Research Proposal

Kinematical Symmetries in Field Theory

My research program deals with field theory and space-time (or kinematical) symmetries, such as Poincaré or Galilei, and their representations, as well as the related topics of Lie algebra contractions and deformations, with a view to concrete physical applications to low-energy systems. Specifically, my research projects aim at describing various non-relativistic systems via the explicit use of Galilean invariance. Although it was superseded by the Poincaré and Lorentz invariance as a fundamental symmetry of nature, the Galilei algebra and its representations is more intricate and very relevant for describing concrete physical systems. This intricacy is illustrated by the fact that the representation theory of the Galilei algebra was worked out in the early 1960s by Lévy-Leblond, even though the symmetry had been identified 300 years before. By comparison, the Poincaré invariance was identified in the late 19th century whereas Wigner worked out its representation theory a mere forty years later. Thus the existence of the Galilean symmetry is very well established, perhaps at a phenomenological rather than a fundamental level, but many of its consequences and applications remain unexplored. Galilean invariance is an active, yet not a mainstream, field of research, and I expect that a proper formulation of this symmetry will help describe non-relativistic phenomena and provide answers to open problems in low-energy physics and many-body systems. The development of a coherent approach is crucial to avoid incorrect conclusions that may result from the naive limits of Lorentz-covariant relativistic equations; e.g. there exist *not one*, *but two* Galilean limits of electromagnetism.

A well-known difficulty with non-relativistic theories is that they are not covariant, since space and time are not treated on an equal footing therein. We and many other authors circumvented this difficulty by defining Lorentz-covariant models on a 4+1 Minkowski manifold with light-cone coordinates, and then projecting these equations onto a 3+1 Galilean space-time. This covariant method is a helpful guide in formulating non-relativistic models for many-body systems. We will use this and other formalisms of Galilean invariance. In the recent years, my collaborators and I have investigated Galilean wave equations for arbitrary-spin fields, fluid models, weak-field gravitational systems with local Galilean covariance, Galilean electromagnetism, and Galilean non-linear equations admitting solitons, just to name a few examples. I plan to study applications of the Lévy-Leblond equation (Galilean version of the Dirac equation). We can also use this five-dimensional manifold in other contexts; for instance, we already applied it to general relativity in the context of brane-based solutions.

The research projects described hereafter revolve mainly around the study of (1) Galilean linear wave equations (à la Bhabha, Dirac, Duffin-Kemmer-Petiau) with various potentials, with and without gauge fields, in commutative and non-commutative spaces; (2) non-linear Galilean equations, integrability and solitons, and (3) other projects such as spin systems, branes in five dimensions and contractions of Lie algebras. My research program depends first and foremost on collaborations, internationally and with members of the U of A Physics Department, because my expertise lies in the Lie-algebras and their representations and appropriate expertise is required for specific physical applications. In the recent years, I have been fortunate to have collaborators and students visiting the University of Alberta; an NSERC grant would further invigorate my research program by allowing me to travel to meet with collaborators and attend conferences. In parallel with this proposal, I am a co-applicant for a separate NSERC Project renewal entitled *The Moedal Experiment at the Large Hadron Collider* (Principal investigator: James Pinfold, Physics, Univ. of Alberta). That proposal has no budget overlap whatsoever with the current proposal. [Hereafter, citations marked with **C** are the numbered publications listed in my CCV.]

Recent Progress

Even though Galilean invariance was superseded by Poincaré invariance as a fundamental symmetry for high-energy phenomena, the former remains present in many active fields of research, motivated by its capability to describe a wide range of low-energy phenomena (e.g. condensed matter physics, fluids and superfluids, nuclear physics, many-body theory, interior of neutron stars, etc.) or by its mathematical intricacies compared to Poincaré invariance. Indeed, in the recent literature only, we can find several investigations pertaining to Galilean invariance, whether in relation to conformal symmetry **[1]**, gravitation and cosmology **[2]**, particle physics **[3]**, condensed matter physics **[4]**, non-commutative geometry **[5]**, supersymmetry **[6]**, representations of the Galilei algebra and its generalizations **[7]**, just to mention a few.

For the past fifteen years, my research has dealt largely with *Galilean covariance*: a description of Galilean, non-relativistic, field theories based on metric tensor methods, as its Lorentz relativistic analogue. In fact, a "Galilean covariant" theory is a Lorentz-covariant theory defined on a 4+1 space-time with light-cone coordinates; then, after a reduction onto a 3+1 space-time, this model describes Galilean systems. In principle, this unifying approach will allow us to apply the elegant and efficient methods of relativistic field theory to non-relativistic problems, hence leading to applications of some high-energy field-theoretical methods to condensed matter physics, nuclear physics, many-body problems, collective phenomena, and so on. My general motivation for studying Galilean invariant field theory is twofold: (1) to construct new models to describe concrete low-energy physical systems (as in Ref. **[8]**), and (2) to clarify some misconceptions that may result from naive low-velocity limits.

As far as I know, Susskind was the first to observe that the Galilean symmetry manifests itself via a dimensional reduction from a relativistic system, in his investigation of the infinite-momentum frame **[9]**. Many authors, such as Duval, Künzle, Horváthy, Bergman and Thorn, Nikitin, and Takahashi, have utilized this dimensional-reduction approach with various perspectives; typically, their interest consisted in the symmetries and other mathematical aspects. Instead, my purpose is oriented to applications such as field interactions and scattering processes, which can be applied and compared with experiments. My initial motivation for studying Galilean covariance stemmed from papers by Takahashi and his collaborators, whose work on Galilean covariance was intended to propose new models of fluids, superfluids and many-body systems **[8]**.

My collaborators, students, and I have exploited Galilean covariance to revisit Lorentz-covariant results and to analyze their Galilean versions: linear wave equations with arbitrary-spin fields in commutative **[C5,10]** and non-commutative **[C7,C8]** spaces, the spin-statistics connection for Galilean fields **[C11]**, fluid and superfluid models **[11]**, Fokker-Planck dynamics **[12,13]**, general Lagrangian formalism and Galilean electrodynamics **[13-17]**, the canonical and path-integral approaches to quantization **[18,19]**, and gravitational theories in the weak-field approximation **[C14]**. We also used the five-dimensional manifold in a non-Galilean context when describing brane-world spherically symmetric solutions for the classic tests of general relativity **[C4]**. This Lorentz-like tensor approach was quite helpful for describing the *two* Galilean limits of electrodynamics **[14,15]**, and it is very promising for the analyses of further interactions expressed in a covariant form. I should mention that the Galilean version of a relativistic model does not only describe the corresponding physical counterpart. For example, our preliminary work on Galilean general relativity models suggests applications to condensed matter physics, rather than to gravitational theories.

Objectives

The bulk of the proposed research program consists in using Galilean invariance for constructing and solving the Galilean Dirac (or Lévy-Leblond) and Duffin-Kemmer-Petiau (DKP), or Bhabha, equations with several potentials, within commutative and non-commutative geometry. The DKP equation, introduced in the 1930s in order to describe mesons, is a first-order linear wave equation for spin-zero and spin-one particles analogous to the Dirac equation. My collaborators and I have already constructed Galilean DKP and Dirac equations in Ref. **[10]**. My former PDF (postdoctoral fellow) Esdras Santos and his collaborators have pursued the study of Galilean DKP equation in Ref. **[20]**. Recently, we have started to work on non-trivial potentials and non-commutative spaces: in Refs. **[C1, C2]**, we have solved the Dirac and DKP equations, respectively, for the oscillator in a non-commutative space. This segment of my research program will be done in collaboration with Brazilian researchers such as E. Santos (now professor in Bahia) and my recent PDF Genilson de Melo. These projects are well defined, consistent with my long-term research program and accessible to a graduate student or PDF. I expect to submit three to four manuscripts on that topic for publication within the next two years. We will begin with commutative spaces, and solve these Galilean wave equations with various potentials (e.g. central, Coulomb, (deformed) Hulthen, step-potential, deformed oscillator, etc.). The priority will be given to concrete physical applications. This will require financial support for an MSc student (a potential candidate, Gustavo Petronilo from Bahia, already contacted me) and for collaborative travel in both ways: Brazil and Canada.

Another objective is to make use of Galilean covariance with our works in **[C12,21]**, and construct new models for spin systems and magnetization. My main collaborators for this are Fuad Saradzhev and E. Santos, and it would be suitable for a graduate student. Motivated by emails from an author of Ref. **[22]**, we will exploit the method of **[21]** for the problem of "diffusive spin transfer" in spintronics **[22]**. An undergraduate student could help with numerical computations. The priority of these projects will be dictated by the pertinence to concrete applications.

Throughout my five-year grant cycle, I also plan to return to my long-standing interest in contractions of Lie algebras. I firmly believe that the most significant accomplishments of these limit procedures are yet to come! I plan to exploit contraction techniques for boson- and fermion-realizations of Lie algebras and introduce contraction parameters within the products of creation and annihilation operators, which are of interest, for instance, in the theory of coherent states or with algebraic models of rotation-vibration spectra of molecules and nuclei (e.g. interacting boson model).

Literature Review

The fact that Eugene Wigner classified the unitary irreducible representations of the Poincaré group in 1939 and recognized their critical importance in quantum theory **[23]**, almost 25 years before Lévy-Leblond performed an equivalent study for the Galilei group **[24]**, although Galilean invariance was recognized centuries prior to Poincaré invariance, indicates that the structure of Galilean kinematics is more intricate than relativistic kinematics. In an elegant paper, which deeply influenced my long-term research interests, Bacry and Lévy-Leblond classified the kinematical groups and utilized contractions to interconnect and interpret them physically **[25]**. The importance of studying Galilean invariance in its own right, as opposed to flippantly looking at it as the $c \to \infty$ limit of relativity, is shown by Lévy-Leblond's argumentation that spin is *not* a purely relativistic concept since it can be entirely understood within a Galilean-invariant framework **[26]**. Another case in point is the existence of *two* Galilean limits of electrodynamics **[27]**. Both limits are used in engineering and fluid mechanics (see Refs. **[15,17]** and references therein). Other researchers who have investigated Galilean invariance (some with the extended space-time approach) include Duval and Künzle **[28]**, Kapuścik **[29]**, Horváthy et al. **[30]**, Elizalde and Gomis **[31]**, Nikitin and Niederle **[32]**, just to mention a few.

My research program will consist mainly in expanding our previous work on the Galilean DKP and Dirac equations **[10,C7,C8]**. The Dirac equation is well known, but for the DKP equation, let me mention the excellent references [1-7] in Ref. **[33]**. The Galilean version of the Dirac equation is sometimes called “Lévy-Leblond equation”, after Jean-Marc Lévy-Leblond’s paper **[34]**, which we formulated with Galilean covariance in Ref. **[14]**. The systems described below were mostly in a relativistic context; we intend to revisit many of them with Galilean covariance. General properties of the DKP equations and the spinless DKP boson in a central field have been discussed in Ref. **[35]**. In a post-mortem paper **[36]**, Gribov formulated an asymptotically free theory that contains both perturbative and non-perturbative phenomena, and he utilized the DKP formalism to replace the usual description of gluons and their interactions. The energy levels of massive spin-one DKP particles in a constant magnetic field were found in Ref. **[37]**. Boumali studied the spin-zero DKP field in a Aharanov–Bohm potential in Ref. **[38]**. The spin-zero DKP field in a deformed Hulthen potential was solved in Ref. **[39]**. I will definitely consider the Galilean DKP equation for spin-zero and spin-one with the Woods-Saxon potential (relativistic solutions in Ref. **[40]**). Cardoso et al. considered the non-minimal vector coupling of a DKP field and the confinement of massive bosons with a linear potential **[41]**. The Dirac equation with position-dependent mass for the Hulthen potential was solved with the asymptotic iteration method (AIM) in Ref. **[42]**. We will utilize the Nikifurov-Uvarov method **[43]** and AIM **[44]** to solve our equations for the energy eigenvalues and the corresponding eigenfunctions.

My collaborators and I studied spin Hamiltonians in **[C12]**, and magnetization damping of spin systems in **[21]**. In a phenomenological analysis of diffusive spin transfer, a group of experimentalists interpreted the action of the relaxing spin of the conduction electrons as an out-of-equilibrium perturbation to the ferromagnetic order parameter **[22]**. Their purpose was to show that the experimental spin-transfer is described by introducing a new Onsager cross-coefficient, which relates the ferromagnetic current to the spins of the conduction electrons.

General references about contractions are given in Ref. **[45]**. I expect that the opposite procedure of contractions, called “deformations” **[46]**, will help achieve a breakthrough by pointing towards new physical models with “simple” (in its Lie-algebraic sense) structure.

Methodology

Galilean covariance consists in writing non-relativistic field equations as Lorentz-like field equations in a 4+1 manifold with light-cone coordinates; a subsequent projection on a 3+1 space-time then leads to Galilean equations. The group-theoretical justification for this embedding of the 3+1 Galilean space-time into a 4+1 Lorentz manifold is that the Galilei algebra is a Lie subalgebra of the Poincaré algebra with one more dimension **[14]**. The method consists in replacing the 4+1 Lorentz metric, diag(+1,+1,+1,−1,+1), by the *Galilean metric* $\begin{pmatrix} 1_{3\times3} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$. Perhaps the simplest example is the reduction from the 4+1 relativistic Klein-Gordon equation to the 3+1 non-relativistic Schrödinger equation. The former equation, with Galilean metric on a complex scalar field ϕ, takes the form: $\partial_\mu \partial^\mu \phi - k^2\phi = \nabla^2\phi + 2\partial_4\partial_5\phi - k^2\phi = 0$ in 4+1 dimensions. If we take $k = 0$ and choose the field such that $\partial_4\phi = \partial_t\phi$ and $\partial_5\phi = \mathrm{i}m\phi$, then the previous equation leads to $\nabla^2\phi + 2m\mathrm{i}\partial_t\phi = 0$, which is the free Schrödinger equation, $\mathrm{i}\partial_t\phi = -\frac{1}{2m}\nabla^2\phi$, in 3+1 dimensions. The reduction to 3+1 space-time is a delicate aspect of the procedure, since it can lead also to relativistic equations. For the main part of my research program, we will construct and solve the Galilean Dirac and DKP equations with various potentials (described in the second paragraph of the literature review) in analogy with Refs. **[35]-[44]**, in (non-)commutative spaces, with the representations found in Refs. **[10,C7,C8]**. We will also study non-linear Galilean differential equations and the existence of solitons. We will also use the 4+1 manifold in a non-Galilean context, as we did in Ref. **[C4]**, and analyze brane-world solutions in relation with gravitation theories.

Also related to Galilean covariance is the diffusive spin transfer, described phenomenologically four years ago by means of a new Onsager cross-coefficient connecting the ferromagnetic current to the spins of electrons **[22]**. The lead author of Ref. **[22]** suggested to me that a proper explanation of their cross-coefficients could result from using the formalism in Ref. **[21]**. In connection with spin and magnetization, Saradzhev and I plan to express equations (e.g. the phenomenological Laudau-Lifschitz-Gilbert equation) in a Galilean covariant form, and thereby explore various generalizations.

The last component of my research program deals with contractions and deformations of Lie algebras. In order to contract boson- and fermion-realizations, we will employ constructions such as those given in Ref. **[47]**. Then the contraction parameters will be introduced directly within the creation and annihilation operators (and combinations thereof), thus describing at once contractions of the Lie algebras and their representations. Finally, let me point out recent interest in the Newton-Hooke kinematics; this "contraction cousin" of the Poincaré algebra may be seen as a Lie subalgebra of the de Sitter algebra with one additional dimension **[48]**. This suggests that an approach similar to Galilean covariance could be applied in this context too, with potential applications to cosmology.

Impact

The principal objective of my research proposal is to exploit a unified framework for non-relativistic and relativistic kinematics, and to broaden the applications of Galilean invariance. As illustrated in my literature review, many researchers have been working on *relativistic* Bhabha wave equations (Dirac, DKP) with various potentials; our non-relativistic treatment should attract the attention of these researchers. At some point, the DKP equation might even warrant an international workshop or conference. In many instances, the Galilean covariance formalism will provide the first fully non-relativistic treatment that would circumvent the need for (sometimes misleading, as mentioned above) low-velocity limits by using instead a Galilean approach throughout the whole calculations. Also, our analysis of the exact wave functions and energy levels of non-commutative Galilean wave equations will allow us to assess the effects of non-commutativity in a Galilean framework, and to compare those effects with their relativistic analogues.

The community of researchers working on symmetries and coherent states, and contractions/deformations of Lie algebras should appreciate our work on boson/fermion realizations. Finally, our original use of extended space-time to describe Newton-Hooke kinematics should attract the attention of cosmologists.

Conclusion

In this proposal, I summarized my proposed research program and its anticipated outcomes. My interest for symmetry and phenomenology stems from my wide-ranging training (MSc on phenomenology of leptoquarks, and PhD on contractions of Lie algebras). I reiterate that collaborations will be crucial, and may even at times reorient my priorities. I have a strong interest in developing international collaborations. My research program is dynamic and appropriate for the training of HQP. I respectfully request that the selection committee strongly support my proposal, so that NSERC will continue to provide support of paramount importance to help me achieve my research goals.

References

[1] A. Bagchi, R. Basu, D. Grumiller, M. Riegler, *Phys. Rev. Lett*. **114** (2015) 111602; N. Aizawa, Z. Kuznetsova, F. Toppan, *J. Math. Phys*. **56** (2015) 031701; I. Masterov, *J. Math. Phys*. **56** (2015) 022902; A. Bagchi, R. Basu, A. Mehra, *JHEP* **11** (2014) 061; M. Henkel, A. Hosseiny, S. Rouhani, *Nucl. Phys. B* **879** (2014) 292; S. Chakraborty, P. Dey, *Mod. Phys. Lett. A* **28** (2013) 1350176; A. Galajinsky and I. Masterov, *Nucl. Phys. B* **866** (2013) 212; A. Bagchi, *JHEP* **5** (2013) 141; J. Gomis and K. Kamimura, *Phys. Rev. D* **85** (2012) 045023

[2] R. Banerjee, A. Mitra, P. Mukherjee, *Class. Quant. Grav*. **32** (2015) 045010; C. Duval, G.W. Gibbons, P.A. Horvathy, *Class. Quant. Grav*. **31** (2014) 085016; N. Bartolo, E. Dimastrogiovanni, M. Fasiello, *J. Cosmol. Astropart. Phys*. **9** (2013) 037; M. Peloso, M. Pietroni, *J. Cosmol. Astropart. Phys.* **5** (2013) 031; M. Setare and V. Kamali, *Class. Quant. Grav.* **28** (2011) 215004; A. Bagchi, *JHEP* **1102** (2011) 091; K. Hotta, T. Kubota, T. Nishinaka, *Nucl. Phys. B* **838** (2010) 358; G. Papageorgiou and B.J. Schroers, *JHEP* **1101** (2011) 20; H. Padmanabhan and T. Padmanabhan, *Phys. Rev. D* **84** (2011) 085018

[3] E. Braaten, *Phys. Rev*. D **91** (2015) 114007; K. Jensen, *JHEP* **4** (2015) 123

[4] R.A. El-Nabulsi, *Nonlin. Dyn*. **81** (2015) 939; S. Moroz, C. Hoyos, L. Radzihovsky, *Phys. Rev. B* **91** (2015) 195409 [erratum 199906]; S. Pieprzyk, D.M. Heyes, S. Mackowiak, A.C. Branka, *Phys. Rev E* **91** (2015) 033312; P. Strack, *Phys. Rev. E* **91** (2015) 032131; V.G. Mazauric, N. Addar, L. Rondot, P.F. Wendling, M.R. Barrault, *IEEE Trans. Magn*. **50** (2014) 7200804; Q. Huang, L.Z. Wang, S.F. Shen, S.L. Zuo, *Physica A Stat. Mec. Appl*. **398** (2014) 25; H.D. Chen, P. Gopalakrishnan, R.Y. Zhang, *Int. J. Mod. Phys. C* **25** (2014) 1450046; A. Sütö, *Phys. Rev. Lett*. **112** (2014) 095301; B.H. Wen, C.Y. Zhang, Y.S. Tu, C.L. Wang, H.P. Fang, *J. Comp. Phys*. **266** (2014) 161; P.J. Dellar, *J. Comp. Phys*. **259** (2014) 270; A. Duran, D. Dutykh, D. Mitsotakis, *Stud. Appl. Math*. **131** (2014) 359

[5] A. Ballesteros, F.J. Herranz, P. Naranjo, *Class. Quant. Grav*. **31** (2014) 245013 ; A. Ngendakumana, J. Nzotungicimpaye, L. Todjihounde, *Int. J. Geom. Meth. Mod. Phys.* 10 (2013) 1350049; M. Daszkiewicz, *Mod. Phys. Lett. A* **27** (2012) 1250083; L. Martina, *Theor. Math. Phys.* **167** (2011) 816; S. Kumar and S. Samanta, *Int. J. Mod. Phys. A* **25** (2010) 3221

[6] J. Lukierski, *Phys. Lett. B* **694** (2011) 478; P.A. Horvathy, M.S. Plyushchay and M. Valenzuela, *Ann. Phys.* **325** (2010) 1931, and *J. Math. Phys.* **51** (2010) 092108; J. Lukierski, I. Próchnicka, P.C. Stichel and W.J. Zakrzewski, *Phys. Lett. B* **639** (2006) 389

[7] M.I. Serov, T.O. Karpaliuk, O.G. Pliukhin, I.V. Rassokha, *J .Math. Anal. Appl*. **422** (2015) 185 ; A. Galajinsky, I. Masterov, *Nucl. Phys. B* **896** (2015) 244; M. Daszkiewicz, *Mod. Phys. Lett. A* **30** (2015) 1550034; R. Lu, V. Mazorchuk, K. Zhao, *J. Pure Appl. Alg*. **218** (2014) 1885

[8] Y. Takahashi, *Fortschr. Phys.* **36** (1988) 63; idem 83; M. Omote, S. Kamefuchi, Y. Takahashi and Y. Ohnuki, *Fortschr. Phys.* **37** (1989) 933

[9] L. Susskind, *Phys. Rev*. **165** (1968) 1535; J.B. Kogut and D.E. Soper, *Phys. Rev. D* **1** (1970) 2901

[10] M. de Montigny, F.C. Khanna, A.E. Santana, E.S. Santos and J.M.D. Vianna, *J. Phys. A: Math. Gen.* **33** (2000) L273**;** M. de Montigny, F.C. Khanna, A.E. Santana and E.S. Santos *J. Phys. A: Math. Gen.* **34** (2001) 8901

[11] M. de Montigny, F.C. Khanna and A.E. Santana, *J. Phys. A: Math. Gen.* **36** (2003) 2009; M. de Montigny, F.C. Khanna and A.E. Santana, *J. Phys. A: Math. Gen.* **34** (2001) 10921

[12] M. de Montigny, F.C. Khanna and A.E. Santana, *Physica A* **323** (2003) 327
[13] J.A. Cardeal, M. de Montigny, F.C. Khanna, T.M. Rocha Filho and A.E. Santana, *J. Phys. A: Math. Gen*. **40** (2007) 13467
[14] E. S. Santos, M. de Montigny, F.C. Khanna and A.E. Santana, *J. Phys. A: Math. Gen.* **37** (2004) 9771
[15] M. de Montigny, F.C. Khanna and A.E. Santana, *Int. J. Theor. Phys*. **42** (2003) 649
[16] M. de Montigny and G. Rousseaux, *Amer. J. Phys*. **75** (2007) 984
[17] M. de Montigny and G. Rousseaux, *Eur. J. Phys*. **27** (2006) 755
[18] E.S. Santos, M. de Montigny and F.C. Khanna, *Ann. Phys. (NY)* **320** (2005) 21; L.M. Abreu, M. de Montigny, F.C. Khanna and A.E. Santana, *Ann. Phys. (NY)* **308** (2003) 244
[19] M. de Montigny, F.C. Khanna and F.M. Saradzhev, *Ann. Phys*. **323** (2008) 1191
[20] L.M. Abreu, F.J.S. Ferreira, E.S. Santos, *Braz. J. Phys*. **40** (2010) 235; E.S. Santos and L.M. Abreu, *J. Phys. A: Math. Theor.* **41** (2008) 075407
[21] F.M. Saradzhev, F.C. Khanna, S.P. Kim and M. de Montigny, *Phys. Rev. B* **75** (2007) 024406
[22] J.E. Wegrowe, S.M. Santos, M.C. Ciornei, H.J. Drouhin and J.M. Rubí, *Phys. Rev. B* **77** (2008) 174408
[23] E. P. Wigner, *Ann. Math.* **40** (1939) 149
[24] J.M. Lévy-Leblond, *J. Math. Phys.* **4** (1963) 776; idem, in *Group Theory and its Applications*, E. Loebl Ed. (Academic Press, NY 1971)
[25] H. Bacry, J. M. Lévy-Leblond, *J. Math. Phys.* **9** (1968) 1605
[26] J.M. Lévy-Leblond, *Comm. Math. Phys.* **6** (1967) 286
[27] M. Le Bellac and J.M. Lévy-Leblond, *Nuov. Cim. B* **14** (1973) 217
[28] C. Duval, G. Burdet, H.P. Künzle and M. Perrin, *Phys. Rev. D* **31** (1985) 1841; C. Duval and H.P. Künzle, *Gen. Rel. Grav.* **16** (1984) 333; H.P. Künzle and C. Duval, in *Semantical Aspects of Spacetime Theories* (1994) Eds. U. Majer and H.J. Schmidt (Mannheim: BI-Wissenschaftsverlag) 113
[29] E. Kapuścik, *Acta Physica Polonica B* **17** (1986) 569; ibid. **12** (1981) 81; ibid. **16** (1985) 937
[30] C. Duval, G.W. Gibbons and P. Horváthy, *Phys. Rev. D* **43** (1991) 3907; C. Duval, P. Horváthy and L. Palla, *Phys. Lett. B* **325** (1994) 39; ibid *Ann. Phys. (NY)* **248** (1996) 265; P. Horváthy, *J. Phys. A: Math. Gen.* **34** (2001) 3079
[31] E. Elizalde and J. Gomis, *J. Math. Phys.* **19** (1978) 1790, and reference therein
[32] J. Niederle and A.G. Nikitin, *J. Phys. A: Math. Theor.* **42** (2009) 105207; ibid *J. Phys. A: Math. Theor*. **42** (2009) 245209; ibid *J. Phys. A: Math. Gen.* **13** (1980) 2319
[33] R.A. Krajcik and M.M. Nieto, *Am. J. Phys.* **45** (1977) 818
[34] J.M. Lévy-Leblond, *Comm. Math. Phys*. **6** (1967) 286
[35] Y. Nedjadi and R.C. Barrett, *J. Phys. G* **19** (1993) 87; ibid *J. Math. Phys.* **35** (1994) 4517
[36] V. Gribov, *Eur. Phys. J. C* **10** (1999) 71
[37] K. Sögüt, A. Havare and I. Acikgöz I, *J. Math. Phys.: Math. Gen.* **43** (2002) 3952
[38] A. Boumali, *Can. J. Phys.* **82** (2004) 67
[39] F. Yasuk, C. Berkdemir, A. Berkdemir and C. Onem, *Phys. Scr.* **71** (2005) 340
[40] B. Boutabia-Chéraitia and T. Boudjedaa, *Phys. Lett. A* **338** (2005) 97
[41] T.R. Cardoso, L.B. Castro and A.S. de Castro, *J. Phys. A: Math. Theor.* **43** (2010) 055306
[42] F. Yasuk, M.K. Bahar, *Phys. Scr.* **85** (2012) 045004
[43] C. Tezcan and R. Sever, *Int. J. Theor. Phys*. **48** (2009) 337; A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics* (1988) Basel: Birkhauser
[44] H. Ciftci, R.L. Hall, N.J. Saad, *J. Phys. A: Math. Theor.* **36** (2003) 11807; *ibid* **38** (2005) 1147; *ibid* **39** (2006) 13445
[45] E. Inönü and E.P. Wigner, *Proc. Nat. Acad. Sci. US* **39** (1953) 510; E.J. Saletan, *J. Math. Phys.* **2** (1961) 1; R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (1974) New York,

Wiley
[46] A. Fialowski and M. de Montigny, *J. Phys. A: Math. Gen*. **38** (2005) 6335
[47] R. Gilmore, D.H. Feng, *Prog. Part. Nucl. Phys*. **9** (1983) 479
[48] J. Brugués, J. Gomis and K. Kamimura, *Phys. Rev. D* **73** (2006) 085011; J. Gomis and J. Lukierski, *Phys. Lett. B* **664** (2008) 107; G.W. Gibbons and C.N. Pope, *Ann. Phys.* **326** (2011) 1760