Fast Wavelet/Framelet Transform for Signal/Image Processing.

The following is based on book manuscript: B. Han, Framelets and Wavelets: Algorithms, Analysis and Applications.

To introduce a discrete framelet transform, we need some definitions and notation. By $l(\mathbb{Z})$ we denote the linear space of all sequences $v = \{v(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ of complex numbers on $\mathbb{Z}$. One -dimensional discrete input data or signal is often treated as an element in $l(\mathbb{Z})$. Similarly, by $l_0(\mathbb{Z})$ we denote the linear space of all sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ on $\mathbb{Z}$ such that $\{k \in \mathbb{Z} \,:\, u(k) \neq 0\}$ is a finite set. An element in $l_0(\mathbb{Z})$ is often regarded as a finite-impulse-response (FIR) filter (also called a finitely supported mask in the literature of wavelet analysis). In this book we often use u for a general filter and v for a general signal or data. It is often convenient to use the formal Fourier series (or symbol) $\widehat{v}$ of a sequence $v = \{v(k)\}_{k\in\mathbb{Z}}$, which is defined as follows:

$$\widehat{v}(\xi) := \sum_{k\in\mathbb{Z}} v(k)e^{-ik\xi}, \qquad \xi \in \mathbb{R}, \tag{1}$$

where i in this book always denotes the imaginary unit. For $v \in l_0(\mathbb{Z})$, $\widehat{v}$ is a 2π-periodic trigonometric polynomial.

A discrete framelet transform can be described using two linear operators—the subdivision operator and the transition operator. For a filter $u \in l_0(\mathbb{Z})$ and $v \in l(\mathbb{Z})$, the *subdivision operator* $\mathscr{S}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$ is defined to be

$$[\mathscr{S}_u v](n) := 2\sum_{k\in\mathbb{Z}} v(k)u(n-2k), \qquad n \in \mathbb{Z} \tag{2}$$

and the *transition operator* $\mathscr{T}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$ is defined to be

$$[\mathscr{T}_u v](n) := 2\sum_{k\in\mathbb{Z}} v(k)\overline{u(k-2n)}, \qquad n \in \mathbb{Z}. \tag{3}$$

The transition operator plays the role of coarsening and frequency-separating the data to lower resolution levels; while the subdivision operator plays the role of refining and predicting the data to higher resolution levels.

In terms of Fourier series, the subdivision operator $\mathscr{S}_u$ in (2) and the transition operator $\mathscr{T}_u$ in (3) can be equivalently rewritten as

$$\widehat{\mathscr{S}_u v}(\xi) = 2\widehat{v}(2\xi)\widehat{u}(\xi), \qquad \xi \in \mathbb{R} \tag{4}$$

and

$$\widehat{\mathscr{T}_u v}(\xi) = \widehat{v}(\xi/2)\overline{\widehat{u}(\xi/2)} + \widehat{v}(\xi/2+\pi)\overline{\widehat{u}(\xi/2+\pi)}, \qquad \xi \in \mathbb{R} \tag{5}$$

for $u, v \in l_0(\mathbb{Z})$, where $\overline{c}$ denotes the complex conjugate of a complex number $c \in \mathbb{C}$.

Let $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_s$ be filters for decomposition. For a positive integer J, *a J-level discrete framelet decomposition* is given by

$$v_j := \tfrac{\sqrt{2}}{2}\mathscr{T}_{\tilde{a}} v_{j-1}, \quad w_{\ell,j} := \tfrac{\sqrt{2}}{2}\mathscr{T}_{\tilde{b}_\ell} v_{j-1}, \qquad \ell = 1,\ldots,s, \quad j = 1,\ldots,J, \tag{6}$$

where $v_0 : \mathbb{Z} \to \mathbb{C}$ is an input signal. The filter $\tilde{a}$ is often called a dual low-pass filter and the filters $\tilde{b}_1, \ldots, \tilde{b}_s$ are called dual high-pass filters. After a J-level discrete framelet decomposition, the original input signal v_0 is decomposed into one sequence v_J of low-pass framelet coefficients and sJ sequences $w_{\ell,j}$ of high-pass framelet coefficients for $\ell = 1,\dots,s$ and $j = 1,\dots,J$. Such framelet coefficients are often processed for various purposes. One of the most commonly employed operations is thresholding so that the low-pass framelet coefficients v_J and high-pass framelet coefficients $w_{\ell,j}$ become $\mathring{v}_J$ and $\mathring{w}_{\ell,j}$, respectively. More precisely, $\mathring{w}_{\ell,j}(k) = \eta(w_{\ell,j}(k)), k \in \mathbb{Z}$, where $\eta : \mathbb{C} \to \mathbb{C}$ is a thresholding function. For example, for a given threshold value $\lambda > 0$, the hard thresholding function η_λ^{hard} and soft-thresholding function η_λ^{soft} are defined to be

$$\eta_\lambda^{hard}(z) = \begin{cases} z, & \text{if } |z| \geqslant \lambda; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta_\lambda^{soft}(z) = \begin{cases} z - \varepsilon\frac{z}{|z|}, & \text{if } |z| \geqslant \lambda; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Another commonly employed operation is quantization, which can be applied after or without thresholding. For example, for a given quantization level $q > 0$, the quantization function $\mathscr{Q} : \mathbb{R} \to q\mathbb{Z}$ is defined to be $\mathscr{Q}(x) := q\lfloor \frac{x}{q} + \frac{1}{2} \rfloor, x \in \mathbb{R}$, where $\lfloor \cdot \rfloor$ is the floor function such that $\lfloor x \rfloor = n$ if $n \leqslant x < n+1$ for an integer n.

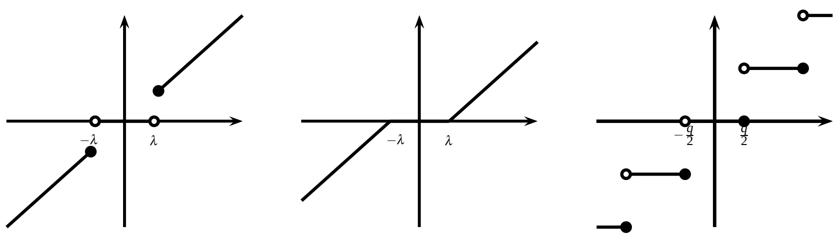

Figure 1: The hard thresholding function η_λ^{hard}, the soft thresholding function η_λ^{soft}, and the quantization function, respectively. Both thresholding and quantization operations are often used to process framelet coefficients in a discrete framelet transform.

Let $a, b_1,\dots,b_s$ be filters for reconstruction. Now *a J-level discrete framelet reconstruction* is

$$\mathring{v}_{j-1} := \frac{\sqrt{2}}{2}\mathscr{S}_a\mathring{v}_j + \frac{\sqrt{2}}{2}\sum_{\ell=1}^{s}\mathscr{S}_{b_\ell}\mathring{w}_{\ell,j}, \qquad j = J,\dots,1. \tag{8}$$

The filter a is often called a primal low-pass filter and the filters $b_1,\dots,b_s$ are called primal high-pass filters.

We say that $(\{\tilde{a};\tilde{b}_1,\dots,\tilde{b}_s\},\{a;b_1,\dots,b_s\})$ is a dual framelet filter bank if it satisfies the perfect reconstruction condition:

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}_1}(\xi) & \cdots & \widehat{\tilde{b}_s}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}_1}(\xi+\pi) & \cdots & \widehat{\tilde{b}_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix}^\star = I_2, \tag{9}$$

$\{a;b_1,\dots,b_s\})$ is called a tight framelet filter bank if $(\{a;b_1,\dots,b_s\},\{a;b_1,\dots,b_s\})$ is a dual framelet filter bank.

If $s=1$, a dual framelet filter bank $(\{a;b\},\{a;b\})$ is called a biorthogonal wavelet filter bank. If $s=1$, a tight framelet filter bank $\{a;b\}$ is called an orthogonal wavelet filter bank.

In the following, let us provide a few examples to illustrate various types of filter banks. For a filter $u=\{u(k)\}_{k\in\mathbb{Z}}$ such that $u(k)=0$ for all $k\in\mathbb{Z}\backslash[m,n]$ and $u(m)u(n)\neq 0$, we denote by $\operatorname{fsupp}(u):=[m,n]$ as its *filter support*. To list the filter u, we shall adopt the following notation throughout the book:

$$u=\{u(m),u(m+1),\ldots,u(-1),\underline{\mathbf{u(0)}},u(1),\ldots,u(n-1),u(n)\}_{[m,n]}, \qquad (10)$$

where we underlined and boldfaced the number $u(0)$ to indicate its position at the origin.

Example 1 *$\{a;b\}$ is an orthogonal wavelet filter bank (called the Haar orthogonal wavelet filter bank), where*

$$a=\{\underline{\mathbf{\tfrac{1}{2}}},\tfrac{1}{2}\}_{[0,1]}, \qquad b=\{\underline{\mathbf{-\tfrac{1}{2}}},\tfrac{1}{2}\}_{[0,1]}. \qquad (11)$$

Example 2 *$(\{\tilde{a};\tilde{b}\},\{a;b\})$ is a biorthogonal wavelet filter bank, where*

$$\tilde{a}=\{-\tfrac{1}{8},\tfrac{1}{4},\underline{\mathbf{\tfrac{3}{4}}},\tfrac{1}{4},-\tfrac{1}{8}\}_{[-2,2]},\ \tilde{b}=\{\underline{\mathbf{-\tfrac{1}{4}}},\tfrac{1}{2},-\tfrac{1}{4}\}_{[0,2]},$$
$$a=\{\tfrac{1}{4},\underline{\mathbf{\tfrac{1}{2}}},\tfrac{1}{4}\}_{[-1,1]},\ b=\{-\tfrac{1}{8},\underline{\mathbf{-\tfrac{1}{4}}},\tfrac{3}{4},-\tfrac{1}{4},-\tfrac{1}{8}\}_{[-1,3]}.$$

Example 3 *$\{a;b_1,b_2\}$ is a tight framelet filter bank, where*

$$a=\{\tfrac{1}{4},\underline{\mathbf{\tfrac{1}{2}}},\tfrac{1}{4}\}_{[-1,1]}, \qquad b_1=\{-\tfrac{\sqrt{2}}{4},\underline{\mathbf{0}},\tfrac{\sqrt{2}}{4}\}_{[-1,1]}, \qquad b_2=\{-\tfrac{1}{4},\underline{\mathbf{\tfrac{1}{2}}},-\tfrac{1}{4}\}_{[-1,1]}.$$

Example 4 *$(\{\tilde{a};\tilde{b}_1,\tilde{b}_2\},\{a;b_1,b_2\})$ is a dual framelet filter bank, where*

$$\tilde{a}=\{\underline{\mathbf{\tfrac{1}{2}}},\tfrac{1}{2}\}_{[0,1]}, \quad \tilde{b}_1=\{-\tfrac{1}{2},\underline{\mathbf{\tfrac{1}{2}}}\}_{[-1,0]}, \quad \tilde{b}_2=\{\underline{\mathbf{-\tfrac{1}{2}}},\tfrac{1}{2}\}_{[0,1]}$$

and

$$a=\{\tfrac{1}{8},\underline{\mathbf{\tfrac{3}{8}}},\tfrac{3}{8},\tfrac{1}{8}\}_{[-1,2]}, \quad b_1=\{-\tfrac{1}{4},\underline{\mathbf{\tfrac{1}{4}}}\}_{[-1,0]}, \quad b_2=\{-\tfrac{1}{8},\underline{\mathbf{-\tfrac{3}{8}}},\tfrac{3}{8},\tfrac{1}{8}\}_{[-1,2]}.$$

At the end of this section, we illustrate a one-level discrete framelet transform using the Haar orthogonal wavelet filter bank in (11). Let

$$v=\{1,0,-1,-1,-4,60,58,56\}_{[0,7]} \qquad (12)$$

be a test input signal. Note that

$$[\mathscr{T}_a v](n)=v(2n+1)+v(2n), \qquad [\mathscr{T}_b v](n)=v(2n+1)-v(2n), \qquad n\in\mathbb{Z}.$$

Therefore, we have the wavelet coefficients:

$$w_0=\tfrac{\sqrt{2}}{2}\{1,-2,56,114\}_{[0,3]}, \qquad w_1=\tfrac{\sqrt{2}}{2}\{-1,0,64,-2\}_{[0,3]}.$$

On the other hand, we have

$$[\mathscr{S}_a \mathring{w}_0](2n) = \mathring{w}_0(n), \qquad [\mathscr{S}_a \mathring{w}_0](2n+1) = \mathring{w}_0(n), \qquad n \in \mathbb{Z}$$

and

$$[\mathscr{S}_b \mathring{w}_1](2n) = -\mathring{w}_1(n), \qquad [\mathscr{S}_b \mathring{w}_1](2n+1) = \mathring{w}_1(n), \qquad n \in \mathbb{Z}.$$

Hence, we have

$$\begin{aligned} \tfrac{\sqrt{2}}{2}\mathscr{S}_a w_0 &= \tfrac{1}{2}\{1,1,-2,-2,56,56,114,114\}_{[0,7]}, \\ \tfrac{\sqrt{2}}{2}\mathscr{S}_b w_1 &= \tfrac{1}{2}\{1,-1,0,0,-64,64,2,-2\}_{[0,7]}. \end{aligned}$$

Clearly, we have the perfect reconstruction of the original input signal v:

$$\tfrac{\sqrt{2}}{2}\mathscr{S}_a w_0 + \tfrac{\sqrt{2}}{2}\mathscr{S}_b w_1 = \{1,0,-1,-1,-4,60,58,56\}_{[0,7]} = v$$

and the following energy-preserving identity

$$\|w_0\|^2_{l_2(\mathbb{Z})} + \|w_1\|^2_{l_2(\mathbb{Z})} = \tfrac{16137}{2} + \tfrac{4101}{2} = 10119 = \|v\|^2_{l_2(\mathbb{Z})}.$$

Next, let us describe how to efficiently implement discrete framelet/wavelet transform.

The subdivision operator and the transition operator in applications are often implemented through the widely used convolution operation in mathematics and engineering. For $u \in l_0(\mathbb{Z})$ and $v \in l(\mathbb{Z})$, the convolution $u * v$ is defined to be

$$[u * v](n) := \sum_{k \in \mathbb{Z}} u(k)v(n-k), \qquad n \in \mathbb{Z}. \tag{13}$$

Note that $\widehat{u*v}(\xi) = \widehat{u}(\xi)\widehat{v}(\xi)$. To implement the subdivision and transition operators using the convolution operation, we also need the upsampling and downsampling operators on sequences in $l(\mathbb{Z})$. The *downsampling (or decimation) operator* $\downarrow \mathsf{d} : l(\mathbb{Z}) \to l(\mathbb{Z})$ and the *upsampling operator* $\uparrow \mathsf{d} : l(\mathbb{Z}) \to l(\mathbb{Z})$ with a sampling factor $\mathsf{d} \in \mathbb{Z}\backslash\{0\}$ are given by

$$[v \downarrow \mathsf{d}](n) := v(\mathsf{d}n) \quad \text{and} \quad [v \uparrow \mathsf{d}](n) := \begin{cases} v(n/\mathsf{d}), & \text{if } n/\mathsf{d} \text{ is an integer;} \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

for $n \in \mathbb{Z}$. For a sequence $v = \{v(k)\}_{k \in \mathbb{Z}}$, we denote its complex conjugate sequence reflected about the origin by $v^\star$, which is defined to be

$$v^\star(k) := \overline{v(-k)}, \qquad k \in \mathbb{Z}.$$

Note that $\widehat{v^\star}(\xi) = \overline{\widehat{v}(\xi)}$. Now the subdivision operator $\mathscr{S}_u$ in (2) and the transition operator $\mathscr{T}_u$ in (3) can be equivalently expressed as follows:

$$\mathscr{S}_u v = 2(v \uparrow 2) * u \quad \text{and} \quad \mathscr{T}_u v = 2(v * u^\star) \downarrow 2. \tag{15}$$

For $u = \{u(k)\}_{k\in\mathbb{Z}}$ and $\gamma \in \mathbb{Z}$, we define the associated *coset sequence* $u^{[\gamma]}$ of u at the coset $\gamma + 2\mathbb{Z}$ by

$$\widehat{u^{[\gamma]}}(\xi) := \sum_{k\in\mathbb{Z}} u(\gamma+2k)e^{-ik\xi}, \text{ i.e., } u^{[\gamma]} = u(\gamma+\cdot)\downarrow 2 = \{u(\gamma+2k)\}_{k\in\mathbb{Z}}. \tag{16}$$

Using the coset sequences of u, we can rewrite (15) as

$$\begin{aligned} &[\mathscr{S}_u v]^{[0]} = 2v * u^{[0]}, \quad [\mathscr{S}_u v]^{[1]} = 2v * u^{[1]}, \\ &\mathscr{T}_u v = 2\big(v^{[0]} * (u^{[0]})^\star + v^{[1]} * (u^{[1]})^\star\big). \end{aligned} \tag{17}$$

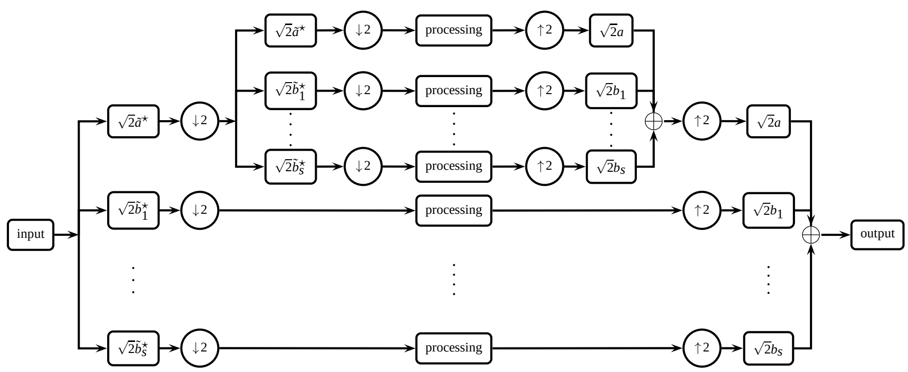

Figure 2: Diagram of a two-level discrete framelet transform employing filter banks $\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}$ and $\{a; b_1, \ldots, b_s\}$. Note that $\frac{\sqrt{2}}{2}\mathscr{T}_{\tilde{b}_\ell} v = \sqrt{2}(v * \tilde{b}_\ell^\star)\downarrow 2$ and $\frac{\sqrt{2}}{2}\downarrow_{b_\ell} v = \sqrt{2}(v\uparrow 2) * \tilde{b}_\ell$ for $\ell = 1, \ldots, s$.