Lecture 6:
Mixed Systems Modeling

M.G. Lipsett
Department of Mechanical Engineering
University of Alberta
http://www.ualberta.ca/~mlipsett/ENGM541/ENGM541.htm
(with thanks to Eric Jackson for additional material)

Solution Methods for Propagation Problems (Review)

- Summary
- Finite difference methods
- Errors introduced by discrete methods
- Step Size Extrapolation
- Recurrence formulae with higher-order truncation errors
  (Runge-Kutta)

Father Michael Heller (is) one of Europe's most-respected thinkers, a physicist, philosopher, cosmologist, mathematician, historian, Catholic priest -- and this year's winner of the $1.7-million Templeton Prize for spiritual study, the world's richest award given to an individual.

"Just as the initial conditions in an equation determine the solution, so it was with my life. It's been an interesting one, for sure."

- Ottawa Citizen, March 15, 2008
(bold italics added for emphasis)
**Mixed Systems**

- To this point, we have considered physical systems with only a single type of constitutive relationship (and energy function)
  - Mechanical (and structural):
  - Electrical:
  - Hydraulic:
  - Thermal:

- We will now consider systems that have combinations of elements, and conversion of energy through special types of elements between different parts of a combined system, also called a mixed system.

- First, however, a quick review of energy relationships…

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**Equilibrium Elements Revisited**

- In most equilibrium models, node variables represent flows. These flows go “through” the element. The loop variables act “across” the elements, according to the constitutive relationships.

<table>
<thead>
<tr>
<th>Systems</th>
<th>Effort (e)</th>
<th>Flow (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mechanical</td>
<td><em>Force</em> (node variable)</td>
<td><em>Velocity</em> (loop variable)</td>
</tr>
<tr>
<td></td>
<td><em>Torque</em> (node variable)</td>
<td><em>Angular velocity</em> (loop)</td>
</tr>
<tr>
<td>Electrical</td>
<td>Voltage</td>
<td>Current</td>
</tr>
<tr>
<td>Hydraulic</td>
<td>Pressure</td>
<td>Volume flow rate</td>
</tr>
<tr>
<td>Thermal</td>
<td>Temperature</td>
<td>Entropy change rate</td>
</tr>
<tr>
<td></td>
<td>Pressure</td>
<td>Volume change rate</td>
</tr>
<tr>
<td>Chemical</td>
<td>Chemical potential</td>
<td>Mole flow rate</td>
</tr>
<tr>
<td></td>
<td>Enthalpy</td>
<td>Mass flow rate</td>
</tr>
<tr>
<td>Magnetic</td>
<td>Magneto-motive force</td>
<td>Magnetic flux</td>
</tr>
</tbody>
</table>
Equilibrium Energy Storage

• In the case of an inductor:

  • The constitutive relationship is
  • The energy storage is

• In the case of a spring:

  • The constitutive relationship is
  • The energy storage is

Mechanical Elements

• Force is the node variable acting through a mechanical element (although it is an “effort”)
• Displacements (& derivatives) are the across variables (although velocity is a “flow”)

• This is different from the standard representation of across variables as efforts (voltage, temperature, pressure, …)
Dynamic Elements

• In an eigenvalue system, energy is being transferred within the system. In a linear eigenvalue system, no energy enters or leaves the system.
• Individual elements, however, may store or release energy.
• In a propagation problem, elements can store, release, or dissipate energy over time. Instantaneously, the power in does not necessarily equal power out (where power is the product of effort and flow)
• The loop law holds: the sum of loop variable changes around a loop sums to zero, although now derivative relationships come into play

Equilibrium Energy Dissipation

• Some elements dissipate energy
• Examples:
Example: Fluid Resistance in a Pipe

- The pressure drop from friction is

- The rate of energy dissipation is

Example: Fluid Resistance in a Pipe (2)

- At the front end of the element, power in is positive:

- At the back end of the element, power out is negative:

- \( P_f > 0 \), and since it acts in the opposite direction of flow \( Q \), energy is lost over time

- Power flows in the direction of flow
**Ideal System Elements**

- **A-type element:** energy equation is written in terms of *Across* variable
- **T-type element:** energy equation is written in terms of *Through* variable
- **D-type element:** **Dissipates** energy

<table>
<thead>
<tr>
<th>System type</th>
<th>Mechanical translational</th>
<th>Mechanical rotational</th>
<th>Electrical</th>
<th>Fluid</th>
<th>Thermal</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-type variable</td>
<td>Velocity, ( v )</td>
<td>Velocity, ( \Omega )</td>
<td>Current, ( i )</td>
<td>Pressure, ( P )</td>
<td>Temperature, ( T )</td>
</tr>
<tr>
<td>Elemental equations</td>
<td>( \frac{dv}{dt} = F )</td>
<td>( \frac{d\Omega}{dt} = \tau )</td>
<td>( \frac{di}{dt} = i )</td>
<td>( \frac{dP}{dt} = F )</td>
<td>( \frac{dT}{dt} = Q )</td>
</tr>
<tr>
<td>Energy stored</td>
<td>Kinetic</td>
<td>Kinetic</td>
<td>Electric field</td>
<td>Potential</td>
<td>Electric field</td>
</tr>
<tr>
<td>Energy equations</td>
<td>( \frac{1}{2}mv^2 )</td>
<td>( \frac{1}{2}I^2 )</td>
<td>( \frac{1}{2}C_i \left( Q_i^2 - \frac{1}{2}V_i^2 \right) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Portality**

- Devices can be referred to by their “portality”
- Energy storage and dissipation devices are one-port devices. They accept an effort or flow as input, and they output the complementary effort or flow.

- Current is input to a capacitor, causing charge build up and a change in output voltage
- Force is input to a mass, causing it to accelerate, with a change in velocity.
### Transducers and Transformers

- Transducers & Transformers are *two-port* devices. They always interconnect two effort/flow pairs.
- A Transformer is any device that transforms an energy flow from one effort/flow ratio to another, in the same medium, with little loss of energy.
- Examples include gears, levers, electrical transformers.
- Energy Flow Relations for an Electrical Transformer:

![Electrical Transformer Diagram]

### Mixed Systems

- Many real physical systems comprise subsystems
- Each subsystem is described by a single set of loop and node variable types (mechanical, thermal, etc.)
- Subsystems connect through particular elements that transform energy (sensors and actuators)
- Descriptions of subsystems and connections form the overall equations of motion
- We analyse each subsystem, then transform variables at the transforming elements (which typically have a well defined constitutive relationship)
- This transformation approach can be done for any relationship where there is energy flow in the system, and even for energy analogues such as money.
Mixed Technological Systems

- Generally, technological systems will have equations for different interactions with a (mostly) common set of variables
- Some examples:
  - Energy requirement (total energy)
  - Cost of parts, labour, etc. (total cost)
  - Time required for activities (total time for unit production)
  - Amount of product to be shipped (production rate, or mass of shipment, or amount of packaging material, or size of shipping area, or amount of waste that has to be disposed, or … )
  - Material transport distance (total kms, or shipping cost, or shipping time, or emissions, or…)
- The remainder of this lecture focuses on mixed physical systems
- But we remember that a system description can comprise a set of equations with different loop or node relationships

Generic Elements

- Regardless of the type of subsystem, there are common characteristics of elements
- Equilibrium: no change in energy storage or release, no dissipation:
- Note: These diagrams show the efforts and flows on the element, not the effort on the environment (nodes)
**Generic Elements (2)**

- Power out is a load when the dot product of through & across vectors is *negative*
- Dynamic conditions:
  1. Power out = power in:
  2. Power out < power in:
  3. Power out > power in:
  4. Power out + dissipation = power in:

**Mechanical Transformers**

- Kinematic relationships for conversion of one type of motion to another
- Example: linear to rotational (rack & pinion)

- Ideally,

- There may be an efficiency loss, e.g., viscous friction (proportional to velocity), in which case

- If there is slip, then
Transducers

- Transducers convert between different types of power, e.g:
  - electric motors
  - piezoelectric actuators
  - propellers/impellers
  - hydraulic cylinders
- Energy Flow Relations for a Hydraulic Cylinder
  - Simply converts between hydraulic pressure & flow and mechanical force & velocity based on the area of the cylinders.
    - Note that cylinders usually have different areas on their two sides.

Example: Electromechanical Elements

- Transmit power, and convert it to another form
- Through and across variables convert as well
- Rotary motors, generators, etc.
  - For positive values of all variables as shown, power in is electrical and power out is mechanical
  - For an ideal motor/generator, \( P_{\text{elec}} = P_{\text{mech}} \), which means
**Ideal Electric Motor**

- Most electric motor designs are inherently bidirectional - they can act as either motors or generators. However, they will usually be optimized for one purpose or the other.
- Each type of electric motor has a specific governing equation. An idealized electric motor is shown here:

![Ideal Electric Motor Diagram]

**Flow and Effort Relationships**

- The relationships between flow (through) and effort (across) variables are:
  
  ![Flow and Effort Relationships Diagram]

- Where $K_r$ is the coupling coefficient that converts an electrical “across” variable to a mechanical one. This relationship can be used to transform an equation in electrical variables to one in terms of mechanical variables (or vice versa)
- For transforming elements involving mechanical systems, be careful to use the right coupling coefficient, because the nature of “across” and “through” is different than “effort” and “flow.”
**Actuators & Sensors**

- Actuators and sensors are made from transducers & transformers. They transform one effort/flow pair to another.
- Sensors are transducers that ideally transmit no power
- They can be modeled as follows:

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**Actuators & Sensors (2)**

- Partial list of transducers & transformers

<table>
<thead>
<tr>
<th>In \ Out</th>
<th>Mechanical Translation</th>
<th>Mechanical Rotation</th>
<th>Fluid</th>
<th>Electrical/Magnetic</th>
<th>Thermal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mechanical Translation</td>
<td>Lever</td>
<td>Rack &amp; Pinion</td>
<td>Hydraulic Cylinder</td>
<td>Piezoelectric</td>
<td>Damper</td>
</tr>
<tr>
<td>Mechanical Rotation</td>
<td>Rack &amp; Pinion</td>
<td>Gears</td>
<td>Gyro</td>
<td>Propeller</td>
<td>Generator/Alternator</td>
</tr>
<tr>
<td>Fluid</td>
<td>Hydraulic Cylinder Wing</td>
<td>Hydraulic Motor</td>
<td>Impeller</td>
<td>Pressure Intensifier</td>
<td>Magneto-hydrodynamic</td>
</tr>
<tr>
<td>Electrical/Magnetic</td>
<td>Solenoid Linear Motor Piezoelectric</td>
<td>Electric Motor</td>
<td>Magneto-hydrodynamic</td>
<td>Transformer</td>
<td>Peltier devices Resistors</td>
</tr>
<tr>
<td>Thermal</td>
<td>Piston</td>
<td>Engine</td>
<td>Heat Pump</td>
<td>Peltier devices</td>
<td></td>
</tr>
</tbody>
</table>
Amplifiers & Modulators

- Amplifiers and modulators are *3-port* devices. Examples:
  - Operational Amplifiers
    - power supply and output are the power ports
    - the + and - input pair is the modulating input port
  - Electric Motor controllers (similar to Op Amp)
  - Hydraulic Servovalves
- Typically the modulating input is not affected by the power consumption of the output, i.e., its impedance is not affected by the impedance of the load connected to the output.

Hydraulic Servovalves

- Servovalves are variable resistance (or conductance) 3-port devices that regulate the flow or output pressure by varying their resistance.
- Consists of a control spool that is moved by an electromagnetic actuator, allowing change in output flow (or P)

(Moog Series 765)
Connecting Subsystems

- One type of physical system, connected by transducer element(s) to another type of physical system, can be expressed in terms of the other subsystem.

State equations can thus be expressed in terms of a single set of variable types.

Example: Electric Motor-Driven Mechanical System

- Consider a mixed system with electrical and mechanical elements.

We differentiate between the electrical subsystem and the mechanical subsystem. Note that a motor has both electrical and mechanical characteristics, which get allocated to the appropriate subsystem.
Example: Electric Motor-Driven Mechanical System (2)

- Choose loop variables and find the governing equation for the electrical subsystem
- Since N=1, there is only one node variable i; and voltages at nodes satisfy admissibility requirements
- Relate loop and node variables using constitutive relationships:
  - Voltage source:
  - Motor resistor:
  - Motor inductor:
  - Motor back emf:

Example: Electric Motor-Driven Mechanical System (3)

- Form loop equation:

- Substitute all but chosen variables to get

- Now, we consider the mechanical subsystem.
- Choose loop variable
- Confirm that admissibility is satisfied (shaft is a node and ground is a node)
- Choose node variables and write
Example: Electric Motor-Driven Mechanical System (4)

• Relate using constitutive relationships

• Substitute & gather terms to get

• Now we use the transducer relationships

• To relate the torque produced electrically to the torque produced mechanically

Example: Electric Motor-Driven Mechanical System (5)

• Electrical

• Mechanical

• Transform into the standard form of state variable equations using and as state variables

• By writing

• and
Example: Electric Motor-Driven Mechanical System (6)

Express in matrix form as

Define non-dimensional variables

And get

Example: Electric Motor-Driven Mechanical System (7)

Move state variable characteristic constants into the matrix:

and do row operations

to get
Example #2: Motor-Driven Pumping System

Example #2: Electrical Subsystem

- Choose node \(<\text{through}>\) variables

- Write admissibility relationships
Example #2: Mechanical Subsystem

- Choose node <through> variables
- Write admissibility relationships

Example #2: Hydraulic Subsystem

- Choose node <through> variables
- Write admissibility relationships
Example #2: Hydraulic Subsystem (2)

Example #2: Governing Equations

• The (nonlinear) governing equations are:
Summary

- Mixed systems have different subsystems that have particular loop and node relationships
- If the subsystems interact, then the subsystems are connected by elements that describe the interaction
- These interactions may be transformations of energy in the case of physical subsystems
- The mixed system may have equations that relate to the system variables, but not using physical relationships, but rather technological attributes such as duration, resource allocation, and cost

Break Time: Number Trivia

- $111,111,111 \times 111,111,111 = 12,345,678,987,654,321$
- There are about $170,000,000,000,000,000,000,000,000,000$ different ways of playing the first ten moves in a game of chess
- The number of legal positions in chess is estimated to be between $10^{43}$ and $10^{50}$, with a game-tree complexity of approximately $10^{123}$.
- The game-tree complexity of chess was first calculated by Claude Shannon as $10^{120}$, a number known as the Shannon number. Typically an average position has thirty to forty possible moves, but there may be as few as zero (in the case of checkmate or stalemate) or as many as 218. Source and further information: http://en.wikipedia.org/wiki/Chess
- Prof Jonathan Schaeffer, FRSC, of the U of A Department of Computing Science, solved the game of checkers in 2007. Checkers has 500 billion billion ($5 \times 10^{20}$) possible situations that could arise while playing. The solution involves backward search from all possible endgame results, and “best first” to prioritize forward searching various positions and lines of play from the current state of the board.
- If all the legos in the world were divided up evenly, we’d get 30 pieces each. (Our basement appears to contain half the world’s inventory.)