Questions

(1) Does the number π^n have first three significant digits 3, 1, and 4 (in that order) for some $n \in \mathbb{N}$? If so, does

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : \pi^n \text{ has first three significant digits 3, 1, and 4}\}}{N}$$

exist?

(2) Assume $T : \mathbb{T} \to \mathbb{T}$ is λ-preserving. Show that the following statements are equivalent:

(i) T is ergodic;

(ii) if $f \circ T(z) = f(z)$ holds for some (measurable, bounded) function $f : \mathbb{T} \to \mathbb{C}$ and λ-almost every $z \in \mathbb{T}$ then f is constant (λ-a.e.).

(3) (i) Is the sequence $(\log_{10} n)$ u. d. mod 1?

(ii) Let $\vartheta \in \mathbb{R}$ be irrational. Is the sequence $(n\vartheta + \log_{10} n)$ u. d. mod 1?

(4) Given a partition $D_1, \ldots, D_9$ of $\mathbb{N}$ into nine *infinite* sets, write $D_j = \{d_{j,1}, d_{j,2}, \ldots\} = \{d_{j,n} : n \in \mathbb{N}\}$ with $d_{j,1} < d_{j,2} < \ldots$, and let $\delta_{j,n} = d_{j,n+1} - d_{j,n}$ for each $j \in \{1, \ldots, 9\}$ and $n \in \mathbb{N}$. Also, let $p = (p_1, \ldots, p_9) \in \mathbb{R}^9$ be a (non-degenerate) probability vector, i.e., $0 < p_j < 1$ and $\sum_{j=1}^9 p_j = 1$. Consider the following three statements about the partition $D_1, \ldots, D_9$:

(**a**) $\lim_{N\to\infty} \dfrac{\#\{1 \le n \le N : n \in D_j\}}{N} = p_j \quad \forall j \in \{1, \ldots, 9\};$

(**b**) $\lim_{N\to\infty} \dfrac{1}{N} \sum_{n=1}^{N} \delta_{j,n} = \dfrac{1}{p_j} \quad \forall j \in \{1, \ldots, 9\};$

(**c**) $\lim_{N\to\infty} \dfrac{\#\{1 \le n \le N : \delta_{j,n} = m\}}{N} = p_j(1-p_j)^{m-1} \quad \forall j \in \{1, \ldots, 9\}$ and $m \in \mathbb{N}$.

Turn (a)♡(b)♠(c) into a true logical statement by replacing ♡, ♠ with either $\Leftarrow$, $\Rightarrow$, or $\Leftrightarrow$. Whenever your choice is $\Leftarrow$ (resp. $\Rightarrow$) rather than $\Leftrightarrow$, give an example for which $\Rightarrow$ (resp. $\Leftarrow$) is false.

(5) Given any sequence (a_n) of positive real numbers, let

$$D_j = \{n \in \mathbb{N} : a_n \text{ has leading (decimal) digit } j\} \quad \forall j \in \{1, \ldots, 9\}.$$

Note that $D_1, \ldots, D_9$ is a partition of $\mathbb{N}$. Choose $p \in \mathbb{R}^9$ appropriately and try to determine which of the statements **(a)**, **(b)**, and **(c)** of Question 4 are correct for this partition, where

(i) $a_n = 6^n$ for all $n \in \mathbb{N}$;

(ii) $a_n = 6^{n^2}$ for all $n \in \mathbb{N}$;

(iii) $a_n = -2a_{n-1} - 3a_{n-2}$ for all $n \ge 3$, and $a_1 = a_2 = 1$.

Recommended reading.

Given the broad and diverse nature of the subject, the literature on dynamical systems is *huge*. Below is but a very short selection of books that you may find helpful when starting out to explore things for yourself. As far as I know, the University of Alberta library has copies of all of them. I'll be happy to provide further references in case you need some.

A. Berger, *Chaos and Chance*, de Gruyter, 2001.

M. Brin and G. Stuck, *Introduction to Dynamical Systems*, Cambridge University Press, 2002.

B. Hasselblatt and A. Katok, *A First Course in Dynamics*, Cambridge University Press, 2003.

A. Katok and B. Hasselblatt, *An Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley, 1974.

K. Petersen, *Ergodic Theory*, Cambridge University Press, 1983.

C.E. Silva, *Invitation to ergodic theory*, American Mathematical Society, 2008.

P. Walters, *An Introduction to Ergodic Theory*, Springer, 1982.