1 Project 1: The Willmore conjecture (theorem)

Recall the *Willmore conjecture* [5]. This conjecture, now a theorem [4], states that for a 2-torus $\mathbb{T}$ in $\mathbb{R}^3$, the *Willmore energy* obeys

$$W := \int H^2 dA \geq 2\pi \ ,$$

and $W = 2\pi$ if and only if $\mathbb{T}$ is the *Willmore torus* given by $x(\theta,\phi) = (a + b\cos\theta)\cos\phi$, $y(\theta,\phi) = (a + b\cos\theta)\sin\phi$, $z(\theta,\phi) = b\sin\theta$. The integral here is a 2-dimensional integral over the torus, and dA is the surface area element (often called the 2-dimensional volume element of the torus or the 2-volume element). The *mean curvature* H is the average of the two *principal curvatures*

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \ ,$$

where r_1 is the radius of the smallest "osculating circle" tangent to the torus at a point and r_2 is the radius of the largest such circle at the same point. (These two circles always lie in orthogonal planes in $\mathbb{R}^3$).
The purpose of this project is to test the conjecture and discover the *minimizer* of the Willmore energy. The minimizer is the torus for which the above inequality becomes an equality $W := \int H^2 dA = 2\pi$. Let's start with a planar circle, say the circle $r^2 := x^2 + y^2 = a^2$ in the $z = 0$ plane of $\mathbb{R}^3$. A tube torus about this circle can be written as the parametrized surface

$$\begin{aligned} x &= (a + b\cos\theta)\cos\phi \\ y &= (a + b\cos\theta)\sin\phi \\ z &= b\sin\theta \ , \end{aligned}$$

Here $\theta, \phi \in [0, 2\pi]$. (Sketch this torus to get the idea. Willmore called this torus an "anchor ring", referring I think to a ring used to tie a ship to an anchor.) In class we will compute the two principal curvatures at each point by finding the radii of the corresponding osculating circles. You should get $\kappa_1 = \frac{1}{b}$, $\kappa_2 = \frac{\cos\theta}{a+b\cos\theta}$ (this depends on how you define θ; those who define their θ to be my θ minus π will get $\kappa_2 = \frac{\cos\theta}{b\cos\theta - a}$). Then we can compute the Willmore energy and find the minimizing torus.
But we don't know if this torus is still the minimizer when we consider more general kinds of torus. Two possible classes for which we might be able to give proofs are the *tube tori* about a general plane curve or the *tori of rotation*. It's an example of a *surface of revolution*. To construct a torus of rotation, first draw a simple closed curve in the (x, z)-plane of $\mathbb{R}^3$. The

curve should lie entirely to the right of the z-axis in this plane. Now rotate this curve about the z-axis. The resulting surface is a *torus of rotation*. To construct a tube torus, first draw a simple closed curve in $\mathbb{R}^3$. Then take all points which lie a fixed distance $b > 0$ from this curve. That's a *tube torus* (in contrast, for a general torus, b would be a function that could vary along the curve). A tube torus can self-intersect. If it does so, we say the torus is *immersed in* $\mathbb{R}^3$. If it doesn't self-intersect, we say the torus is *embedded in* $\mathbb{R}^3$. It can be rather hard to figure out the mean curvature for these tori unless you know some theory beyond what we discussed in the lectures, but for some of these special cases, perhaps you can prove the conjecture. A derivation of the mean curvature of a surface of revolution can be found at [2].

For a review of the Willmore conjecture, see [3]. For biographies of TJ Willmore, who posed the conjecture, and Fernando Codá Marques and André Neves, who proved it, see their Wikipedia pages.

References

[1] J Jost, *Partial differential equations*, Graduate texts in Mathematics GTM 214 (Springer, New York, 2002).

[2] https://conf.math.illinois.edu/ kapovich/423-14/surfacesofrevolution.pdf

[3] FC Marques, *The Willmore conjecture*, [https://arxiv.org/abs/1409.7664].

[4] FC Marques and A Neves, *Min-max theory and the Willmore conjecture*, Ann Math 179 (2014) 683–782.

[5] TJ Willmore, *Note on embedded surfaces*, An Stiint Univ "Al I Cuza" Iasi Ia Mat (1965) 493—496. Can be found at https://www.math.uaic.ro/ annalsmath/pdf-uri

[6] TJ Willmore, *Mean curvature of Riemannian immersions.*