2 Project 2: The Curve Shortening Flow

During the first lecture, I displayed a webpage [1] which evolved closed curves under the so-called *curve shortening flow*

$$\frac{\partial \vec{X}}{\partial t} = H\vec{\nu} \ ,$$

where $\vec{X}(s,t) = \vec{X}_t(s) = (x_t(s), y(s))$ is a parametrized curve with parameter s at each time t. Here t is the flow time, $H = H_t(s)$ is the mean curvature of the curve at $(x_t(s), y(s))$ at time t, and $\vec{\nu} = \vec{\nu}_t(s)$ is the unit normal vector to the curve pointing in the direction of the centre of the osculating circle at $(x_t(s), y(s))$. The code is available on GitHub.
Here are some directions to pursue. You don't have to pursue all of them. In fact, you don't have to pursue any of them. You may have your own ideas.

1. Can you modify the code, or write your own code? Suggested modifications are
   - the area-preserving CSF (see [2]) $\frac{\partial \vec{X}}{\partial t} = \kappa - \frac{\int_X \kappa ds}{\int_X ds}$ (difficulty: the program would have to compute numerical integrals at each step, which is computationally expensive).
   - the inverse curve shortening flow ICSF $\frac{\partial \vec{X}}{\partial t} = -\frac{\vec{\nu}}{\kappa}$ (difficulty: how do you handle curvatures near zero?).
   - flows of the form $\frac{\partial \vec{X}}{\partial t} = (\kappa^p + c)\nu$ for constants c and p.
   - CSF for curves with (fixed) endpoints.
   - a program that accepts the formula for a parametric closed curve as the initial condition.
2. The curve shortening flow has several interesting properties. Starting from an initial closed curve at time $t = 0$ then:
   - if the initial curve does not self-intersect, the flowing curve never self-intersects,
   - the flow remains smooth until the curve disappears, and
   - the curve always shrinks to zero arclength and disappears in finite time.

   How many of these properties can you prove analytically?

3. The curve shortening flow can be used to prove theorems. You could try one of the following:

   - What is the *isoperimetric inequality* for closed curves in $\mathbb{R}^2$? Can you prove it using curve shortening flow?
   - During my first lecture, one of you asked if the CSF can be thought of as a "flow of contour curves" of a surface S in $\mathbb{R}^3$, where the flowing curve $X(t)$ is the intersection of S with the plane $z = t$. What properties must S have if this is true? (I don't know the answer.) Can this be used to define CSF for curves that are not closed (e.g., if S is not compact but extends to infinity perhaps)?

The Wikipedia page on curve shortening flow has an excellent discussion [3]. For ideas about writing a computer program, see the section on *Numerical Approximations* on that page.

References

[1] Website of A Carapetis http://a.carapetis.com/csf/

[2] E Mäder-Baumdicker, https://arxiv.org/pdf/1503.05814.pdf.

[3] https://en.wikipedia.org/wiki/Curve-shortening_flow