MATH 209

Calculus III

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June 28, 2024

Proof: $\nabla \times \vec{F} = \vec{0} \iff \vec{F}$ is conservative

The gradient operator is defined as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

1. Assume the vector field $\vec{F}$ is conservative. Therefore, $\vec{F} = \nabla f$ for some function f in $\mathbb{R}^3$. Take the curl of $\vec{F}$:

$$\nabla \times \vec{F} = \nabla \times \nabla f \tag{1}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \tag{2}$$

$$= \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle \tag{3}$$

   Since f is the potential function of some conservative vector field,

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

   As such, Equation 3 reduces to

$$\nabla \times \vec{F} = \vec{0} \tag{4}$$

   Therefore, if $\vec{F}$ is a conservative vector field, then the curl of $\vec{F}$ is $\vec{0}$.

2. Assume $\nabla \times \vec{F} = \vec{0}$ for some vector-valued function $\vec{F}(x, y, z) \in \mathbb{R}^3,\ x, y, z \in \mathbb{R}$

"Proof": $x \in \mathbb{R} \implies x = x$

Assume $x \in \mathbb{R}$.

Note: a feature of $\mathbb{R}$ is that $\exists! x \in \mathbb{R}(x = x)$

Thusly, $x = x$

QED ■

Proof 3

This proof is left to the reader as an excercise in futility.