 3c and 3d show the $f$-$k$ spectra of data in Figures 3a and 3b, respectively. Figure 4 shows the plot of $M(p)$ function computed from Figure 3c for the dip range $-8 \leq p \leq 8$ with the sampling interval of 0.08. One can clearly detect that there are distinctive peaks in Figure 4 that indicate the dip information of three linear events in the original data. Figure 5 shows the mask function built using the 3 dominant dips in Figure 4. It is clear that this mask function accurately matches the $f$-$k$ spectra of the original data in Figure 2b. Figure 6a shows the de-noised data using Canales (1984) $f$-$x$ de-noising method. Figure 6b shows the $f$-$k$ spectra
of Figure 6a. In contrast to the method being proposed here, the Canales f-x de-noising method is separately applied to each frequency. Therefore, in the presence of strong noise, it actually struggles to de-noise most of the frequencies. This shortcoming of the Canales f-x de-noising method is effectively overcome by using all frequencies to find the dominant dips in the data.

Synthetic interpolation example

The proposed method in this article can be used, without any changes, to interpolate seismic data. Figure 7a shows a section of seismic data after randomly eliminating almost 75% of the traces. Figure 7d shows the f-k representation of the section with missing traces. The artifacts have large amplitudes due to the high percentage of missing traces. Figures 7b and 7c show the reconstructed data using the proposed method and the MWNI method, respectively. Figures 7e and 7f show the f-k representation of the reconstructed data in Figure 7b and 7c, respectively. Despite the high percentage of missing traces, the dominant dips of the original data have been successfully restored. In contrast, the MWNI method seeks the high amplitude wavenumbers in each frequency slice independently and therefore unsuccessful in identifying and attenuating high amplitude artifacts.

Figure 8a shows the synthetic seismic section with a gap of traces in the middle of the
Figure 3: a) Noisy data obtained from original data in Figure 2a by adding random noise (SNR=1). b) De-noised data using the proposed method. c) and d) are the $f-k$ spectra of a and b, respectively.
Figure 4: The distribution of energy for dip range $-8 \leq p \leq 8$, computed from $f$-$k$ domain of noisy data (Figure 3b).

Figure 5: The $f$-$k$ domain mask function, $W$, obtained from the dominant dips in Figure 4.
Figure 6: a) De-noised data using Canales $f$-$x$ method. b) The $f$-$k$ spectra of a.
Figure 7: a) Data with missing traces. b) Reconstructed data using the proposed method. c) Reconstructed data using the MWNI method. d-f) are the f-k spectra of a-c, respectively.
section. Figure 8c shows the f-k spectra of data in Figure 8a. The gap of traces in the t-x domain produces repetition of the spectrum in the vicinity of the original spectrum of data. Figures 8b and 8d show the t-x and f-k domains of the reconstructed data using the proposed method, respectively. Figure 9a shows the synthetic data with missing traces on the sides of the section which require extrapolation. Figure 9b shows the reconstructed data using the proposed method. Figures 9c and 9d show the f-k spectra of the Figures 9a and 9b, respectively.

It is interesting to analyze the performance of the proposed method for the interpolation of noisy data. Figure 10a shows the data in Figure 2a after adding random noise with SNR = 2. Figure 10b shows the data after regularly setting to zero two traces out of three. Figure 10c shows the result of simultaneous interpolation and de-noising of data in Figure 10b. The interpolated section shows noise reduction as well as successful reconstruction of missing traces. Figures 10d-f show the f-k spectra of data in Figures 10a-c, respectively.

The f-k method proposed in this article can also be utilized to de-alias seismic data in the spatial direction. Figure 11a shows the section of seismic data after interleaving 4 zero traces between each pair of traces in Figure 2a. Figure 11c shows the f-k spectra of data in Figure 11a. It is clear that the regular interleaving of zero traces has produced replicas of the original spectrum of data. Figure 11b and 11d show the interpolated data using the proposed method in this paper and its f-k spectra, respectively. Note that in this example we have searched for the dominant dips in the range \(-1 \leq p \leq 1\) with the sampling rate of 0.02.

To apply the proposed method to data with curved seismic events, one needs to deploy a spatial windowing strategy with proper overlapped areas. Sufficiently small spatial windows satisfies the assumption of linear events. Figure 12a shows a synthetic seismic section composed of hyperbolic events with conflicting dips. Figure 12b shows the data after replacing every other trace of the original data with zero traces. Figure 12c depicts the reconstructed data using the proposed method. The reconstruction was applied with the spatial windows of 10 traces with the trace spacing equal to 25 m. The spatial windows were designed to have 6 overlapped traces with the adjacent spatial windows. Figures 12d-f show the f-k spectra of the data in Figures 12a-c, respectively. Notice that the events with strong curvatures occasionally show dimmer amplitudes in the reconstructed section. This problem might be alleviated by adapting a time windowing strategy in order to increase the sparseness in the wavenumber domain.

**Real data interpolation example**

Figure 13a shows a real shot gather from the Gulf of Mexico data set. The original data were interpolated using the proposed method in this paper and the result was shown in Figure 13b. The interpolation was carried out in small spatial windows (7 traces per window) with 3 trace overlaps between each consecutive spatial window (with the trace spacing of 100 m). Figures 13c and 13d show the f-k spectra of Figures 13a and 13b, respectively. Figures 14a and 14b show a time window of data in Figures 13a and 13b between 2 to 3 Seconds. The interpolated data show good continuity of events and preservation of amplitude variation with offset (AVO) responses.
Figure 8: a) Data with large spatial gap. b) Reconstructed data using the proposed method. c) and d) are the $f$-$k$ spectra of a and b, respectively.
Figure 9: a) Data with missing traces. b) Reconstructed data using the proposed method. c) and d) are the f-k spectra of a and b, respectively.
Figure 10: a) Data in Figure 2a after adding random noise with $SNR = 2$. b) Data with two missing traces between adjacent traces. c) Reconstructed and de-noised data using the proposed method. d)-f) are the $f$-$k$ spectra of a-c, respectively.
Figure 11: a) Data with missing traces. This section is obtained by adding 4 zero traces between each of the traces in Figure 2a. b) Reconstructed data using the proposed method. c) and d) are the f-k spectra of a and b, respectively.
Figure 12: a) Original data composed of hyperbolic events with conflicting dips. b) Data after replacing every other trace of original data with zero traces. c) Reconstructed data using the proposed method. d)-f) are the $f$-$k$ spectra of a-c, respectively.
Figure 13: a) Original shot gather from Gulf of Mexico. b) Interpolated shot gather using the proposed method. c) and d) are the f-k spectra of a and b, respectively.
Figure 14: a) Time window of original shot gather in Figure 13a. b) Time window of interpolated shot gather in Figure 13b.
DISCUSSION

One of the common assumptions when designing interpolation or de-noising methods is the linearity of seismic events (events with constant dips) in the t-x domain. This assumption is usually validated for non-linear seismic events by analyzing the data in small spatial windows. The presence of linear events in the t-x domain implies the concentration of energy, at a limited number of wavenumbers, at each given frequency in the f-k domain. This is the cornerstone of most de-noising (Canales, 1984; Soubaras, 1994; Trickett, 2003) methods which aim to eliminate the noise by preserving the limited number of wavenumbers with the highest energies at each frequency. The same principle is applied for interpolation methods (Spitz, 1991; Porsani, 1999; Gulunay, 2003; Naghizadeh and Sacchi, 2007, 2009) except that information is extracted from low frequencies. However, all of the aforementioned methods tend to ignore an important property of linear events. In the f-k domain linear events start from the origin (zero frequency and zero wavenumber) and spread according to their associated dips. This property states that one can look for a sparse number of dips by summing over all frequencies in order to bring more resolution to the de-noising and interpolation methods. The proposed method amply exploits this property by summing along angular rays associated with a range of dips.

The proposed method not only allows extraction of information from low to high frequencies but also from high to low frequencies. This, of course, is feasible if the stringent condition of dealing with linear events is satisfied. Also, this method offers more flexibility in choosing the band of frequencies to estimate the dominant dips. Therefore, By using any band of frequencies, one can estimate the location of a desired signal on other frequencies. A similar strategy has been previously adapted by Nichols (1992) and Moore and Kostov (2002) in the $\tau - p$ and Radon domains. In addition, the proposed method in this article can be easily extended to higher dimensions. This can be achieved by deploying angular search paths that start from the origin of multiple wavenumber axes and spread in the f-k data volume. While the generalization to the multidimensional case is straightforward, in practice the angular search in multidimensional cubes can be very demanding computationally.

The proposed method uses a least-squares fitting algorithm to directly match the dominant dips in the f-k domain to the data in the t-x domain. Notice that the mask function is quite strong and will eliminate energy which resides under its zero values. Therefore, any desirable event which might have a very low energy level and that can not be detected by the angular search will be eliminated. On the other hand, methods such as POCS and ALFT might be able to recover these low amplitude events in their later iterations. Also, notice that the proposed method in this article is only suitable for the data that are randomly sampled from a regular grid, while methods such as ALFT can handle purely irregular sampling scenarios. The proposed method in this paper also leads to an amplitude-preserved and artifact-free de-noised, or interpolated, seismic section. The direct mapping between the t-x and f-k domains avoids introducing noisy features which are akin to f-x methods. However, the strict linearity condition makes this method intolerant to any curved or dispersed events. Therefore, it is safer to apply this method on very small spatial and time windows. Nevertheless, the smoothing of the mask mentioned above will provide a certain amount of tolerance and robustness of the method in the presence of moderately curved events. For dispersive events one can also adopt a strategy which deploys independent dip searches for various frequencies ranges. Finally, note that the mask design strategy introduced in this article can be utilized for other seismic data processing tasks such as
bandwidth extension, coherent noise attenuation, and multiple subtraction.

CONCLUSIONS

In this article, a new \( f-k \) domain interpolation and de-noising method was introduced. In the first step, the proposed method entails an angular search in the \( f-k \) domain to determine the dips with dominant energy. Next, the identified dominant dips were used to design a mask function in the \( f-k \) domain. Finally, a least-squares fitting routine was deployed to map the Fourier coefficients under the mask function to the data in \( t-x \) domain. The proposed method is considered as a generalization of the \( f-x \) and \( f-k \) interpolation methods in order to fully exploit the information from all frequencies rather the just using the lower frequencies. The proposed method shows effective performances on synthetic and real data examples.

ACKNOWLEDGMENTS

I thank the sponsors of the Signal Analysis and Imaging Group (SAIG) at the University of Alberta. I would also like to thank Dr. Mauricio Sacchi for his inspiring discussions and useful comments.

REFERENCES

Tikhonov, A. N. and A. V. Goncharsky, 1987, Ill-posed problems in the natural sciences: MIR Publisher.