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Sampling Considerations for Band-limited Fourier Reconstruction of Aliased Seismic Data

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SUMMARY

The multidimensional discrete sampling theory is investigated from the interpolation point of view. We define a sampling operator and its associated wavenumber signature, the scaling function, and investigate sampling conditions for optimal band-limited Fourier reconstruction. We present a case where a 2D signal (band-limited in one spatial dimension) can be recovered by designing an acquisition grid that minimizes the mixing between the unknown spectrum of the well-sampled signal and aliasing artifacts. The analysis can be easily extended to higher dimensions and used to define potential strategies for acquisition-guided Fourier reconstruction.
Introduction

The sampling theorem plays an important role in signal processing. It defines the ideal conditions, known as Nyquist criteria, for the conversion of an analog signal to a discrete one (Unser, 2000). It is important to understand under which conditions a given discrete sampling function (Naghizadeh and Sacchi, 2008) can lead to accurate reconstruction strategies. In particular, band-limitation in one or more spatial dimensions is an important constraint for accurate seismic data reconstruction using Fourier methods (Duijndam et al., 1999; Liu and Sacchi, 2004). We investigate the sampling problem for a 2D spatial reconstruction problem and show how one can accurately reconstruct data using Fourier methods when sampling and spatial bandwidth are properly examined.

Multidimensional sampling function and Fourier reconstruction

We start our analysis by considering a 2D discrete signal. The analysis can be easily extended to the multidimensional case. The discrete signal, $u$, of size $N_x \times N_y$, lives on a regular lattice

$$u(n_x,n_y), \quad n_x \in 0:N_x-1, \quad n_y \in 0:N_y-1.$$  (1)

We consider that only $M$ grid points contain data. We indicate the location of the $M$ available samples by the set

$$H = \{(h_x(m), h_y(m)) | m \in 0:M-1\}.$$  (2)

where $h_x$ and $h_y$ indicate the location of the available samples in the 2D grid. The Discrete Fourier Transform (DFT) of the original data, $U$, is given by:

$$U(k_x,k_y) = \frac{1}{N_x \times N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} u(n_x,n_y) e^{-\frac{2\pi i n_x k_x}{N_x}} e^{-\frac{2\pi i n_y k_y}{N_y}}.$$  (3)

It is easy to show that the DFT of the data with missing observations is given by the following expression

$$U_s = U * Q,$$  (4)

where the symbol $*$ represents 2D convolution. The operator $Q$ is the 2D scaling function given by

$$Q(k_x,k_y) = \frac{1}{N_x \times N_y} \sum_{m=0}^{M-1} e^{-\frac{2\pi i m h_x(k_x)}{N_x}} e^{-\frac{2\pi i m h_y(k_y)}{N_y}}.$$  (5)

The scaling function $Q(k_x,k_y)$ is a 2D complex-valued function. In our numerical examples we will plot the amplitude of the scaling function. Figure 1a shows a 2D regular lattice where each grid point is occupied by an observation (black filled circles). Figure 1b shows the amplitude of the scaling function associated to Figure 1a. The scaling function is a single spike in the center of the 2D spectrum and therefore, $U_s = U$. In Figure 1c the data was decimated in the $Y$ direction (the symbol $\times$ indicates an empty grid point). Figure 1d shows the amplitude of the scaling function of the decimated data in Figure 1c. The scaling function now has two impulses located at the normalized wavenumbers $(k_x, k_y) = (0,0)$ and $(k_x, k_y) = (0,-0.5)$. Figure 1e shows the 2D grid but now decimated with a chessboard pattern. Figure 1f shows the scaling function of Figure 1e. In comparison to Figure 1d, the unit impulse at the normalized wavenumber $(0,-0.5)$ has moved to $(-0.5,-0.5)$.

This example serves to highlight an interesting phenomenon and a potential strategy to couple acquisition design with Fourier reconstruction methods. Band-limitation in one or more spatial dimensions is an important constraint for credible seismic data reconstruction using Fourier methods (Duijndam et al., 1999; Liu and Sacchi, 2004). It is also important to stress that the amount of spectral mixing (alias) for a given acquisition grid will depend not
only on the scaling function $Q(k_x, k_y)$ but also on the characteristics of the unknown well-sampled signal $U$. This is illustrated with the following example.

Figure 1: a) Full sampling function c) Decimation in $Y$ direction. e) Chessboard pattern sampling. b), d), and f) are the scaling functions of a), c), and e), respectively. In a), c), and e) filled circles and symbol $\times$ represent the available and missing samples, respectively. In a), c), and e) filled circles and symbol $\times$ represent the available and missing samples, respectively.

Figure 2a shows a synthetic cube of curved events that are aliased in the $Y$ direction and well-sampled in the $X$ direction. Figure 2b shows slice views of the original data at time 0.35 (s) (top view), $Y$ slice 17 (side view), and $X$ slice 21 (front view). Figure 2c shows the data after eliminating every other slice of data in the $Y$ direction. The sampling operator is equivalent to the one depicted in Figures 1c and 1d. Figure 2d shows the result of band-limited Fourier reconstruction by applying band-limitation on the normalized wavenumber in the $X$ direction with reconstruction band $k_x \in [-0.15, 0.15]$. Aliased energy is inside the band chosen for reconstruction and consequently the reconstruction failed. Figure 2e shows the cube of data with missing traces but this time we used the chessboard acquisition pattern (Figures 1c and 1d). Figure 2f shows the band-limited Fourier reconstruction (Duijndam et al., 1999) of the data in Figure 2e using the reconstruction band $k_x \in [-0.15, 0.15]$. For the chessboard pattern, the aliased energy falls outside the reconstruction band and, as a result, band-limited Fourier reconstruction can accurately recover the missing traces. Figures 3a, 3b, and 3c show the $f$-$k$ panel of front views in Figures 2b, 2e, and 2f, respectively.

Figure 4a shows slice views of 19 consecutive shots from the Gulf of Mexico. Figure 4b shows the $f$-$k$ panel of the front view (shot 10) in Figure 4a. There is a significant amount of aliased energy in the offset ($Y$) direction. On the other hand, there is less variability in the $X$ direction and therefore the $X$-axis is well-sampled. Figure 4c shows the data in Figure 4a after decimation using the chessboard pattern. Figure 4e displays the results of band-limited Fourier reconstruction of Figure 4c. The reconstruction was carried out in the intervals $k_x = [-0.2, 0.2]$ and $k_y = [-0.5, 0.5]$. Figures 4d and 4f show the $f$-$k$ panels of the front view of Figures 4c and 4e, respectively. It is clear now that the 3D gather was precisely recovered.
Conclusion

We have reviewed the sampling of discrete signals and examined the wavenumber domain signature of the sampling operator. It is well known that Fourier reconstruction methods heavily rely on band-limiting assumptions. Our numerical examples have shown that signals that are band-limited in one spatial dimension could be accurately reconstructed by Fourier methods provided that the acquisition is carried out on a chessboard pattern. The extension of this analysis to the 3D or 4D spatial cases should lead to a good understanding of the
conditions under which regularization methods will succeed or fail in recovering unrecorded data.

Figure 4: a) Original data from the Gulf of Mexico. b) Decimated data with a chessboard pattern. c) Reconstructed data using band-limited Fourier reconstruction. b), d), and f) are the f-k panels of the front views a), c), and e), respectively.

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References