CISC 882, Class ??

Singular Value Decomposition of a 2x2 Matrix

Suppose that $A_{2\times 2}$ is a real matrix. It has a singular value decomposition (SVD) of the form

$$A = U_{2\times 2}\ \Sigma_{2\times 2}\ V_{2\times 2}^T \tag{3.1}$$

where U and V are orthogonal, and Σ is a diagonal matrix of the singular values, ordered as $\sigma_1 \geq \sigma_2 \geq 0$.

The matrices U and V do not necessarily have to be pure rotations,i.e., they may not have a unit determinant. Assume that U is a rotation and V is not; we will work with U, an interim rotation W, and an orthogonal correction matrix C such that

$$A = U\Sigma V^T = U\Sigma C^T W^T \tag{3.2}$$

The matrices in Equation 3.2 are:

$$U = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix} \tag{3.3}$$

$$W = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} C_\phi & -S_\phi \\ S_\phi & C_\phi \end{bmatrix} \tag{3.4}$$

$$C = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \tag{3.5}$$

Using Equation 3.2 we can expand $A^T A$ as

$$\begin{aligned} AA^T &= U\Sigma C^T W^T W C \Sigma^T U^T = U\Sigma\Sigma^T U^T \\ &= UDU^T = \begin{bmatrix} J_u & L_u \\ L_u & K_u \end{bmatrix} \end{aligned} \tag{3.6}$$

and likewise

$$\begin{aligned} A^T A &= WC\Sigma^T U^T U \Sigma C^T W^T = W\Sigma\Sigma^T W^T \\ &= WDW^T = \begin{bmatrix} J_w & L_w \\ L_w & K_w \end{bmatrix} \end{aligned} \tag{3.7}$$

Let the diagonal matrix of Equation 3.6 be

$$D = \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \tag{3.8}$$

Expanding UDU^T in terms of D and U gives

$$\begin{aligned} UDU^T &= U \left[\begin{array}{cc} C_\theta E & S_\theta E \\ -S_\theta F & C_\theta F \end{array} \right] \\ &= \left[\begin{array}{cc} C_\theta^2 E - S_\theta^2 F & C_\theta S_\theta (E - F) \\ C_\theta S_\theta (E - F) & S_\theta^2 E + C_\theta^F \end{array} \right] \\ &= \left[\begin{array}{cc} J_u & L_u \\ L_u & K_u \end{array} \right] \end{aligned} \quad (3.9)$$

and likewise

$$\begin{aligned} WDW^T &= W \left[\begin{array}{cc} C_\phi E & S_\phi E \\ -S_\phi F & C_\phi F \end{array} \right] \\ &= \left[\begin{array}{cc} C_\phi^2 E - S_\phi^2 F & C_\phi S_\phi (E - F) \\ C_\phi S_\phi (E - F) & S_\phi^2 E + C_\phi^F \end{array} \right] \\ &= \left[\begin{array}{cc} J_w & L_w \\ L_w & K_w \end{array} \right] \end{aligned} \quad (3.10)$$

The angles in Equation 3.9 and Equation 3.10 can be found using the trigonometric double-angle identities. For an angle ψ,

$$\begin{aligned} \cos(2\psi) &= \cos^2(\psi) - \sin^2(\psi) \\ \sin(2\psi) &= 2\cos(\psi)\sin(\psi) \end{aligned}$$

Using UDU^T from Equation 3.9,

$$\begin{aligned} J_u + K_u &= [C_\theta^2 + S_\theta^2](E + F) &&= (E + F) \\ J_u - K_u &= [C_\theta^2 - S_\theta^2](E - F) &&= \cos(2\theta)(E - F) \\ 2L_u &= [C_\theta S_\theta + C_\theta S_\theta](E - F) &&= \sin(2\theta)(E - F) \end{aligned} \quad (3.11)$$

so

$$\begin{aligned} \tan(2\theta) &= 2L_u/(J_u - K_u) \\ \equiv \theta &= 0.5 * \operatorname{atan2}(2L_u, J_u - K_u) \end{aligned} \quad (3.12)$$

and likewise

$$\begin{aligned} \tan(2\phi) &= 2L_w/(J_w - K_w) \\ \equiv \phi &= 0.5 * \operatorname{atan2}(2L_w, J_w - K_w) \end{aligned} \quad (3.13)$$

The diagonal elements of D can be found from UDU^T. Using Equation 3.11, we can find the sum and the differences of the elements of D. Assuming that $d_{11} \geq d_{22}$, i.e., that $E \geq F$,

$$\begin{aligned} J_u + K_u &= E + F \\ \sqrt{(J_u^2 - K_u)^2 + 4L_u^2} &= E - F \end{aligned}$$

so

$$E = 0.5 * \left(J_u + K_u + \sqrt{(J_u^2 - K_u)^2 + 4L_u^2} \right) \tag{3.14}$$

$$F = 0.5 * \left(J_u + K_u - \sqrt{(J_u^2 - K_u)^2 + 4L_u^2} \right) \tag{3.15}$$

and thus

$$\begin{aligned} \sigma_1 &= \sqrt{E} \\ \sigma_2 &= \sqrt{F} \end{aligned} \tag{3.16}$$

from which we assemble the singular-value matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \tag{3.17}$$

Next, we must find the correction matrix C that rotates or reflects W to V. (This is because our derivations have been in terms of the squares of the matrix A.) We can derive a matrix S from Equation 3.2

$$\Sigma = U^T AWC = (U^T AW)C = SC \tag{3.18}$$

The correction matrix C will be the diagonal matrix that is the signs of the diagonal elements of S, i.e., it maps S to Σ only by changing signs and not magnitudes. This can be written algebraically or in terms of the "sign" function as

$$\begin{aligned} C &= \begin{bmatrix} S_{11}/|S_{11}| & 0 \\ 0 & S_{22}/|S_{22}| \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{sgn}(S_{11}) & 0 \\ 0 & \mathrm{sgn}(S_{22}) \end{bmatrix} \end{aligned} \tag{3.19}$$

Finally, we form the orthogonal matrix V as

$$V = WC \tag{3.20}$$

which, with U and Σ, completes the singular-value decomposition of A.

Matlab Code for 2x2 SVD

```
function [U,SIG,V] = svd2x2(A)
% [U,SIG,V] = svd2x2(A) finds the SVD of 2x2 matrix A
% where U and V are orthogonal, SIG is diagonal,
% and A=U*SIG*V’

% Find U such that U*A*A’*U’=diag

Su = A*A’;
phi = 0.5*atan2(Su(1,2)+Su(2,1), Su(1,1)-Su(2,2));
Cphi = cos(phi);
Sphi = sin(phi);
U = [Cphi - Sphi ; Sphi Cphi];

% Find W such that W’*A’*A*W=diag

Sw = A’*A;
theta = 0.5*atan2(Sw(1,2)+Sw(2,1), Sw(1,1)-Sw(2,2));
Ctheta = cos(theta);
Stheta = sin(theta);
W = [Ctheta, -Stheta ; Stheta, Ctheta];

% Find the singular values from U

SUsum = Su(1,1)+Su(2,2);
SUdif = sqrt((Su(1,1)-Su(2,2))ˆ2 + 4*Su(1,2)*Su(2,1));
svals = [sqrt((SUsum+SUdif)/2) sqrt((SUsum-SUdif)/2)];
SIG = diag(svals);

% Find the correction matrix for the right side

S = U’*A*W;
C = diag([sign(S(1,1)) sign(S(2,2))]);
V = W*C;
```