V042

Robust Reconstruction of Aliased Data Using Autoregressive Spectral Estimates

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SUMMARY

Autoregressive modeling is used to estimate the multi-dimensional spectrum of aliased data. A region of spectral support is determined by identifying the location of peaks in the estimated spectrum of the data. This information is used to pose a Fourier reconstruction problem that inverts for a few dominant wavenumbers that are required to model the data. Synthetic and real data examples are used to illustrate the method. In particular, we show that the proposed method can accurately reconstruct aliased data and data with gaps.
Introduction

Spitz (1991) showed how one could extract prediction filters from spatial data at low frequencies to reconstruct aliased spatial data. This idea was expanded by Naghizadeh and Sacchi (2007) and used to reconstruct data with irregular distribution of traces on a grid. The latter is named Multi-Step Auto-Regressive (MSAR) reconstruction. The MSAR reconstruction method is a combination of a Fourier reconstruction method (Sacchi and Liu, 2004) and f-x interpolation (Spitz, 1991). MSAR can be summarized as follows:

1) The low frequency (unaliased) portion of data is reconstructed using Minimum Weighted Norm Interpolation (MWNI) (Liu and Sacchi, 2004).
2) Prediction filters of all frequencies are extracted from already regularized low frequency spatial data.
3) The estimated prediction filters are used to reconstruct the missing spatial samples in the aliased portion of the spectrum.

Stages 1) and 2) are estimation stages and 3) is the reconstruction stage. In this paper we propose a new and robust method to solve the reconstruction stage. In the original formulation of MSAR the reconstruction stage uses prediction filters harvested from low frequencies to reconstruct spatial data in the aliased band (Spitz, 1991). In this article we propose to use the autoregressive (AR) spectrum of data to define a region of spectral support. Once the region of spectral support (areas of unaliased energy in the f-k plane) is defined we turn the reconstruction problem into a Fourier reconstruction algorithm that solves for unknown spectral components using the least-squares method (Duijndam et al., 1999). Synthetic examples illustrate that the proposed method can handle gaps and extrapolation problems much better than our original formulation of MSAR (Naghizadeh and Sacchi, 2007).

Theory

Let \( d(x_h, f) \) represent data in the f-x domain where \( x_h \) indicates the given spatial positions. Suppose that a small band of low frequencies from \( f \in [f_{min}, f_{max}] \) has been regularized and we have access to equally spaced data \( x_n = (n - 1)\Delta x, n = 1, 2, \ldots, N \). In other words we used \( d(x_h, f) \) to estimate \( d(x_n, f) \) for the frequencies \( f \in [f_{min}, f_{max}] \). This is not a difficult task and can be achieved via various methods including MWNI (Liu and Sacchi, 2004). Now, the prediction filters of all frequencies can be estimated using the MSAR method (Naghizadeh and Sacchi, 2007) via forward and backward prediction equations

\[
d(x_n, f) = \sum_{k=1}^{M} P_k(\alpha f)d(x_{n+\alpha k}, f), \quad n = \alpha M + 1, \ldots, N, \quad (1)
\]

\[
d^*(x_n, f) = \sum_{k=1}^{M} P_k(\alpha f)d^*(x_{n+\alpha k}, f), \quad n = 1, \ldots, N - \alpha M, \quad (2)
\]

Where the symbol * indicates the conjugate, \( P_k \) is the k-th component of the prediction filter and \( \alpha = 1, 2, \ldots, \alpha_{max} \) is the step factor used to extract the prediction filter for frequency \( \alpha f \) from frequency \( f \). The parametric AR spectrum of the data can be computed from the prediction filter via the following expression (Marple, 1987):

\[
S_{\alpha k}(\kappa, f) = \frac{\sigma_e^2}{1 - \sum_{m=1}^{M} P_m(f)e^{-i2\pi mf}}^2, \quad -\frac{1}{2} \leq \kappa \leq \frac{1}{2}, \quad (3)
\]

where, \( \kappa \) is the normalized wavenumber and \( \sigma_e^2 \) is the noise variance. These formulas can be easily extended to the multidimensional case (Kumaresan and Tufts, 1981). The parametric AR spectrum (equation 3) of the data is a smoothly varying function with peak values at the dominant wavenumbers. While the location and number of peaks are accurate, spectral amplitudes are not a direct estimate of the amplitude of the signal in the f-x domain. The key
information needed for data reconstruction, however, is the location and number of dominant wavenumbers (Xu and Pham, 2004). A basic sample-by-sample comparison algorithm can identify the location of the peaks in the parametric AR spectrum. Considering a discrete axis of wavenumbers \( \kappa = (k_1, k_2, \ldots, k_q) \), the location of the spectral peaks are found by the following algorithm

\[
\Lambda(k_j, f) = \begin{cases} 
1 & \text{if } S_{AR}(k_{j-1}, f) < S_{AR}(k_j, f) > S_{AR}(k_{j+1}, f) \\
0 & \text{otherwise}
\end{cases}
\]  

(4)

Uncertainties in the estimation of the prediction filter are handled by widening the position of the spectral peaks via convolution with a 2D boxcar function: \( B(k, f) \ast \Lambda(k, f) \). Then the region of spectral support is defined by the set of wavenumbers \( k_j(f), j = 1, \ldots, NK(f) \), where \( B(k, f) \ast \Lambda(k, f) = 1 \). We can now consider the problem of finding the Fourier coefficients that represent the available data

\[
d(x_h, f) = \sum_{j=1}^{NK(f)} D(k_j(f), f)e^{ik_jx_h}.
\]  

(5)

The latter is solved by minimizing the cost function:

\[
J(f) = \sum_{h} \left| d(x_h, f) - \sum_{j=1}^{NK(f)} D(k_j(f), f)e^{ik_jx_h} \right|^2.
\]  

(6)

The minimization of equation (6) leads to the standard least-squares Fourier reconstruction solution (Duijndam et al., 1999). It is important to stress that we are inverting for a few wavenumbers and therefore, in general, the problem is over-determined and stable. The proposed algorithm can be summarized as follows:

1. Transform the data from \( t-x \) to \( f-x \) domain and reconstruct the low frequencies of spatial data using Fourier inversion.
2. Use the MSAR algorithm to extract the prediction filters of all frequencies from the low frequency reconstructed spatial data.
3. Compute the parametric AR spectrum of the data using expression (3).
4. Identify the location of spectral peaks and define the region of spectral support.
5. Solve Equation (6) to estimate \( D(k, f) \) and to predict data at new spatial locations.
6. Use the inverse Fourier to transform \( D(k, f) \) to \( d(x, f) \).
7. Transform the \( f-x \) data to the \( t-x \) domain.

Examples

Figure 1a shows an example of decimated data that, in addition, contains a gap. Figure 1e shows the \( f-k \) panel of Figure 1a. The decimation process causes overlapped repetition of the spectrum of the original data. The existence of a gap in the middle of the section produces small artifacts around the spectrum of the decimated data. Figures 1b, 1c, and 1d show the reconstructed data using MWN I, MSAR (original formulation), and MSAR with the reconstruction stage proposed in this paper, respectively. The \( f-k \) panel of Figures 1b, 1c, and 1d are depicted in Figures 1f, 21g, and 1h, respectively. It is interesting that MWN I is only able to recover the decimated data and still cannot fill the regularly missing traces. Meanwhile, the MSAR method (in the original form proposed by Naghizadeh and Sacchi, 2007) successfully fills the decimated data outside the gap area but struggles to recover the data in the gap. When the MSAR method, however, is used to define regions of spectral support, the method overcomes the shortcomings seen in Figures 1b and 1c. The spectrum of prediction filter and the region of support are shown in Figure 2 for the normalized frequency 0.3.
Figure 1 a) Section of data containing missing traces. b), c), and d) are the reconstruction of (a) using the MWNI, MSAR, and MSAR with the new proposed reconstruction, respectively. e), f), g), and h) are the f-k panel of a), b), c), and d), respectively.

Figure 2 Parametric AR spectrum computed from the prediction filter at normalized frequency 0.3. The region of spectral support (dashed line) computed by first identifying the spectral peaks and expanding each peak location via convolution with a boxcar function.

To continue testing the proposed algorithm in higher dimensions, we chose a real data set from the Gulf of Mexico. We picked 19 consecutive shots each with 91 traces. We have randomly eliminated 60% of the traces. Figure 3a shows the location of available and missing traces. The filled circles indicate the available traces and symbol × represents the missing traces. A window of data from shots 1488 to 1491 and from time 3.0 s to 5.0 s is chosen for illustration purposes. Figures 3b and 3c show the original and missing data. Figure 3d shows the result of the reconstruction using the spectrum of 2D prediction filters.

Conclusions
Parametric spectral analysis can accurately determine the location of dominant wavenumbers at each single frequency of the seismic data. The latter permits us to define regions of spectral support that can be used to define a stable and robust Fourier reconstruction. The proposed reconstruction method overcomes the shortcomings of ordinary Fourier reconstruction methods in the case of aliased and regularly decimated data. In addition, our examples show
that it improves the original MSAR method at the time of reconstructing large gaps of missing data.

![Source and receiver positions of the data from the Gulf of Mexico. Filled circles show the available traces and symbol × represents the missing traces. b) Original data. c) Section of missing data. d) Reconstructed data using the proposed method.](image)

**Figure 3** a) Source and receiver positions of the data from the Gulf of Mexico. Filled circles show the available traces and symbol × represents the missing traces. b) Original data. c) Section of missing data. d) Reconstructed data using the proposed method.

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**References**


