Compound Matrices and Applications

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Abstract

This is a brief introduction to the general theory of compound matrices emphasizing those aspects which are applicable to differential equations. Much of the original work is joint with Michael Li.

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1. COMPOUND MATRICES

1.1. Minors, Cofactors and Adjugates

Consider a $m \times n$ matrix: $A = \left[a_i^j\right]$, $1 \le i \le m$, $1 \le j \le n$. We introduce some notation for minors of A and, in the case $m = n$, for their cofactors.

Definition 1. $a_{r_1 \ldots r_p}^{s_1 \ldots s_p} = \det \left[a_{r_i}^{s_j}\right]$, $1 \le i, j \le p$, *is the minor of A determined by the rows $r_1, ..., r_p$ and the columns $s_1, ..., s_p$*

For example, $a_{11}^{12} = 0$, $\quad a_{12}^{12} = \begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix}$, $a_{13}^{12} = \begin{vmatrix} a_1^1 & a_1^2 \\ a_3^1 & a_3^2 \end{vmatrix}$,

$a_{122}^{123} = 0$, $a_{123}^{123} = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{vmatrix}$, $a_{123}^{124} = \begin{vmatrix} a_1^1 & a_1^2 & a_1^4 \\ a_2^1 & a_2^2 & a_2^4 \\ a_3^1 & a_3^2 & a_3^4 \end{vmatrix}$.

Definition 2. *When $p < m = n$, $A_{r_1 \ldots r_p}^{s_1 \ldots s_p}$ denotes the cofactor of $a_{r_1 \ldots r_p}^{s_1 \ldots s_p}$; i.e. it is the minor determined by the rows complementary to rows $r_1, ..., r_p$ and by the columns complementary to columns $s_1, ..., s_p$ multiplied by $(-1)^{r_1+s_1+\ldots+r_p+s_p}$ .If $p = n$,define $A_{12 \ldots n}^{12 \ldots n} = 1$*

When $A = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix}$, then we have

$A_1^1 = a_{23}^{23} = \begin{vmatrix} a_2^2 & a_2^3 \\ a_3^2 & a_3^3 \end{vmatrix}$, $A_2^3 = -a_{13}^{12} = -\begin{vmatrix} a_1^1 & a_1^2 \\ a_3^1 & a_3^2 \end{vmatrix}$, $A_{12}^{12} = a_3^3$, $A_{13}^{12} = -a_2^3$ and $A_{123}^{123} = 1$. Note that in this case we have

$$\det A = a_i^1 A_i^1 + a_i^2 A_i^2 + a_i^3 A_i^3,\ i = 1, 2, 3$$

and

$$0 = a_i^1 A_r^1 + a_i^2 A_r^2 + a_i^3 A_r^3, \text{ if } i \neq r.$$

Analogous expressions hold with i, r as superscripts. In general we have the following expansions of $\det A$ due to *Laplace*.

Theorem 3. *If A is a $n \times n$ matrix, then*

$$\det A = \sum_{j=1}^{n} a_i^j A_i^j = \sum_{i=1}^{n} a_i^j A_i^j$$

Moreover $0 = \sum_{j=1}^{n} a_i^j A_r^j$ *,if* $r \neq i$ *and* $0 = \sum_{i=1}^{n} a_i^j A_i^r$*, if* $r \neq j$*.*

Definition 4. *The cofactor matrix of a square matrix A is*

$$\operatorname{cof} A = \left[A_i^j\right], \ i, j = 1, ..., n$$

and the adjugate (or classical adjoint) matrix of A is

$$\operatorname{adj} A = (\operatorname{cof} A)^T .$$

From Theorem 5 and Definition 6 we find that

$$A (\operatorname{adj} A) = (\operatorname{adj} A) A = (\det A) I \tag{1.1}$$

so that we obtain the following expression for A^{-1}. If $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A. \tag{1.2}$$

Moreover, from (1.1),

$$\det (\operatorname{cof} A) = \det (\operatorname{adj} A) = (\det A)^{n-1} \tag{1.3}$$

The following expression for $\det A$ is also due to *Laplace* and generalizes Theorem 3.

Theorem 5. *If A is a $n \times n$ matrix and $1 \leq k \leq n$, then*

$$\det A = \sum_{1 \leq s_1 < ... < s_k \leq n} a_{r_1 ... r_k}^{s_1 ... s_k} A_{r_1 ... r_k}^{s_1 ... s_k}, \ \text{if } 1 \leq r_1 < ... < r_k \leq n$$

$$\det A = \sum_{1 \leq r_1 < ... < r_k \leq n} a_{r_1 ... r_k}^{s_1 ... s_k} A_{r_1 ... r_k}^{s_1 ... s_k}, \ \text{if } 1 \leq s_1 < ... < s_k \leq n$$

Moreover $0 = \sum_{(s)} a_{r_1 ... r_k}^{s_1 ... s_k} A_{t_1 ... t_k}^{s_1 ... s_k}$*, if* $(r) \neq (t)$*, and* $0 = \sum_{(r)} a_{r_1 ... r_k}^{s_1 ... s_k} A_{r_1 ... r_k}^{t_1 ... t_k}$*, if* $(s) \neq (t)$*.*

Proof. This Theorem is obviously true when $n = 1$. If it holds for $(n-1)\times(n-1)$ matrices, the result for $n \times n$ matrices follows by applying this to the matrices obtained by replacing the first row of A by rows of the form $(0, ..., 1, ..., 0)$ and using the linearity of $\det A$ in the rows of A. ■

Corollary 6. *If $C = AB$ where A is $n \times p$ and B is $p \times n$, then*

$$\det C = \begin{cases} \sum_{(t)} a_{12....n}^{t_1 t_2 ... t_n} b_{t_1 t_2 ... t_n}^{12....n}, \text{ if } n \le p \\ 0, \text{ if } n > p \end{cases} \tag{1.4}$$

Proof. Let $\delta = \det C$. Since

$$\begin{bmatrix} I_{nn} & -A_{np} \\ O_{pn} & I_{pp} \end{bmatrix} \begin{bmatrix} O_{nn} & A_{np} \\ B_{pn} & I_{pp} \end{bmatrix} = \begin{bmatrix} -AB & O \\ B_{pn} & I_{pp} \end{bmatrix},$$

it follows that

$$\det \begin{bmatrix} O & A \\ B & I \end{bmatrix} = \det \begin{bmatrix} -AB & O \\ B & I \end{bmatrix}$$

Apply the Laplace expansion $(k = n)$ by the minors from the first n rows to both matrices. From the second we find that the value of the determinant is $(-1)^n \delta$. From the first the value is 0, if $n > p$, since every $n \times p$ minor from the first n rows contains at least one zero column; $\delta = 0$ in this case as asserted. When $n \le p$, the first determinant is $(-1)^n \sum_{(t)} a_{12....n}^{t_1 t_2 ... t_n} b_{t_1 t_2 ... t_n}^{12....n}$ so that this equals $(-1)^n \delta$ and the corollary follows. ■

1.2. Multiplicative Compounds

For any $m \times n$ matrix A and $1 \le k \le \min\{m, n\}$,the $k-th$ *multiplicative compound* (or k-th *exterior power*) $A^{(k)}$ of A is the $\binom{m}{k} \times \binom{n}{k}-$dimensional matrix defined as follows.

Definition 7. *If $1 \le r \le \binom{m}{k}$ and $1 \le s \le \binom{n}{k}$, then the entry in the $r-th$ row and the $s-th$ column of $A^{(k)}$ is $a_{r_1 ... r_k}^{s_1 ... s_k}$, where $(r) = (r_1, ..., r_k)$ is the $r-th$ member of the lexicographic ordering of the integers $1 \le r_1 < r_2 < ... < r_k \le m$ and $(s) = (s_1, ...s_k)$ is the s-th member in the lexicographic ordering of all k-tuples of the integers $1 \le s_1 < s_2 < ... < s_k \le n$.*

Thus, if $A=\begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ \vdots & \vdots \\ a_m^1 & a_m^2 \end{bmatrix}$, a $m\times 2$ matrix, then $A^{(2)}=\begin{bmatrix} a_{12}^{12} \\ a_{13}^{12} \\ \vdots \\ a_{m-1,m}^{12} \end{bmatrix}$, a $\binom{m}{2}\times 1$ matrix. It is useful as usual to visualize the columns of A as representing a pair of oriented line segments with the entry of any column in the i-th row representing the projection of the line segment onto the i-th coordinate axis. In this picture we can then consider $A^{(2)}$ as the oriented 2- dimensional parallelogram determined by the columns of A; the entry in the r-th column, $(r)=(r_1,r_2)$, is the projection of this area onto the (r_1,r_2)-coordinate plane.

Note that, if $A^{(k)}\neq O$ and $A^{(k+1)}=O$, then the rank of A is k.

The term *multiplicative compound* arises because of the *Binet-Cauchy Theorem.*

Theorem 8. *If $AB=C$, then $A^{(k)}B^{(k)}=C^{(k)}$,where A,B are $n\times p$, $p\times n$ matrices respectively.*

Proof. Note that $c_r^s=\sum_t a_r^t b_t^s$, so that the submatrix $\left[c_{r_i}^{s_j}\right]$, $i,j=1,...,k$, of C is the product of $\left[a_{r_i}^t\right]$, $i=1,...,k,\ t=1,...,p$ and $\left[b_t^{s_j}\right]$, $t=1,...,p,\ j=1,...,n$ and , from Corollary 6, its determinant is

$$c_{r_1\ldots r_k}^{s_1\ldots s_k}=\sum_{(t)} a_{r_1\ldots r_k}^{t_1\ldots t_k} b_{t_1\ldots t_k}^{s_1\ldots s_k} \tag{1.5}$$

The left-hand side of (1.5) is the entry in the r-th row and s-th column of $C^{(k)}$,where $(r)=(r_1,...,r_k)$ and $(s)=(s_1,...,s_k)$, while the right-hand side is the product of the $r-$th row of $A^{(k)}$ and the s-th column of $B^{(k)}$. Thus $C^{(k)}=A^{(k)}B^{(k)}$ as asserted. ■

Analogously to cof$A$ and adj$A$ we define $\mathrm{cof}^{(k)}A$ and $\mathrm{adj}^{(k)}A$ (not to be confused with $\mathrm{cof}\left(A^{(k)}\right)$ or $(\mathrm{cof}\ A)^{(k)}$ and $\mathrm{adj}\left(A^{(k)}\right)$ or $(\mathrm{adj}A)^{(k)}$; *cf.* Jacobi's Theorem for the precise relationships).

Definition 9. *Let $(r)=(r_1,...,r_k)$ and $(s)=(s_1,...,s_k)$. The entry in the r-th row and s-th column of $\mathrm{cof}^{(k)}A$ is $A_{r_1\ldots r_k}^{s_1\ldots s_k}$ and $\mathrm{adj}^{(k)}A=(\mathrm{cof}^{(k)}A)^T$.*

Thus each entry in $\mathrm{cof}^{(k)}A$ is the signed minor **with respect to A** of the corresponding entry in $A^{(k)}$.

$$\text{e.g. } A^{(3)}=\begin{bmatrix} a_{123}^{123} & a_{123}^{124} & a_{123}^{125} & . \\ a_{124}^{123} & a_{124}^{124} & a_{124}^{125} & . \\ a_{125}^{123} & a_{125}^{124} & a_{125}^{125} & . \\ . & . & . & . \end{bmatrix},\quad \mathrm{cof}^{(3)}A=\begin{bmatrix} A_{123}^{123} & A_{123}^{124} & A_{123}^{125} & . \\ A_{124}^{123} & A_{124}^{124} & A_{124}^{125} & . \\ A_{125}^{123} & A_{125}^{124} & A_{125}^{125} & . \\ . & . & . & . \end{bmatrix}$$

The entries in $\mathrm{cof}^{(k)}A$ are clearly also the minors of order $n-k$ of A; in fact $\mathrm{cof}^{(k)}A = U^T A^{(n-k)} U$, where U is the unitary matrix defined by $u_i^j = (-1)^{j+1}$ if $i+j = \binom{n}{k}+1$ and $u_i^j = 0$ otherwise. Thus all entries in U are 0 except those on the skew diagonal which are alternately ± 1.

The following result, known as *Sylvester's Theorem,* relates the determinants of $A^{(k)}$ and $\mathrm{cof}^{(k)}A$ to the determinant of A.

Theorem 10.

$$\det A^{(k)} = (\det A)^{\binom{n-1}{k-1}}\ , \tag{1.6}$$

$$\det\left(cof^{(k)}A\right) = \det\left(adj^{(k)}A\right) = (\det A)^{\binom{n-1}{k}}\,. \tag{1.7}$$

Proof. Note that $\det A$ is a polynomial of degree n in the entries of A and that it is prime, i.e. it is not the product of two such non-constant polynomials. As in (1.1) it follows from Theorem 5 that, with I denoting the identity in dimension $\binom{n}{k}$,

$$A^{(k)}(\mathrm{adj}^{(k)}A) = (\det A)\, I \tag{1.8}$$

and therefore

$$\det(A^{(k)})\det\left(\mathrm{adj}^{(k)}A\right) = (\det A)^{\binom{n}{k}} \tag{1.9}$$

Now $\det\left(A^{(k)}\right)$ and $\det\left(\mathrm{adj}^{(k)}A\right)$ are also polynomials in the entries of A and (1.9) shows that they factor the polynomial $(\det A)^{\binom{n}{k}}$. Thus, since the polynomial $\det A$ is prime, there exist constants α, β, γ such that $\alpha+\beta = \binom{n}{k}$ and $\det A^{(k)} = \gamma\,(\det A)^{\alpha}$, $\det\left(\mathrm{adj}^{(k)}A\right) = \frac{1}{\gamma}\,(\det A)^{\beta}$. Consideration of a pure diagonal matrix A shows that $\alpha = \binom{n-1}{k-1}$, $\beta = \binom{n-1}{k}$ and $\gamma = 1$ as asserted. ■

The following result, *Jacobi's Theorem,* relates $\mathrm{adj}(A^{(k)})$ and $(\mathrm{adj}A)^{(k)}$ to $\mathrm{adj}^{(k)}A$.

Theorem 11.

$$adj\left(A^{(k)}\right) = (\det A)^{\binom{n-1}{k-1}-1}\, adj^{(k)}A \tag{1.10}$$

$$(adjA)^{(k)} = (\det A)^{k-1}\, adj^{(k)}A \tag{1.11}$$

Proof. From (1.8) we see that $\left(A^{(k)}\right)^{-1} = \frac{1}{\det A}\mathrm{adj}^{(k)}A$; but we also know from (1.2) that $\left(A^{(k)}\right)^{-1} = \frac{1}{\det A^{(k)}}\left(\mathrm{adj}A^{(k)}\right)$. Comparing these two expressions and using Sylvester's Theorem, we find (1.10). Also, from (1.1) $A\left(\mathrm{adj}A\right) = \left(\det A\right)I$,using the Binet-Cauchy Theorem, we find $A^{(k)}\left(\mathrm{adj}A\right)^{(k)} = \left(\det A\right)^{k} I^{(k)}$. Observing that $I^{(k)}$ is also the identity matrix of dimension $\binom{n}{k}$, it follows that $\left(\mathrm{adj}A\right)^{(k)} = \left(\det A\right)^{k}\left(A^{(k)}\right)^{-1} = \left(\det A\right)^{k-1}\mathrm{adj}^{(k)}A$, which is (1.11). ■

Observe that, when A is singular, all minors of order 2 in adj$A$ are zero. This follows from $k = 2$ in (1.11).

To determine the spectrum of the k-th multiplicative compound $A^{(k)}$ of a $n \times n$ matrix A, recall that there exists a non-singular matrix T such that

$$AT = T\Lambda \tag{1.12}$$

where Λ is lower triangular and the diagonal elements are $\lambda_1, ..., \lambda_n$, the eigenvalues of A repeated according to multiplicity. Conversely, if (1.12) is satisfied for some non-singular T and lower triangular Λ, then the diagonal entries of Λ are the eigenvalues of A. From the Binet-Cauchy Theorem, we find that

$$A^{(k)}T^{(k)} = T^{(k)}\Lambda^{(k)} \tag{1.13}$$

First the diagonal element in the s-th row and column of $\Lambda^{(k)}$ is $\lambda^{s_1...s_k}_{s_1...s_k} = \lambda_{s_1}\lambda_{s_2}...\lambda_{s_k}$ where $(s) = (s_1, ...s_k)$. Moreover $\Lambda^{(k)}$ is lower diagonal; to see this consider the element $\lambda^{s_1...s_k}_{r_1...r_k}$ in the r-th row and s-th column of $\Lambda^{(k)}$. This element is above the main diagonal provided $r < s$: i.e. $r_i < s_i < s_{i+1} < ... < s_k$ for some i, $1 \leq i \leq k$, and $r_j = s_j,\ j = 1, ..., k-1$, if $k > 1$. Then every $k \times k$ minor from the first i rows of the matrix $\left[\lambda^{s_q}_{r_p}\right]$, $p, q = 1, ..., k$ is zero since there is at least on column of zeros in each minor. Summarizing, we find the following theorem.

Theorem 12. *The eigenvalues of* $A^{(k)}$ *are* $\lambda_{s_1}\lambda_{s_2}...\lambda_{s_k}$, $1 \leq s_1 < s_2 < ... < s_k \leq n$, *where* $\lambda_1, ..., \lambda_n$ *are the eigenvalues of* A.

Proof. By considering $n \times k$ matrices T whose columns are eigenvectors corresponding to eigenvalues $\lambda_1, ..., \lambda_k$, we find that (1.12) is satisfied by $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_k)$. Then (1.13) is satisfied again and $\Lambda^{(k)} = \lambda_1\lambda_2...\lambda_k$ so that $T^{(k)}$ is an eigenvector of $A^{(k)}$. The generalized eigenvectors of $A^{(k)}$ may also be discovered in this way and, although they are related to the generalized eigenvectors of A, the relationship is not as simple as this. ■

The *singular values* of a $n \times n$ matrix A are the non-negative real numbers $\sigma_1, ..., \sigma_n$ such that $(\sigma_1)^2, ..., (\sigma_n)^2$ are the eigenvalues of AA^*. Since $A^{(k)}A^{(k)*} = (AA^*)^{(k)}$, it follows as in the preceding discussion that the singular values of $A^{(k)}$ are the numbers $\sigma_{s_1}\sigma_{s_2}...\sigma_{s_k}$, $1 \leq s_1 < s_2 < ... < s_k \leq n$.

1.3. Additive Compounds

Let A be a $n \times n$ matrix and let $1 \leq k \leq n$. Then the $k-th$ *additive compound* $A^{[k]}$ of A is a $\binom{n}{k} \times \binom{n}{k}$ matrix defined as follows.

Definition 13.

$$A^{[k]} = \left.\frac{d}{dt}(I + tA)^{(k)}\right|_{t=0} = \lim_{h \to 0} h^{-1}[(I + hA)^{(k)} - I^{(k)}] \tag{1.14}$$

It follows that the entry b_r^s in $B = A^{[k]}$ is:

$$b_r^s = \begin{cases} a_{r_1}^{r_1} + \cdots + a_{r_k}^{r_k}, & \text{if } (r) = (s) \\ (-1)^{i+j} a_{r_i}^{s_j}, & \text{if exactly one entry } r_i \text{ in } (r) \text{ does not occur} \\ & \text{in } (s) \text{ and } s_j \text{ does not occur in } (r) \\ 0, & \text{if } (r) \text{ differs from } (s) \text{ in two or more entries} \end{cases} \tag{1.15}$$

In the special cases $k = 1$, $k = n$, we find

$$A^{[1]} = A, \quad A^{[n]} = \mathrm{Tr}A.$$

The term *additive compound* arises since

$$(A + B)^{[k]} = A^{[k]} + B^{[k]} \tag{1.16}$$

and indeed the map $A \mapsto A^{[k]}$ is linear. This may be deduced from the Binet-Cauchy Theorem and the definition (1.14) since

$$(I + tA)^{(k)}(I + tB)^{(k)} = ((I + tA)(I + tB))^{(k)} = \left(I + t(A + B) + t^2AB\right)^{(k)}.$$

Alternatively, (1.16) may be deduced directly from (1.15). Since (1.12) implies

$$(I + tA)T = T(I + t\Lambda) \tag{1.17}$$

we find from the Binet-Cauchy formula that

$$(I+tA)^{(k)}\,T^{(k)} = T^{(k)}\,(I+t\Lambda)^{(k)} \tag{1.18}$$

and therefore

$$A^{[k]}T^{(k)} = T^{(k)}\Lambda^{[k]} \tag{1.19}$$

so that the eigenvalues of $A^{[k]}$ are $\lambda_{s_1} + \lambda_{s_2} + \ldots + \lambda_{s_k}$ where $\lambda_1, \ldots\lambda_n$ are the eigenvalues of A. The eigenvectors of $A^{[k]}$ are the same as those of $A^{(k)}$. The important formula

$$(\exp(A))^{(k)} = \exp\left(A^{[k]}\right) \tag{1.20}$$

may also be derived from (1.14) directly but is most readily obtained in the context of differential equations. First recall that the matrix

$$X(t) = \exp(tA) = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots$$

is the unique function which satisfies $\dot{X} = AX$, $X(0) = I$ so that $X(t+h) = (I+hA)\,X(t) + o(h)$ and therefore, from the Binet-Cauchy formula, $X^{(k)}(t+h) = (I+hA)^{(k)}\,X^{(k)}(t) + o(h)$ so that $Y(t) = X^{(k)}(t)$, from (1.14), satisfies $\dot{Y} = A^{[k]}Y$, $Y(0) = I^{(k)}$ and hence $Y(t) = \exp\left(tA^{[k]}\right)$. In particular, choosing $t = 1$ shows $(\exp(A))^{(k)} = \exp\left(A^{[k]}\right)$ as asserted.

Another identity which may be proved directly from the definition but is most easily established in a differential equations context is

$$-\left(A^{[k]}\right)^T = U^T A^{[n-k]} U - (\mathrm{Tr}A)\,I, \tag{1.21}$$

where $U = \left[u_i^j\right]$, $u_i^j = (-1)^{j+1}$, if $i + j = \binom{n}{k} + 1$ and $u_i^j = 0$ otherwise. This will be proved in the next chapter.

2. LINEAR DIFFERENTIAL EQUATIONS

2.1. Compound Equations

Motivated by the preceding considerations, we now consider a general first order system of linear differential equations

$$\dot{x} = A(t)\,x \tag{2.1}$$

where $t \mapsto A(t)$ is a continuous real or complex $n \times n$ matrix-valued function on $[0, \infty)$. A solution is uniquely determined by its value $x(t_0)$ for any $t_0 \in [0, \infty)$. A *matrix solution* of (2.1) is a $n \times m$ matrix-valued function $t \mapsto X(t)$ such that $\dot{X} = A(t)\,X$. A *fundamental matrix* is a non-singular $n \times n$ matrix solution $X(t)$. Thus $x(t)$ is a solution of (2.1) if and only if

$$x(t) = X(t)\,X^{-1}(t_0)\,x(t_0) \tag{2.2}$$

where $X(t)$ is any fundamental matrix.

Conversely any continuously differentiable matrix $X(t)$ which is non-singular for each $t \in [0, \infty)$ uniquely determines a differential equation (2.1) for which it is a fundamental matrix; here $A(t) = \dot{X}(t)\,X^{-1}(t)$. From Sylvester's Theorem (1.6), $X^{(k)}(t)$ is non-singular if $X(t)$ is. Therefore, if $X(t)$ is a fundamental matrix for (2.1), $Y(t) = X^{[k]}(t)$ is a fundamental matrix for a system of dimension $\binom{n}{k}$. If we do the computation directly, we find that $\dot{Y}(t)\,Y^{-1}(t) = A^{[k]}(t)$ as specified in (1.15) so that $Y(t)$ is a fundamental matrix for (2.3) below. We may also deduce this from the definition of $A^{[k]}$.

If $X(t)$ is a $n \times m$ matrix solution, $X(t+h) = (I + hA(t))\,X(t) + o(h)$ implies $X^{(k)}(t+h) = (I + hA(t))^{(k)}\,X(t) + o(h)$ and therefore $Y(t) = X^{(k)}(t)$ satisfies $\dot{Y} = A^{[k]}(t)\,Y$ so that $Y(t)$ is a matrix solution of

$$\dot{y} = A^{[k]}(t)\,y \tag{2.3}$$

This is the k-th *compound equation* associated with (2.1). When $k = 1$, (2.3) is the original equation (2.1); when $k = n$, it is the scalar equation $\dot{y} = \mathrm{Tr}A(t)\,y$. This is the *Liouville-Jacobi* equation whose general solution is $y(t) = c \exp\left(\int_0^t \mathrm{Tr} A\right)$. In particular, it is satisfied by $y(t) = \det X(t)$ when $X(t)$ is a $n \times n$ matrix solution of (2.1).

If $x^1(t), ..., x^k(t)$ are solutions of (2.1), let $y(t) = x^1(t) \wedge ... \wedge x^k(t) = X^{(k)}(t)$, where $X(t) = \left[x_i^j(t)\right]$, $i = 1, ...n$, $j = 1, ...k$; $y(t)$ is a solution of (2.3). The solution space of (2.3) is the linear span of all such exterior products of k-tuples of solutions of (2.1).

The formula (1.21) arises naturally in a discussion of adjoint differential equations. When $X(t)$ is non-singular, so also is $X^{-1}(t)$. It is an easy exercise to check that $Z(t) = X^{-1}(t)^T = \frac{1}{\det A} \mathrm{cof} X(t)$ is a fundamental matrix for the *adjoint equation* of (2.1),

$$\dot{z} = -A^T(t) z, \tag{2.4}$$

whenever $X(t)$ is a fundamental solution of (2.1). Now, since $\mathrm{cof}^{(k)} X = U^T X^{(n-k)T} U$, it follows that $W(t) = \mathrm{cof}^{(k)} X(t)$ satisfies $\dot{W} = U^T A^{[n-k]} U W$. Therefore, since $\frac{d}{dt} \det X = (\mathrm{Tr} A) \det X$, $Z(t) = \left[X^{(k)}\right]^{-1}(t) = \frac{1}{\det X(t)} \mathrm{cof}^{(k)} X(t) = \frac{1}{\det X(t)} W(t)$ satisfies $\dot{Z} = \left[U^T A^{[n-k]} U - (\mathrm{Tr} A) I\right] Z$. But $Z(t)$ is a fundamental solution for the adjoint of (2.3), so $\dot{Z} = -\left[A^{[k]}(t)\right]^T Z$. It follows that

$$-\left[A^{[k]}\right]^T = U^T A^{[n-k]} U - (\mathrm{Tr} A) I,$$

which is (1.21).

2.2. Stability and Asymptotic Behaviour

Let $X(t)$ be a fundamental matrix for (2.1) and let $|\cdot|$ denote any matrix norm. We may assume without loss of generality that the norm is induced by a vector norm. The equation is said to be

(i) *Stable* if there is a constant K such that $|X(t)| \leq K$, $0 \leq t < \infty$.

(ii) *Asymptotically stable* if $|X(t)| \to 0$, as $t \to \infty$.

(iii) *Uniformly stable* if there exists a constant K such that $|X(t) X^{-1}(t_0)| \leq K$, $\quad 0 \leq t_0 \leq t < \infty$.

(iv) *Uniformly asymptotically stable* if there exist constants K, $\alpha > 0$ such that $|X(t) X^{-1}(t_0)| \leq K e^{-\alpha(t-t_0)}$, $0 \leq t_0 \leq t < \infty$.

Since $x(t) = X(t) X^{-1}(t_0) x(t_0)$, the equation is stable (asymptotically stable) provided solutions are bounded (tend to zero). It is uniformly stable provided solutions satisfy $|x(t)| \leq K |x(t_0)|$ and uniformly asymptotically stable provided $|x(t)| \leq K e^{-\alpha(t-t_0)} |x(t_0)|$, $0 \leq t_0 \leq t < \infty$.

Before developing concrete conditions for the various stability types, we illustrate with the following results how the compound equations (2.3) may be used to give a more refined analysis of the asymptotic behaviour of the solutions of (2.1) The first result is due to *Macki and Muldowney.*

Theorem 14. *Suppose that* (2.1) *is uniformly stable. Then there is at least one non-trivial solution* $x_0(t)$ *of* (2.1) *such that* $\lim_{t\to\infty} x_0(t) = 0$ *if and only if*

$$\lim_{t\to\infty} \int_0^t \mathrm{Tr}A = -\infty. \tag{2.5}$$

Interpreted in the context of the present discussion, this theorem states that, for a uniformly stable linear differential equation (2.1), the dimension of the linear subspace of the solution space which consists of those solutions which have limit zero at infinity is at least 1 if and only if the $n-$ th multiplicative compound equation, $\dot{y} = (\mathrm{Tr}A(t))\, y$, a scalar equation, is asymptotically stable. This is seen from the fact that its solutions are $y(t) = c\exp\left(\int_0^t \mathrm{Tr}A\right)$ and $\lim_{t\to\infty} y(t) = 0$ provided (2.5) holds. The theorem is generalized by the following result of *Muldowney.*

Theorem 15. *Suppose that* (2.1) *is uniformly stable. Then the set of solutions* $x_0(t)$ *satisfying* $\lim_{t\to\infty} x_0(t) = 0$ *has dimension at least* $n-k+1$ *if and only if the* k*-th compound equation* (2.3) *of* (2.1) *is asymptotically stable.*

Proof. The necessity of the condition is seen by considering a fundamental matrix $X(t)$ of (2.1) so that $Y(t) = X^{(k)}(t)$ is a fundamental matrix for (2.3). Without loss of generality, it may be assumed that the first $n-k+1$ columns of $X(t)$ have limit 0; moreover all columns are bounded. Since every k-dimensional sub-matrix of $X(t)$ therefore has a zero-tending column, it follows that $\lim_{t\to\infty} X^{(k)}(t) = 0$ and (2.3) is asymptotically stable as asserted. To prove the sufficiency, again let $X(t)$ be a fundamental matrix for (2.1). Since this is a bounded function, we may choose a sequence $t_i \to \infty$ such that the $n \times n$ matrix $M = \lim_{i\to\infty} X(t_i)$ exists. Moreover $M^{(k)} = \lim_{i\to\infty} X^{(k)}(t_i) = O$, from the asymptotic stability of (2.3), so that the rank of M is at most $k-1$. The set of solutions c of $Mc = 0$

then has dimension at least $n-k+1$. The solutions $x(t)=X(t)c$ must then satisfy $\lim_{i\to\infty} x(t_i)=Mc=0$ and the dimension of this set of solutions is at least $n-k+1$. In fact these solutions satisfy $\lim_{t\to\infty} x(t)=0$ since $|x(t)|\le K|x(t_i)|$, if $t\ge t_i$. ■

One relatively general approach to a stability analysis of (2.1) is a special case of Lyapunov's direct method. It gives rise to the concept of the Lozinskiĭ measure or logarithmic norm of a matrix. Let $|\cdot|$ denote any vector norm in $\mathbb{R}^n$and the matrix norm which it induces. Then the *Lozinskiĭ measure* $\mu(A)$ of a $n\times n$ matrix A is defined by

$$\mu(A)=D_+|I+tA|_{t=0}=\lim_{h\to 0+} h^{-1}\left[|I+hA|-1\right] \tag{2.6}$$

For some standard vector norms, the matrix norms and the Lozinskiĭ measures are given in the table.

norm: $\quad |x|$	$|A|$	$\mu(A)$
$l^2:\ \sqrt{x^*x}$	σ_1	λ_1
$l^1:\ \sum_i |x_i|$	$\max_j \sum_i \left|a_i^j\right|$	$\max_j \left\{\operatorname{Re} a_j^j+\sum_{i\ne j}\left|a_i^j\right|\right\}$
$l^\infty:\ \max_i |x_i|$	$\max_i \sum_j \left|a_i^j\right|$	$\max_i \left\{\operatorname{Re} a_i^i+\sum_{j\ne i}\left|a_i^j\right|\right\}$

Here σ_1 is the largest singular value of A and λ_1 is the largest eigenvalue of $\frac{1}{2}(A^*+A)$

(2.7)

More generally, we have the following expressions for $k=1,...,n$.

$$\mu\left(A^{[k]}\right)=\begin{cases}\lambda_1+\cdots+\lambda_k\\ \max_{(j)}\left\{\operatorname{Re}\left(a_{j_1}^{j_1}+\cdots+a_{j_k}^{j_k}\right)+\sum_{i\notin(j)}\left(\left|a_i^{j_1}\right|+\cdots+\left|a_i^{j_k}\right|\right)\right\}\\ \max_{(i)}\left\{\operatorname{Re}\left(a_{i_1}^{i_1}+\cdots+a_{i_k}^{i_k}\right)+\sum_{j\notin(i)}\left(\left|a_{i_1}^{j}\right|+\cdots+\left|a_{i_k}^{j}\right|\right)\right\}\end{cases} \tag{2.8}$$

where $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$ are the eigenvalues of $\frac{1}{2}(A^*+A)$.

The importance of the Lozinskiĭ measure arises when we use $|x|$ as a Lyapunov function for (2.1). Then $|x(t+h)|=|x(t)+h\dot{x}(t)+o(h)|=|(I+hA(t))x(t)|+o(h)\le|I+hA(t)||x(t)|+o(h)$ and therefore $D_+|x(t)|\le\mu(A(t))|x(t)|$ so that $|x(t)|\exp\left(-\int_{t_0}^t \mu(A)\right)$ is decreasing. Similarly we find that $|x(t)|\exp\left(\int_{t_0}^t \mu(-A)\right)$ is increasing. It follows that

$$|x(t)|\exp\left(-\int_{t_0}^t \mu(A)\right)\le|x(t_0)|\le|x(t)|\exp\left(\int_{t_0}^t \mu(-A)\right),\ t_0\le t \tag{2.9}$$

Proposition 16. *The equation* (2.1) *is:*

(i) Stable if $\int_0^t \mu(A)$ is bounded $0 \leq t < \infty$.

(ii) Asymptotically stable if $\int_0^t \mu(A) \to -\infty$ as $t \to \infty$.

(iii) Uniformly stable if $\int_{t_0}^t \mu(A) \leq K$, $0 \leq t_0 \leq t < \infty$, where K is independent of t_0, t.

(iv) Uniformly asymptotically stable if $\int_{t_0}^t \mu(A) \leq -\alpha(t - t_0) + \beta$, $0 \leq t_0 \leq t < \infty$, where $\alpha, \beta > 0$ are independent of t_0, t.

Proof. This follows directly from (2.8) since $|x(t)| \leq |x(t_0)| \exp\left(\int_{t_0}^t \mu(A)\right)$, $t_0 \leq t$. ■

Remark 1. *If we replace* $\mu(A)$ *by* $\mu\left(A^{[k]}\right)$*the proposition, then gives stability criteria for the compound equations* (2.3) *. Thus we can conclude from Theorem 15 that equation*(2.1) *has a* $(n-k+1)$ *−dimensional set of solutions* $x_0(t)$ *satisfying* $\lim_{t\to\infty} x_0(t) = 0$ *if* $\int_0^t \mu(A)$ *is bounded and* $\lim_{t\to\infty} \int_0^t \mu\left(A^{[k]}\right) = -\infty$.

Remark 2. *A result of Geršgorin states that every eigenvalue* λ *of a* $n \times n$ *matrix* A *lies in one of the* n *discs* $\{z : |z - a_i^i| \leq \delta(i)\}$ *, where* $\delta(i) = \sum_{j \neq i} \left|a_i^j\right|$ *. While not every one of these discs contains an eigenvalue, it is nevertheless the case that every connected component of their union contains* m *eigenvalues if it is the union of* m *of these discs. It is of interest to note that the third expression for* $\mu(A)$ *in* (2.7) *is the upper bound on* $\operatorname{Re}\lambda$ *implied by the Geršgorin discs. Every* $\mu(A)$ *is in fact such an upper bound; nevertheless, it is well known that Proposition 16 is no longer valid if* $\mu(A)$ *is replaced by* $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{sp} A\}$ *. The row sums in the definition of* $\delta(i)$ *may be replaced by column sums.*

Remark 3. *Applying Geršgorin's result to* $A^{[k]}$*, any sum of* k *of the eigenvalues of* A *is contained in one of the* $\binom{n}{k}$ *discs* $\left\{z : \left|z - a_{i_1}^{i_1} - \cdots - a_{i_k}^{i_k}\right| \leq \rho(i)\right\}$ *, where* $\rho(i) = \sum_{j \notin (i)} \left(\left|a_{i_1}^j\right| + \cdots + \left|a_{i_k}^j\right|\right)$ *. Moreover, a set of* m *of these discs having no points in common with the remaining* $\binom{n}{k} - m$ *discs contains exactly* m *sums of* k *eigenvalues of* A*. Of course, an analogous statement for column sums is valid.*

3. NONLINEAR DIFFERENTIAL EQUATIONS

3.1. The Variational Equation and its Compounds

Consider a non-linear system

$$\dot{x} = f(x), \tag{3.1}$$

where f is a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}^n$. A solution $x = \phi(t, x_0)$ is uniquely determined by its initial value, the map $(t, x_0) \to \phi(t, x_0)$ is a C^1 diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ and satisfies the group property $\phi(0, x_0) = x_0$, $\phi(t, \phi(s, x_0)) = \phi(t+s, x_0)$ as long as these expressions exist. If $f(x_0) = 0$, then $\phi(t, x_0) = x_0$ is a constant solution, an *equilibrium.* The sets$\{\phi(t, x_0) : t \in \mathbb{R}\}$ and $\{\phi(t, x_0) : t \geq 0\}$ are the *orbit* and *positive semi-orbit* of x_0 respectively.

For a given solution $\phi(t) = \phi(t, x_0)$, the associated linearization of (3.1) at this solution is the equation

$$\dot{y} = \frac{\partial f}{\partial x}(\phi(t))\, y, \tag{3.2}$$

the variational equation. The $n \times n$ matrix $Y(t) = \frac{\partial \phi}{\partial x_0}(t, x_0)$ satisfies (3.2) and $Y(0) = I$ and is therefore a fundamental matrix for (3.2). If x_0 is an equilibrium, then $\frac{\partial f}{\partial x}(\phi(t)) = \frac{\partial f}{\partial x}(x_0)$, a constant matrix, and $\frac{\partial \phi}{\partial x_0}(t, x_0) = \exp\left(t\frac{\partial f}{\partial x}(x_0)\right)$. It is also important to notice that $y = \dot{\phi}$ is always a solution of (3.2) and is non-trivial if x_0 is not an equilibrium.

Here we will be primarily concerned with the compound equations associated with (3.2), $k = 1, ..., n$,

$$\dot{z} = \frac{\partial f}{\partial x}^{[k]}(\phi(t))\, z \tag{3.3}$$

for which a fundamental matrix is $Y^{(k)}(t) = \frac{\partial \phi}{\partial x_0}^{(k)}(t, x_0)$ a matrix whose entries are Jacobian determinants of the form $\partial\left(\phi_{i_1}, ..., \phi_{i_k}\right) / \partial\left(x_{0j_1}, ..., x_{0_{j_k}}\right)(t, x_0)$.

When $k = 1$, this is equation (3.2) and when $k = n$, since $\frac{\partial f}{\partial x}^{[n]} = \operatorname{div} f$ it is the *Liouville Equation*

$$\dot{z} = \operatorname{div} f\left(\phi\left(t\right)\right) z \tag{3.4}$$

a scalar equation; a solution is

$$\partial\left(\phi_1, ..., \phi_n\right) / \partial\left(x_{01}, ..., x_{0n}\right)\left(t, x_0\right) = \det \frac{\partial \phi}{\partial x_0}\left(t, x_0\right) = \exp\left(\int_0^t \operatorname{div} f\left(\phi\left(s\right)\right) ds\right).$$

This is the Jacobian of the map $x_0 \mapsto \phi\left(t, x_0\right)$ so, for example, if $\operatorname{div} f < 0$ throughout $\mathbb{R}^n$, then $n-$dimensional volumes decrease under this map.

More generally, the equations (3.3) may be used to study the evolution in time of measures of k-dimensional surface content under the dynamics of (3.1). Let $\{dx_{01}, ..., dx_{0n}\}$ be a basis for the vector space of differential $1-$forms in $\mathbb{R}^n$. Under the map $x_0 \mapsto x = \phi\left(t, x_0\right)$, this basis is transformed into the basis $\{dx_1, ..., dx_n\}$ given by

$$dx_i = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_{0j}}\left(t, x_0\right) dx_{0j},\ i = 1, ..., n. \tag{3.5}$$

Thus the basis evolves in time as a solution of the variational equation (3.2). The corresponding lexicographically ordered basis $\{dx_{i_1} \wedge ... \wedge dx_{i_k} : 1 \leq i_1 < ... < i_k \leq n\}$ for the differential $k-$forms in $\mathbb{R}^n$ therefore satisfies

$$dx_{i_1} \wedge ... \wedge dx_{i_k} = \sum_{(j)} \frac{\partial(\phi_{i_1}, ..., \phi_{i_k})}{\partial\left(x_{0j_1}, ..., x_{0j_k}\right)}\left(t, x_0\right) dx_{0j_1} \wedge ... \wedge dx_{0j_k} \tag{3.6}$$

and evolves in time as a solution of (3.3).

Consider now a function $u \mapsto \psi_0\left(u\right)$, $u = \left(u_1, ..., u_k\right) \in D \subset \mathbb{R}^k$, $\psi_0(u) \in \mathbb{R}^n$. Such a function will be considered a *k-surface* in $\mathbb{R}^n$; it is a *smooth k-surface* if it is C^1.A measure of the k-content of this surface is

$$\sigma_k\left(\psi_0\right) = \int_D \left|\frac{\partial \psi_0}{\partial u_1} \wedge \cdots \wedge \frac{\partial \psi_0}{\partial u_k}\right| = \int_D \left|\frac{\partial \psi_0}{\partial u}^{(k)}\right|; \tag{3.7}$$

it is assumed that D is such that the integral makes sense. Different norms$|\cdot|$ give rise to different measures σ_k the most common being that generated by the l^2 norm: $\sigma_k\left(\psi_0\right) = \int \sqrt{\sum_{(i)} \left(\frac{\partial\left(\psi_{0i_1}, ..., \psi_{0i_k}\right)}{\partial\left(\ u_1, ..., u_k\right)}\right)^2}$. While all such measures are equivalent, the choice of norm for the applications may often be critical and it may even be useful to consider more general expressions than (3.7). Let $\psi_t(u) = \phi(t, \psi_0(u))$; as long as it exists this function is also a k-surface in $\mathbb{R}^n$ and $\sigma_k(\psi_t) = \int_D \left| \frac{\partial \psi_t}{\partial u}^{(k)} \right|$ from (3.7). Now $\frac{\partial \psi_t}{\partial u} = \frac{\partial \phi}{\partial x_0}(t, \psi_0) \frac{\partial \psi_0}{\partial u}$ is a $n \times k$ matrix solution of the variational equation (3.2), so $\frac{\partial \psi_t}{\partial u}^{(k)} = \frac{\partial \phi}{\partial x_0}^{(k)}(t, \psi_0) \frac{\partial \psi_0}{\partial u}^{(k)}$ is a solution of (3.3) for each $\psi_0 = \psi_0(u)$, $u \in D$. From (2.9), if $t \geq t_0$,

$$\left| \frac{\partial \psi_t}{\partial u}^{(k)} \right| \leq \left| \frac{\partial \psi_{t_0}}{\partial u}^{(k)} \right| \exp \left[\int_{t_0}^{t} \mu \left(\frac{\partial f}{\partial x}^{[k]} (\phi(s, \psi_0)) \right) ds \right] \tag{3.8}$$

and

$$\left| \frac{\partial \psi_t}{\partial u}^{(k)} \right| \geq \left| \frac{\partial \psi_{t_0}}{\partial u}^{(k)} \right| \exp \left[- \int_{t_0}^{t} \mu \left(- \frac{\partial f}{\partial x}^{[k]} (\phi(s, \psi_0)) \right) ds \right]. \tag{3.9}$$

From (3.8) [(3.9)], $\sigma_k(\psi_t)$ decreases [increases] as t increases if the trace of the k-surface ψ_t lies in a region where $\mu\left(\frac{\partial f}{\partial x}^{(k)}\right) \leq 0$ $\left[\mu\left(-\frac{\partial f}{\partial x}^{(k)}\right) \leq 0\right]$. Thus, for example, the usual k-dimensional measure $\sigma_k(\psi_t)$ decreases in time as long as solutions $\phi(t, x_0)$ with initial values $x_0 = \psi_0(u)$, $u \in D$, remain in a region where $\lambda_1 + \cdots + \lambda_k \leq 0$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\frac{1}{2}\left(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x}\right)$. This the first of the expressions for $\mu\left(A^{(k)}\right)$, $A = \frac{\partial f}{\partial x}$, given in (2.8). The other row and column expressions in (2.8) are in fact easier to calculate; however the specific choice of a Lozinskiĭ measure may be none of these and may be dictated to some extent by the problem under consideration.

3.2. Bendixson's Condition

A solution $\phi(t)$ of (3.1) is *periodic* if there exists $\omega > 0$ such that $\phi(t + \omega) = \phi(t)$; ω is a period. Obviously an equilibrium is a periodic solution; a periodic solution is called non-trivial if it is not an equilibrium. Conditions which guarantee the existence or non-existence of periodic solutions are of considerable interest and are known in great generality for 2-dimensional systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{3.10}$$

where $f : (x, y) \mapsto (P(x, y), Q(x, y))$ is C^1. The Poincaré-Bendixson theory allows us to conclude the existence of a periodic solution of (3.10) from the existence of a bounded solution. Bendixson's condition $\operatorname{div} f = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ in a simply connected region D allows us to conclude that there exists no non-trivial periodic solution whose orbit lies in D. The elegant classical proof is as follows:

> Suppose that $(x, y) = (\phi_1(t), \phi_2(t))$ is a periodic solution of least period $\omega > 0$ and orbit C. Then Green's Theorem implies that $\int_C (Pdy - Qdx) = \pm \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy \neq 0$; but (3.10) implies that $\int_C (Pdy - Qdx) = \int_0^\omega (P\dot{y} - Q\dot{x})\, dt = \int_0^\omega (PQ - QP)\, dt = 0$, a contradiction. Thus no such solution can exist.

This proof does not lend itself readily to higher dimensional generalizations of the Bendixson result. The following proof does give some insight into this situation:

> Let $\phi(t; x_0, y_0)$ denote the solution of (3.10) such that $\phi(0; x_0, y_0) = (x_0, y_0)$. Then the Jacobian determinant $z = \frac{\partial(\phi_1, \phi_2)}{\partial(x_0, y_0)}(t; x_0, y_0)$ is a solution of the Liouville equation (3.4) and therefore, if D_0 is any region in the plane, the area of the corresponding region $\phi(t; D_0)$ is
>
> $$\begin{aligned} \int_{\phi(t, D_0)} dxdy &= \int_{D_0} \frac{\partial(\phi_1, \phi_2)}{\partial(x_0, y_0)}(t; x_0, y_0)\, dx_0 dy_0 \\ &= \int_{D_0} \exp\left[\int_0^t \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) |_{\phi(s; x_0, y_0)}\, ds \right] dx_0 dy_0. \end{aligned}$$
>
> Evidently this area is strictly decreasing or increasing depending on the sign of $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$. But the area $\phi(t, D_0)$ is constant if D_0 is the region enclosed by a periodic orbit C. This contradiction again establishes the Bendixson result. In fact it shows that no simple closed curve C can be *invariant* with respect to (3.10), $\phi(t, C) = C$, since this would also imply that the area of $\phi(t, D_0)$ is constant. This also follows from the classical proof although it is not so immediate.

The preceding argument works because C is the boundary of a region D_0 which is invariant under the dynamics of the differential equation, $\phi(t, D_0) = D_0$. In a space of dimension higher than 2, let C be a simple closed curve which is invariant with respect to (3.1) and let D_0 be the trace of a 2-surface whose boundary is C and whose surface area is a minimum.. Since C is invariant, $\phi(t, D_0)$ is also the trace of a 2-surface with boundary C; its area can not be less than that of D_0. But if (3.1) is such that 2-surface areas decrease (increase) strictly, the area of $\phi(t, D_0)$ is strictly less than that of D_0, $t > 0\,(t < 0)$ contradicting the minimality of the area of D_0. If $|\cdot|$ is any norm in $\mathbb{R}^{\binom{n}{2}}$, then (3.7) defines a measure of 2-surface area in $\mathbb{R}^n$. From $(3, 8)$ and (3.9), this area of $\phi(t, D_0)$ is strictly decreasing (increasing) as long as $\phi(t, D_0)$ is in a region where

$$\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0 \quad \left[-\mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) > 0\right]. \tag{3.11}$$

Either of these is a higher dimensional version of Bendixson's condition; when $n = 2$, the first is $\operatorname{div} f < 0$ and the second is $\operatorname{div} f > 0$. For the l^2, l^1, l^∞ norms in $\mathbb{R}^{\binom{n}{2}}$respectively, we have from (2.8) the following concrete expressions for these Lozinskiĭ measures:

$$\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) = \begin{cases} \lambda_1 + \lambda_2 \\ \max_{r \neq s}\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right)\right\} \\ \max_{r \neq s}\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right)\right\} \end{cases} \tag{3.12}$$

$$-\mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) = \begin{cases} \lambda_{n-1} + \lambda_n \\ \min_{r \neq s}\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right)\right\} \\ \min_{r \neq s}\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right)\right\} \end{cases} \tag{3.13}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\frac{1}{2}(\partial f/\partial x^* + \partial f/\partial x)$. A Bendixson condition is satisfied if any of the expressions in (3.12) is negative or if any of those in (3.13) is positive. With some other technical restrictions, any such condition should imply the non-existence of invariant closed curves, in particular the non-existence of non-constant periodic solutions.

Broadly speaking, the preceding argument is correct. There are however technical questions which arise:

- How do we know that, for a simple closed curve C in the region D where one of (3.11) is satisfied, there is a 2-surface with boundary C for which the corresponding measure of area is a minimum? This is a deep geometric question. Fortunately, it will not be necessary for us to answer it here. We will see that our requirements will be met if we can ensure that a minimizing sequence of surfaces with boundary C exists in the region where (3.11) holds.

- What can be said if the geometry of D is such that no minimizing sequence of surfaces exists in D? In this situation we will still sometimes be able to conclude the non-existence of invariant closed curves C if we can establish that there is a positive lower bound for the areas of those surfaces which have a given C as their boundary.

The first result of this type seems to be due to *RA Smith* who showed that, if (3.1) is dissipative and $\lambda_1 + \lambda_2 < 0$, then there is no simple closed curve which is invariant. The proof shows that under these conditions the Hausdorff dimension of the global attractor is less than 2 and the desired contradiction is obtained by showing that the dimension of the attractor is at least 2 if it contains a simple closed curve. There is an error in this part of the proof but the general approach is valid and highly original; the result as stated is correct. Smith's approach can also be used here for dissipative systems as it can be shown that the general condition $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0$ also implies that the Hausdorff dimension of the attractor is less than 2. However the approach which we use here relies on evolution of areas rather than estimations of Hausdorff dimension.

We will give two general approaches to the question developed by *Muldowney* and *Li & Muldowney*:

- The first one shows that in any open region D in which a measure of 2-surface area decreases strictly, there is no invariant closed curve which is the boundary of a minimizing sequence of $2-$surfaces for this area. This imposes strong restrictions on the shape of D. This is not so restrictive on the differential equation; it does not, for example require that solutions originating in D exist for all time t but only for t close to 0.

- The second approach only requires that D be simply connected. Then any simple closed curve in D is homotopic to a point in D and is therefore the boundary of a $2-$surface in D. By showing that the areas of all such surfaces have a positive lower bound which depends only on the boundary, requiring that solutions exist for all time and imposing a condition that implies the area of such a surface tends to 0 as $t \to \infty$, we again reach a contradiction if the closed curve is invariant. This approach relaxes the restrictions on D but has stronger requirements on the differential equation in terms of global existence of solutions.

Let $D \subset \mathbb{R}^n$ and let $U = B^2(0,1)$, the open unit disc in $\mathbb{R}^2$, $\overline{U}$ is its closure and ∂U is its boundary. A function $\psi \in \mathrm{Lip}\left(\overline{U} \to D\right)$ is a *simply connected rectifiable 2-surface* in D; a function $\gamma \in \mathrm{Lip}(\partial U \to D)$ is a *closed rectifiable curve* in D and is *simple (a Jordan curve)* if it is one-to-one. If γ is the restriction of ψ to ∂U, then γ is the *boundary* of ψ, which is denoted $\gamma = \partial\psi$. The sets $\psi\left(\overline{U}\right), \gamma(\partial U)$ are the *traces* of ψ, γ respectively.

Proposition 17. *(a) If γ is a Jordan curve in $\mathbb{R}^n$, there is a simply connected rectifiable 2-surface ψ in $\mathbb{R}^n$ such that $\gamma = \partial\psi$.*

(b) Let C be the trace of a Jordan curve γ in $\mathbb{R}^n$ and let σ_2 be a measure of 2-surface area corresponding to a norm $|\cdot|$ in $\mathbb{R}^{\binom{n}{2}}$ as defined in (3.7) There exists $m > 0$ which depends only on C such that

$$m \leq \sigma_2(\psi) \tag{3.14}$$

for every simply connected rectifiable 2-surface ψ such that $\gamma = \partial\psi$ is a Jordan curve with trace C.

Proof. The proof of Part (a) is left as an exercise. To prove Part (b), it is sufficient to consider only the case $|y| = (y^*y)^{1/2}$, since all norms in $\mathbb{R}^{\binom{n}{2}}$ are equivalent. We will consider only curves γ which are C^2; less smooth curves can be handled by approximation. It is also sufficient to consider only surfaces ψ whose trace lies in the convex hull of C. This follows from the fact that if Π is any $(n-1)$-dimensional hyperplane which does not intersect C we can replace ψ, by orthogonal projection of sections onto Π if necessary, by a surface ψ_0 such that $\partial\psi_0 = \partial\psi = \gamma$ and $\sigma_2(\psi_0) \leq \sigma_2(\psi)$. Let $x \mapsto a\ (x)$, be a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}^n$an let α be the 1-form defined by $\alpha = \sum_i a_i(x)\,dx_i$. Then Stokes' Theorem, $\alpha(\partial\psi) = d\alpha(\psi)$, implies

$$\int_{\gamma(\partial U)} \sum_i a_i(x)\,dx_i = \int_{\psi\left(\overline{U}\right)} \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right) dx_i \wedge dx_j. \tag{3.15}$$

The expression on the left of (3.15) does not depend on the particular parametrization γ of C chosen. Now, for some particular γ, if $x = \gamma(\cos\theta, \sin\theta)$, let

$a(x) = \gamma_\theta(\cos\theta, \sin\theta) = \frac{d}{d\theta}\gamma(\cos\theta, \sin\theta)$ and extend a as a continuously differentiable function to $\mathbb{R}^n$. The length l of C satisfies, by the Cauchy-Bunyakowski-Schwarz inequality,

$$\frac{1}{2\pi}l^2 \leq \int_0^{2\pi} \gamma_\theta^* \gamma_\theta \, d\theta = \int_{\gamma(\partial U)} \sum_i a_i(x)\, dx_i. \tag{3.16}$$

With this choice of a,

$$\int_{\psi(\overline{U})} \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right) dx_i \wedge dx_j = \int_{\overline{U}} z^*(u)\, y(u)\, du, \tag{3.17}$$

where $z_i(u) = \left(\frac{\partial a_{i_2}}{\partial x_{i_1}} - \frac{\partial a_{i_1}}{\partial x_{i_2}}\right)|_{x=\psi(u)}$, $y_i(u) = \frac{\partial\left(\psi_{i_1}, \psi_{i_2}\right)}{\partial(u_1, u_2)}$ and $(i) = (i_1, i_2)$. Since $\psi(u)$ is in the convex hull of C, there is a constant M such that $|z(u)| \leq M$ and so

$$\int_{\overline{U}} z^*(u)\, y(u)\, du \leq M\sigma_2(\psi). \tag{3.18}$$

Combining $(3.15) - (3.18)$, we find $\frac{l^2}{2\pi M} \leq \sigma_2(\psi)$ so that we may choose $m = \frac{l^2}{2\pi M}$. ■

Let $|\cdot|$ be a norm in $\mathbb{R}^{\binom{n}{2}}$. A subset D of $\mathbb{R}^n$ has the *minimum property* with respect to the area defined by $|\cdot|$ if each Jordan curve with trace C in D is such that there exists a sequence of simply connected rectifiable 2-surfaces $\{\psi^k\}$ in D, $\{\psi^k(\overline{U}) : k = 1, 2, ...\}$ has compact closure in D and $\lim_{k\to\infty} \sigma_2(\psi^k) = m_0$, the infimum of the surface areas of all such 2-surfaces ψ with $\psi(\partial U) = C$. We note that any convex open set has the minimum property with respect to the area defined by the l^2 norm.

Theorem 18. *Suppose that (a) D is an open subset of $\mathbb{R}^n$ which has the minimum property with respect to the area defined by a norm $|\cdot|$.*

(b) Either $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0$ or $\mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) < 0$ in D where μ is the Lozinskiĭ measure corresponding to $|\cdot|$.

Then there is no Jordan curve in D which is invariant with respect to (3.1).

Proof. Suppose $C \subset D$ is the trace of a Jordan curve γ which is invariant with respect to (3.1), *ie.* $\phi(t,C) = C$, and that $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0$ in D. Let $\{\psi^k\}$ be a sequence of simply connected 2-surfaces in D such that $\psi_0^k(\partial U) = C$, ψ_0^k is one-to-one on ∂U and $\lim_{k\to\infty} \sigma_2\left(\psi_0^k\right) = m_0$, the infimum of the areas of all 2-surfaces which have boundary C. Since $\left\{\psi_0^k\left(\overline{U}\right) : k = 1, 2, ...\right\}$ is a compact subset of D, every solution $\phi\left(t, \psi_0^k(u)\right) = \psi_t^k(u)$ exists for $t \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ and $u \in \overline{U}$. Also, since $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0$ in D, $\mu\left(\frac{\partial f}{\partial x}^{[2]}\left(\psi_t^k(u)\right)\right) \leq -\delta < 0$, if $t \in [0, \varepsilon]$. Therefore (3.7) and (3.8) imply $\sigma_2\left(\psi_\varepsilon^k\right) \leq e^{-\delta\varepsilon}\sigma_2\left(\psi_0^k\right)$, $k = 1, 2, ...$ But the surfaces ψ_ε^k also have C as their boundary since C is invariant with respect to (3.1) so that $m_0 \leq \sigma_2\left(\psi_\varepsilon^k\right)$ and hence $m_0 \leq e^{-\delta\varepsilon} m_0 < m_0$, since $m_0 > 0$. This contradiction shows that C can not be invariant. A similar argument applies by considering the surfaces ψ_t^k, $t \in [-\varepsilon, 0]$ when $\mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) < 0$. ■

Given a family $\mathcal{M}$ of $n \times n$ matrix functions A, we consider the corresponding differential equations $\dot{x} = A(t)\,x$, $A \in \mathcal{M}$. These equations are *equi-asymptotically stable* if the fundamental matrices $X(t)$ satisfy $\lim_{t\to\infty} \left|X(t)\,X^{-1}(0)\right| = 0$ uniformly with respect $A \in \mathcal{M}$. Equivalently, if $\varepsilon > 0$ there exists $T > 0$ such that all solutions $x(t)$ of the equations satisfy $|x(t)| \leq \varepsilon\,|x_0|$ if $t \geq T$. A sufficient condition for equi-asymptotic stability is $\lim_{t\to\infty} \int_0^t \mu(A) = -\infty$ uniformly with respect to $A \in \mathcal{M}$.

Theorem 19. *Suppose that (a) D is an open simply connected subset of $\mathbb{R}^n$.*

(b) If $x_0 \in D$, $\phi(t, x_0)$ exists for all $t \geq 0$.

(c) The equations $\dot{y} = \frac{\partial f}{\partial x}^{[2]}(\phi(t, x_0))\,y$, $x_0 \in S$, are equi-asymptotically stable if S is a compact subset of D.

Then there is no Jordan curve in D which is invariant with respect to (3.1).

Proof. Suppose that $C \subset D$ is the trace of a Jordan curve γ which is invariant with respect to (3.1). Let ψ_0 be a simply connected rectifiable surface in D which is one-to-one on ∂U and $\psi_0(\partial U) = C$. Then $u \mapsto \psi_t(u) = \phi(t, \psi_o(u))$ is also a simply connected rectifiable 2-surface which is one-to-one on ∂U and $\psi_t(\partial U) = C$. Then, since $\frac{\partial \psi_t}{\partial u}(u) = \frac{\partial \phi}{\partial x_0}(t, \psi_0(u))\,\frac{\partial \psi_0}{\partial u}(u)$ implies $y = \frac{\partial \psi_t}{\partial u}^{(2)}(u) = \frac{\partial \phi}{\partial x_0}^{(2)}(t, \psi_0(u))\,\frac{\partial \psi_0}{\partial u}^{(2)}(u)$ is a solution of $\dot{y} = \frac{\partial f}{\partial x}^{[2]}(t, x_0)\,y$, $x_0 = \psi_0(u)$, $u \in \overline{U}$.

These equations are equi-asymptotically stable and $\lim_{t\to\infty}\left|\frac{\partial\phi_t}{\partial u}^{(2)}(t,\psi_0(u))\right| = 0$ uniformly with respect to $u\in\overline{U}$. Therefore $\lim_{t\to\infty}\left|\frac{\partial\psi_t}{\partial u}^{(2)}(u)\right| = 0$ uniformly with respect to $u\in\overline{U}$ so that $\lim_{t\to\infty}\sigma_2(\psi_t) = \int_{\psi_t(\overline{U})}\left|\frac{\partial\psi_t}{\partial u}\right| = 0$. But Proposition 17 shows that $m\le\sigma_2(\psi_t)$ for some $m>0$. Thus C can not be invariant. ■

Corollary 20. *Suppose that D is open and simply connected, $\phi(t,x_0)$ exists for all $t>0$ $[t<0]$ and*

$$\lim_{t\to\infty}\int_0^t\mu\left(\frac{\partial f}{\partial x}^{[2]}(\phi(t,x_0))\right) = -\infty\ \left[\lim_{t\to-\infty}\int_0^t\mu\left(-\frac{\partial f}{\partial x}^{[2]}(\phi(t,x_0))\right) = \infty\right]$$

uniformly with respect to $x_0\in S$, if S is any compact subset of D, then there is no Jordan curve in D which is invariant with respect to (3.1).

The first of these conditions implies the equi-asymptotic stability requirement (c) of Theorem 19 and the corollary may be deduced by applying Proposition 16(ii) to $A(t) = \frac{\partial f}{\partial x}^{[2]}(\phi(t,x_0))$; the second may be inferred from this by considering behaviour as $t\to-\infty$ instead of $t\to\infty$.

Remark 4. *Corollary 20 illustrates the fact that, in contrast to Theorem 18, Theorem 19 does not require that Bendixson's condition hold everywhere but rather in some averaged sense along orbits. For example, we have seen that if $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}<0$ in a simply connected subset D of the plane then* (3.10) *has no non-trivial periodic orbits in D. The corollary shows that the same conclusion can be drawn if instead we assume that solutions $\phi(t;x_0,y_0)$ exist for all $t\ge 0$ when $(x_0,y_0)\in D$ and $\lim_{t\to\infty}\int_0^t\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\big|_{\phi(s;x_0,y_0)}\,ds = -\infty$ uniformly with respect to (x_0,y_0) in compact subsets of D.*

A generalization of Bendixson's condition for planar systems (3.10) is *Dulac's Condition*: $\frac{\partial}{\partial x}(\alpha P)+\frac{\partial}{\partial y}(\alpha Q)\neq 0$, where $(x,y)\mapsto\alpha(x,y)$ is real-valued. This condition can also be shown to preclude an invariant Jordan curve C in a simply connected region D by considering $\int_C(\alpha P)\,dy-(\alpha Q)\,dx$ in the classical proof. Alternatively, this may also be demonstrated by showing that this condition implies $\int_{\phi(t,D_0)}\alpha(x,y)\,dxdy$ is either strictly increasing or decreasing, contradicting the fact that it is a constant if D_0 is the region enclosed by an invariant Jordan curve C. The introduction of this arbitrary function α is in fact just a change in the 2−measure under consideration and adds considerable versatility to the criterion. This can be extended to higher dimensions by considering, instead of $\sigma_2(\psi_t)$, more general functionals of the form $\int_{\overline{U}} \left| A(\psi_t) \frac{\partial \psi_t}{\partial u}^{(2)} \right|$, where A is any $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function, and formulating conditions under which this decreases or increases as in the preceding paragraphs. This adds great versatility to the criterion: the matrix A represents $\binom{n}{2}^2$ arbitrary functions which may be used to test the system.

Many physical and biological systems have *first integrals:* functions which are constant along solutions such as conservation laws. An important observation of *M Li* is that, if (3.1) has m independent first integrals, then the conditions given above on the system (3.3) with $k = 2$ may be relaxed to conditions on (3.3) with $k = m + 2$. Thus, where $\lambda_1 + \lambda_2 < 0$ is, with appropriate restrictions on the shape of the region, a Bendixson condition for (3.1): when (3.1) has m independent first integrals, this condition may be relaxed to $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+2} < 0$. For example, when $n = 3$, $\operatorname{div} f < 0$ on $\mathbb{R}^3$ does not preclude non-trivial periodic solutions of (3.1) in general but does so when the system has a first integral.

3.3. Orbital Stability of Periodic Solutions

The preceding section was primarily concerned with investigating the implications of the compound equations for the behaviour of measures in the dynamics of (3.1). Here we will be more concerned with algebraic aspects of the equations and the manner in which they reveal the stability of periodic orbits. Suppose that $x = \phi(t)$ is a periodic solution of (3.1) with least period $\omega > 0$: $\phi(t + \omega) = \phi(t)$, with orbit C, a simple closed curve in $\mathbb{R}^n$. This solution is *orbitally stable* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_0, C) < \delta$ implies $d(\phi(t, x_0), C) < \varepsilon$ for all $t \geq 0$. It is *orbitally asymptotically stable* if it also satisfies $\lim_{t\to\infty} d(\phi(t, x_0), C) = 0$ and *orbitally asymptotically stable with asymptotic phase* if, additionally, there is a $\delta_0 > 0$ and whenever $d(x_0, C) < \delta_0$ there is a real number $\tau(x_0)$ such that $\lim_{t\to\infty} |\phi(t + \tau(x_0), x_0) - \phi(t)| = 0$.

A useful approach to the stability problem is by means of the *Poincaré map.* If $x_0 \in C$, let $\Pi = \{x : (x - x_0)^* f(x_0) = 0\}$, the $(n-1)$-dimensional hyperplane at x_0 perpendicular to the vector field of (3.1) at that point. It is an exercise in the implicit function theorem to show that there is a unique real-valued C^1 function $x \mapsto \omega(x)$, $\omega(x_0) = \omega$, such that $\phi(\omega(x), x) \in \Pi$. The Poincaré map $\mathcal{P}$ is the restriction $\mathcal{P}(x) = \phi(\omega(x), x)|_{x\in\Pi}$. Evidently the fixed point x_0 of this C^1 map corresponds to the periodic solution ϕ and the stability character of C may be determined by studying the stability of x_0 with respect to iterations of the map $\mathcal{P}$. In particular, if $\lim_{n\to\infty} \mathcal{P}^n(x) = x_0$ for all $x \in \Pi$ near x_0, where $\mathcal{P}^0(x) = x$ and $\mathcal{P}^n(x) = \mathcal{P} \circ \mathcal{P}^{n-1}(x)$, $n = 1, 2, \cdots$, the fixed point x_0 is said to be asymptotically stable with respect to iterations of the map $\mathcal{P}$. This is a necessary and sufficient condition for the solution ϕ to be orbitally asymptotically stable with asymptotic phase. Finally, a sufficient condition for the asymptotic stability of the fixed point x_0 with respect to the Poincaré map and hence for the orbital stability of ϕ is that all eigenvalues ν of the operator $\mathcal{DP}(x_0)$ satisfy $|\nu| < 1$.

The linearization of (3.1), $\dot{x} = f(x)$, with respect to the ω-periodic solution $x = \phi(t)$ is the differential equation (3.2), $\dot{y} = \frac{\partial f}{\partial x}(\phi(t))\, y$, a linear system with periodic coefficient matrix. Therefore the Floquet theory applies and, if $Y(t)$ is a fundamental matrix, there exists a non-singular $n \times n$ matrix V such that $Y(t+\omega) = Y(t)\, V$ and hence $Y(t + n\omega) = Y(t)\, V^n$, $n = 1, 2, \cdots$. Thus the stability of (3.2) is determined by the eigenvalues of V. These eigenvalues are called the Floquet multipliers of the system; the system is asymptotically stable,for example, if and only if every Floquet multiplier λ satisfies $|\lambda| < 1$. Further (3.2) has a non-trivial periodic solution if and only if $\lambda = 1$ is a Floquet multiplier. Since $y = \dot{\phi}(t)$ is a non-trivial periodic solution of (3.2), it follows that at least one multiplier of the system (3.2) equals 1. Since $Y(t) = \frac{\partial}{\partial x}\phi(t, x)$, $x = x_0$, is a fundamental matrix for (3.2) with $Y(0) = I$ it follows that $V = \frac{\partial}{\partial x}\phi(\omega, x)$, $x = x_0$, in this case and that $f(x_0)$ is an eigenvector of V corresponding to the multiplier 1. The remaining Floquet multipliers are the eigenvalues of $\mathcal{DP}(x_0)$.

Thus, from the preceding two paragraphs, the periodic solution $\phi(t)$ is orbitally asymptotically stable with asymptotic phase if all but one of the Floquet multipliers λ of the system (3.2) satisfy $|\lambda| < 1$. This condition is difficult to check in practice. However, we have the following orbital stability condition of *Poincaré:*

> When $n = 2$, a ω-periodic solution $\phi(t)$ of (3.1) is orbitally asymptotically stable with asymptotic phase if $\int_0^\omega \operatorname{div} f(\phi(s))\, ds < 0$. To see this, recall that $z(t) = \det \frac{\partial}{\partial x}\phi(t, x_0)$ is the solution of $\dot{z} = \operatorname{div} f(\phi(t))\, z$, $z(0) = 1$ so that
>
> $$\det V = \det \frac{\partial}{\partial x}\phi(\omega, x_0) = \exp\left(\int_0^\omega \operatorname{div} f(\phi(s))\, ds\right) < 1.$$
>
> But $\det V = \lambda_1 \lambda_2$ and one of these Floquet multipliers $\lambda_1 = 1$ so $0 < \lambda_2 < 1$ and the result follows. Recall that, when $n = 2$, $\operatorname{div} f = \frac{\partial f}{\partial x}^{[2]}$

so the Poincaré stability criterion has the following generalization to higher dimensions.

Theorem 21. *The periodic solution $\phi(t)$ of $\dot{x} = f(x)$ is orbitally asymptotically stable with asymptotic phase if*

$$\dot{z} = \frac{\partial f}{\partial x}^{[2]}(\phi(t))\, z \tag{3.19}$$

is asymptotically stable.

Proof. The equation (3.19) is also a system with periodic coefficient matrix and $Z(t) = Y(t)^{(2)}$ is a fundamental matrix if $Y(t)$ is a fundamental matrix for (3.2). Therefore $Y(t+\omega) = Y(t)V$ implies $Y^{(2)}(t+\omega) = Y^{(2)}(t)V^{(2)}$ so that $Z(t+\omega) = Z(t)V^{(2)}$and $\lambda_i\lambda_j$, $1 \leq i < j \leq n$, are the Floquet multipliers of (3.19) if $\lambda_1, \cdots, \lambda n$ are the Floquet multipliers for (3.2). Therefore the stability of (3.19) implies $|\lambda_i\lambda_j| < 1$. But $\lambda_1 = 1$ is a multiplier for (3.2) since $z = \dot{\phi}(t)$ is a periodic solution and thus $|\lambda_i| < 1$, $i = 2, ..., n$ and the result follows. ■

A more concrete generalization of the Poincaré criterion is given in the corollary where the given condition implies the asymptotic stability of (3.19) by Proposition16(ii).

Corollary 22. *The periodic solution $\phi(t)$ of $\dot{x} = f(x)$ is orbitally asymptotically stable with asymptotic phase if*

$$\int_0^\omega \mu\left(\frac{\partial f}{\partial x}^{[2]}(\phi(s))\right) ds < 0, \tag{3.20}$$

where μ is a Lozinskiĭ measure.

Analogous to Li's result on generalized Bendixson criteria, if the system (3.1) has m independent first integrals, then $\frac{\partial f}{\partial x}^{[2]}$ may be replaced by $\frac{\partial f}{\partial x}^{[m+2]}$ in Theorem 21 and Corollary 22 and a similar result on orbital stability with respect to solutions in the integral manifold containing the periodic orbit may be proved.

3.4. General Orbital Stability

In this section we will consider the implications of Poincaré's stability condition if the assumption of periodicity of $\phi(t)$ is relaxed to one of boundedness and it is still assumed that (3.19) is uniformly asymptotically stable. Note that the qualification 'uniformly' is not included in the statement of Theorem 21 since that is implied when $\phi(t)$ is periodic. It turns out that this has strong implications for the omega limit set Ω of $\phi(t)$, where Ω is the set of points x such that $\lim_{n\to\infty}\phi(t_n) = x$ for some sequence $t_n \to \infty$. It will be shown that the cited stability condition on (3.19) implies that either Ω contains an equilibrium or it is a stable hyperbolic periodic orbit. '*Stable hyperbolic*' here means that the Floquet multipliers λ of the periodic orbit, with the exception of a single multiplier which equals 1, all satisfy $|\lambda| < 1$. The importance of this type of result lies in the fact that it infers the existence of a periodic orbit, a limit cycle, from the stability of a general orbit. The most well-known result of this type is the Poincaré-Bendixson Theorem which shows that, in a 2-dimensional system, if $\phi(t)$ is a bounded solution then its omega limit set is either a periodic orbit if it does not contain an equilibrium.

The main results of this section are special cases of results for general semi-flows in metric spaces by Li and Muldowney. These results generalize theorems of Pliss and of Sell among others and are outlined in Section 3.4.3. We will first consider the situation of equilibria.

3.4.1. Stable Hyperbolic Equilibria

If $f(x_0) = 0$, then $\phi(t, x_0) = x_0$ and x_0 is an equilibrium of (3.1), $\dot{x} = f(x)$. Then the variational equation with respect to this solution is

$$\dot{y} = \frac{\partial f}{\partial x}(x_0)\, y. \tag{3.21}$$

The equation is uniformly asymptotically stable if and only if every eigenvalue λ of $\frac{\partial f}{\partial x}(x_0)$ satisfies $\operatorname{Re}\lambda < 0$. This is equivalent to $\left|\frac{\partial}{\partial x}\phi(t, x_0)\right| \le Ke^{-\alpha t}, t \ge 0$, where $K, \alpha > 0$. Then x_0 is said to be a *stable hyperbolic equilibrium* of the non-linear system (3.1) and x_0 attracts all nearby points exponentially: $|\phi(t, x) - x_0| \le Ke^{-\alpha t}$ for some $K, \alpha > 0$ and all x in a neighbourhood of x_0.

We assume for simplicity that the domain of f is $\mathbb{R}^n$ and investigate the consequences of the stability of (3.21) if the equilibrium x_0 is replaced by a bounded solution $\phi(t)$.

Theorem 23. *Let $\phi(t)$ be a bounded solution of* (3.1), $\dot{x} = f(x)$. *Then the omega limit set Ω of ϕ is a stable hyperbolic equilibrium if and only if* (3.2), $\dot{y} = \frac{\partial f}{\partial x}(\phi(t))\, y$, *is uniformly asymptotically stable.*

Proof. Let $C_+ = \{\phi(t) : t \geq 0\}$, the positive semi-orbit of ϕ. First, suppose that (3.2) is uniformly asymptotically stable: this is equivalent to

$$\left| \frac{\partial}{\partial x} \phi(t, x) \right| \leq K e^{-\alpha t},\ 0 \leq t < \infty, \tag{3.22}$$

for all $x \in C_+$, where K, α are positive constants. By continuity, (3.22) is satisfied also at every omega limit point of ϕ. Next, since $y = \dot{\phi}(t)$ is a solution of (3.2), it follows that $\lim_{t\to\infty} \dot{\phi}(t) = 0$ and hence $\lim_{t\to\infty} f(\phi(t)) = 0$ so that every omega limit point x of ϕ is an equilibrium. From (3.22), it is a hyperbolic stable equilibrium and therefore isolated; Ω is thus a single stable hyperbolic equilibrium. Conversely, suppose that the omega limit set $\Omega = \{x_*\}$, where x_* is a stable hyperbolic equilibrium. Then (3.22) is satisfied at $x = x_*$. First, choose $\beta,\ L, T > 0$ such that $\beta < \alpha$, $L > K$ and T sufficiently large that $L \leq e^{\frac{\beta}{2}T}$ and hence $Le^{-\beta T} \leq e^{-\frac{\beta}{2}T}$. Then, by continuity, there is a neighbourhood U of x_* such that

$$\left| \frac{\partial}{\partial x} \phi(t, x) \right| \leq L e^{-\beta t},\ 0 \leq t \leq T \tag{3.23}$$

and $\phi(t) \in U$ for t sufficiently large. Let $t_k = kT$ and $x_k = \phi(t_k, x_0) \in U$, $k = 0, 1, 2, \cdots$. Suppose that $t \in [t_N, t_{N+1})$. Then $\phi(t+s, x) = \phi(t, \phi(s, x))$ implies $\phi(t, x) = \phi(t - t_N, \phi(t_N - t_{N-1}, \cdots, \phi(t_1 - t_0, x) \cdots))$ and hence

$$\frac{\partial}{\partial x}\phi(t, x_0) = \frac{\partial}{\partial x}\phi(t - t_N, x_N)\, \frac{\partial}{\partial x}\phi(t_N - t_{N-1}, x_{N-1}) \cdots \frac{\partial}{\partial x}\phi(t_1 - t_0, x_0).$$

Therefore $\left|\frac{\partial}{\partial x}\phi(t, x_0)\right| \leq Le^{-\beta(t-t_N)} \left(Le^{-\beta T}\right)^N \leq Le^{-\frac{\beta}{2}(t-NT)} \left(e^{-\frac{\beta}{2}T}\right)^N = Le^{-\frac{\beta}{2}t}$, from (3.23), and it follows that (3.2) is uniformly asymptotically stable. ■

Corollary 24. *A sufficient condition for the omega limit set of a bounded solution $\phi(t)$ of* (3.1) *to be a stable hyperbolic equilibrium is*

$$\int_{t_0}^{t} \mu\left(\frac{\partial f}{\partial x}(\phi(s))\right) ds \leq -\alpha(t - t_0) + \beta,\ 0 \leq t_o \leq t < \infty$$

where $\alpha > 0, \beta$ are constants and μ is any Lozinskiĭ measure.

Proof. This follows from Proposition 16(iv). ■

Remark 5. *The condition of Corollary 24 is satisfied if* $\mu\left(\frac{\partial f}{\partial x}(x)\right) \leq -\alpha < 0$ *for all* $x \in \operatorname{cl} C_+$, *the closure of the semi-orbit of* $\phi(t)$.

3.4.2. Stable Hyperbolic Periodic Orbits

Theorem 25. *Let* $\phi(t)$ *be a bounded solution of* (3.1), $\dot{x} = f(x)$, *and let* Ω *be its omega limit set. If* Ω *contains no equilibrium, then it is a stable hyperbolic periodic orbit if and only if* (3.21), $\dot{z} = \frac{\partial f}{\partial x}^{[2]}(\phi(t))\, z$, *is uniformly asymptotically stable.*

The proof depends on the following result on dichotomies for a linear system. Let $X(t)$ be a fundamental matrix for a system (2.1), $\dot{x} = A(t)\, x$; then $Y(t) = X^{(2)}(t)$ is a fundamental matrix for its second compound equation $\dot{y} = A^{[2]}(t)\, y$.

Proposition 26. *Suppose that:*

(a) There exist constants K, L, β *such that*

$$|x(t)| \leq K\, |x(s)|\, e^{\beta(t-s)},\ 0 \leq s \leq t < \infty$$

for all solutions $x(t)$ *of* (2.1) *and*

$$|x_1(t)| \leq L\, |x_1(s)|\,,\ 0 \leq s, t < \infty$$

for some non-zero solution $x_1(t)$ *of* (2.1).

(b)The second compound equation of (2.1) *is uniformly asymptotically stable.*

Then (2.1) *is uniformly stable and there exist supplementary projections* P_1, P_2 *on* $\mathbb{R}^n$ *where* $\operatorname{rk} P_1 = 1$, $\operatorname{rk} P_2 = n-1$, *and constants* $C, \alpha > 0$ *such that*

$$\begin{aligned} \left|X(t)\, P_1 X^{-1}(s)\right| &\leq C,\ 0 \leq s, t < \infty \\ \left|X(t)\, P_2 X^{-1}(s)\right| &\leq Ce^{-\alpha(t-s)},\ 0 \leq s \leq t < \infty \end{aligned} \tag{3.24}$$

In particular, (2.1) *is uniformly stable.*

Condition (a) requires that all solutions of (2.1) grow no faster than exponentially (true if $A(t)$ is bounded) and that it has a 1-dimensional strongly stable subspace (roughly, a solution which is bounded and bounded away from zero). The conclusion then is that there is also a $(n-1)-$dimensional subspace which is uniformly asymptotically stable and that the angle between the two subspaces is bounded away from zero provided that the second compound equation is uniformly asymptotically stable. A proof is given by Li and Muldowney.

Proof of Theorem 25(sketch). First, since $\phi(t)$ is bounded, $A(t) = \frac{\partial f}{\partial x}(\phi(t))$ is bounded and the solution $x_1(t) = \dot{\phi}(t)$ is bounded and bounded away from 0 since Ω contains no equilibrium so condition (a) of Proposition 26 is satisfied by the variational equation $\dot{x} = \frac{\partial f}{\partial x}(\phi(t))\, x$. Suppose that (b) is also satisfied: $\dot{z} = \frac{\partial f}{\partial x}^{[2]}(\phi(t))\, z$ is uniformly asymptotically stable. Then the proposition implies that the solution space of the variational equation splits into two subspaces as described in (3.24). Moreover, the conditions (a), (b) are sufficiently robust that they are also satisfied if $\phi(t)$ is replaced by any solution in the omega limit set Ω which is invariant. We will therefore assume that $\phi(t) = \phi(t, x_0)$ where $x_0 \in \Omega$ and so $\phi(t) \in \Omega$, $0 \leq t < \infty$. Now let $z = x - \phi(t)$; then (3.1), $\dot{x} = f(x)$, is equivalent to

$$\dot{z} = \frac{\partial f}{\partial x}(\phi(t))\, z + F(t,z), \qquad F(t,z) = f(\phi(t) + z) - f(\phi(t)) - \frac{\partial f}{\partial x}(\phi(t))\, z, \tag{3.25}$$

where $F(t,0) = 0$ and $F(t,z) = o(|z|)$ uniformly with respect to $t \geq 0$ when $|z|$ is small. Consider the Banach space $\mathbf{B}_\gamma = \{z(\cdot) \in C([0,\infty) \to \mathbb{R}^n, \|z(\cdot)\| < \infty\}$, where $\|z(\cdot)\| = \sup_{t\geq 0} e^{\gamma t}|z(t)|$. If $0 < \gamma < \alpha$ and $\xi \in \mathbb{R}^n$, $P_1\xi = 0$, then (3.24) with $X(t) = \frac{\partial}{\partial x}\phi(t, x_0)$ implies that the map $z(\cdot) \mapsto \mathcal{T}_\xi z(\cdot)$ defined by

$$\mathcal{T}_\xi z(t) = X(t)\xi + \int_0^t X(t) P_2 X^{-1}(s) F(s, z(s))\, ds - \int_t^\infty X(t) P_1 X^{-1}(s) F(s, z(s))\, ds \tag{3.26}$$

on a sufficiently small neighbourhood of $0 \in \mathbf{B}_\gamma$ is a contraction and the contraction constant may be chosen independent of ξ if $|\xi|$ is small. From the Uniform Contraction Mapping Principle, there is a unique fixed point $z(\cdot)$, $z(t) = z(t,\xi)$, and the map $\xi \mapsto z(\cdot,\xi)$ is C^1. Moreover (3.26) implies that $\phi(t) + z(t,\xi)$ is a solution (3.1):

$$\phi(t,x) = \phi(t) + z(t,\xi), \quad x = z(0,\xi) = x_0 + \xi + G(\xi) \tag{3.27}$$

where $G(\xi) = P_1 \int_0^\infty X^{-1}(s) F(s, z(s))\, ds$. From (3.24), (3.25) and (3.26) and the fact that $\phi(\cdot)$ is bounded, $G(\xi) = o(|\xi|)$ when $\xi \to 0$. The linearization at 0 of the map $\xi \mapsto x_0 + \xi + G(\xi)$ is therefore the identity map $\xi \mapsto \xi$. For ρ

sufficiently small, the set of all x satisfying (3.27), $|\xi| < \rho$, $P_1\xi = 0$ is a manifold S_ρ of dimension $n-1$ and $x \in S_\rho$ implies

$$\lim_{t\to\infty} |\phi(t,x) - \phi(t)| = 0; \tag{3.28}$$

S_ρ is transverse to the flow at x_0 since $P_1\xi = 0$, $P_1 f(x_0) = f(x_0)$ and $P_1 + P_2 = I$. Thus every orbit which comes close to x_0 crosses S_ρ. Since $x_0 \in \Omega$, there exist $x_1, x_2 \in S_\rho$ such that $\phi(t_1, x_1) = x_2$, $t_1 > 0$ and a sequence $t_k \to \infty$, $\phi(t_k, x_1) \to x_0$, $k \to \infty$. Now (3.28) with $x = x_1$, $x = x_2$ implies $\lim_{t\to\infty} |\phi(t,x_1) - \phi(t,x_2)| = 0$ so that

$$\lim_{t\to\infty} |\phi(t,x_1) - \phi(t_1, \phi(t,x_1))| = 0 \tag{3.29}$$

since $\phi(t,x_2) = \phi(t,\phi(t_1,x_1)) = \phi(t+t_1,x_1) = \phi(t_1,\phi(t,x_1))$. Letting $t = t_k$ in (3.29) we find $\lim_{k\to\infty} |\phi(t_k,x_1) - \phi(t_1,\phi(t_k,x_1))| = 0$ so that $x_0 = \phi(t_1,x_0)$; thus $\phi(0) = \phi(t_1)$ and $\phi(\cdot)$ is periodic with period t_1. From Theorem 25, this orbit is stable hyperbolic and therefore attracts all nearby orbits; its orbit is thus the whole set Ω. Conversely, if Ω is a stable hyperbolic periodic orbit, Theorem 21 implies that $\left|\frac{\partial\phi}{\partial x}^{(2)}(t,x)\right| \le Ke^{-\alpha t}$, if $x \in \Omega, 0 \le t < \infty$, where $K, \alpha > 0$. As in the proof of Theorem 23, if $L > K, T > 0$ and $0 < \beta < \alpha$, there exists a neighbourhood U of Ω such that

$$\left|\frac{\partial\phi}{\partial x}^{(2)}(t,x)\right| \le Le^{-\beta t}, \text{ if } x \in U, 0 \le t \le T. \tag{3.30}$$

Since $\phi(t+s,x) = \phi(t,\phi(s,x))$ implies $\frac{\partial\phi}{\partial x}(t+s,x) = \frac{\partial\phi}{\partial x}(t,\phi(s,x))\frac{\partial\phi}{\partial x}(s,x)$, it follows that $\frac{\partial\phi}{\partial x}^{(2)}(t+s,x) = \frac{\partial\phi}{\partial x}^{(2)}(t,\phi(s,x))\frac{\partial\phi}{\partial x}^{(2)}(s,x)$ by the Binet-Cauchy Theorem. By choosing T sufficiently large and using (3.30) analogously to (3.23) in the proof of Theorem 23, we find that $\left|\frac{\partial\phi}{\partial x}^{(2)}(t,x)\right| \le Le^{-\beta t}$, if $x \in U, 0 \le t < \infty$. Thus, if $\phi(t)$ is any solution of (3.1) with omega limit set Ω, the equation $\dot{z} = \frac{\partial f}{\partial x}^{[2]}(\phi(t))z$ is uniformly asymptotically stable. ■

3.5. A Note on Semiflows

Both Theorem 23 and Theorem 25 may be generalized to a semiflow in a metric space $\{X, d\}$. If $t \in \mathbb{R}_+$, $x \in X$, the map $(t,x) \mapsto \phi(t,x) \in X$ is a *semiflow* on X if:

(*i*) $\phi(0,x) = x$

(*ii*) $\phi(t+s,x) = \phi(t,\phi(s,x))$

(*iii*) $(t,x) \mapsto \phi(t,x)$ is continuous

(a) For any $x \in X$, the *positive orbit* of x is $C_+(x) = \bigcup_{t\geq 0}\phi(t,x)$ and the *omega limit* set is $\Omega(x) = \bigcap_{s\geq 0}\operatorname{cl}\bigcup_{t\geq s}\phi(t,x)$, where cl denotes the topological closure.

(b) $C_+(x)$ is *periodic* with period ω if $\phi(t+\omega,x) = \phi(t,x)$ for some $\omega > 0$.

(c) The semiflow is *Lagrange stable* at x if $\operatorname{cl} C_+(x)$ is compact.

(d) The semiflow is *Lyapunov stable* at $S \subset X$ if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x_0 \in S$ and $d[x_0,x] < \delta$ implies $d[\phi(t,x_0),\phi(t,x)] < \varepsilon$. When $S = C_+$, an orbit, this is the usual concept of uniform Lyapunov stability of C_+.

(e) The semiflow is *asymptotic* at $S \subset X$ if there exists $\rho > 0$ such that $x_0 \in S$ and $d[x_0,x] < \rho$ implies

$$\lim_{t\to\infty} d[\phi(t,x_0),\phi(t,x)] = 0.$$

(f) The semiflow is *phase asymptotic* at $S \subset X$ if there exist $\rho,\eta > 0$ such that, for each $x_0 \in S$ there is a real-valued function $(x_0,x) \mapsto h(x_0,x)$ with $|h(x_0,x)| < \eta$ and $d[x_0,x] < \rho$ implies

$$\lim_{t\to\infty} d[\phi(t,x_0),\phi(t+h,x)] = 0.$$

Note: The quantities ρ,η are independent of $x_0 \in S$ here. However the *phase function* h in general depends on (x_0,x). This dependence is suppressed in the notation.

In the following theorems, the phrases in square brackets may either be included or excluded. Theorem 28 generalizes results of Pliss(1966 English translation) and Sell(1966).

Theorem 27. *Suppose that the semiflow ϕ is Lagrange stable at x_*. Then ϕ is asymptotic [and Lyapunov stable] at $C_+(x_*)$ if and only if $\Omega(x_*)$ is an equilibrium at which ϕ is asymptotic [and Lyapunov stable].*

Proof. Suppose that ϕ is asymptotic at $C_+(x_*)$. First we will show that ϕ is asymptotic at $\Omega(x_*)$. Let $x_0 \in \Omega(x_*)$; if $d[x,x_0] < \rho/2$, then there exists $x_1 \in C_+(x_*)$ such that $d[x_0,x_1] < \rho/2$. Thus also $d[x_1,x] < \rho$ and hence

$\lim_{t\to\infty} d\left[\phi\left(t,x_0\right),\phi\left(t,x_1\right)\right] = 0$ and $\lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(t,x\right)\right] = 0$. It follows that $x_0 \in \Omega\left(x_*\right)$ and $d\left[x_0,x\right] < \rho/2$ implies $\lim_{t\to\infty} d\left[\phi\left(t,x_0\right),\phi\left(t,x\right)\right] = 0$ and so ϕ is asymptotic at $\Omega\left(x_*\right)$ with ρ replaced by $\rho/2$ in (e). To see that $\Omega\left(x_*\right)$ is a single equilibrium, observe that $x_1 \in \Omega\left(x_*\right),\ d\left[x_0,x_1\right] < \rho/2$ implies $d\left[x_0,\phi\left(s,x_1\right)\right] < \rho/2,\ 0 \le s \le \varepsilon,$ for some $\varepsilon > 0$, and therefore $\lim_{t\to\infty} d\left[\phi\left(t,x_0\right),\phi\left(t,\phi\left(s,x_1\right)\right)\right] = 0,\ 0 \le s \le \varepsilon$. Thus

$$\lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(s,\phi\left(t,x_1\right)\right)\right] = \lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(t,\phi\left(s,x_1\right)\right)\right] = 0,\ 0 \le s \le \varepsilon. \tag{3.31}$$

With $x_1 \in C_+\left(x_*\right)$, we may choose $t_k \to \infty$ such that $\phi\left(t_k,x_1\right) \to x_0$, as $k \to \infty$, and (3.31) implies $\lim_{k\to\infty} d\left[\phi\left(t_k,x_1\right),\phi\left(s,\phi\left(t_k,x_1\right)\right)\right] = 0$; thus $x_0 = \phi\left(s,x_0\right)$, if $0 \le s \le \varepsilon$, and therefore for $0 \le s < \infty$. Thus x_0 is an equilibrium. Since it attracts all nearby orbits, $\Omega\left(x_*\right) = \left\{x_0\right\}$. Conversely, if ϕ is asymptotic at an equilibrium x_0, it is asymptotic at every orbit attracted to x_0. Finally, the phrase in square brackets may be included throughout this argument. ■

Theorem 28. *Suppose that the semiflow ϕ is Lagrange stable at x_*. Then ϕ is phase asymptotic [and Lyapunov stable] at $C_+\left(x_*\right)$ if and only if $\Omega\left(x_*\right)$ is a periodic orbit at which ϕ is phase asymptotic [and Lyapunov stable].*

Proof. Suppose that ϕ is phase asymptotic at $C_+\left(x_*\right)$. We will show that ϕ is phase asymptotic at $\Omega\left(x_*\right)$. Let $x_0 \in \Omega\left(x_*\right)$; if $d\left[x_0,x\right] < \rho/2$, then there exists $x_1 \in C_+\left(x_*\right)$ such that $d\left[x_0,x_1\right] < \rho/2$. Thus also $d\left[x_1,x\right] < \rho$ and both $\lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(t+h_0,x_0\right)\right] = 0$, $\lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(t+h,x\right)\right] = 0$ are satisfied. It follows that, if $x_0 \in \Omega\left(x_*\right)$ and $d\left[x_0,x\right] < \rho/2$, then

$$\lim_{t\to\infty} d\left[\phi\left(t+h_0,x_0\right),\phi\left(t+h,x\right)\right] = 0$$

so that

$$\lim_{t\to\infty} d\left[\phi\left(t,x_0\right),\phi\left(t+h-h_0,x\right)\right] = 0.$$

Thus ϕ is phase asymptotic at $\Omega\left(x_*\right)$ with ρ,η of the definition (f) replaced by $\rho/2, 2\eta$ and the phase h replaced by $h-h_0$, where $\left|h-h_0\right| < 2\eta$. To see that $\Omega\left(x_*\right)$ is a periodic orbit, observe that, if $x_0 \in \Omega\left(x_*\right)$, there exist $x_1,x_2 \in C_+\left(x_*\right)$ such that $d\left[x_0,x_i\right] < \rho,\ i = 1,2,\ x_2 = \phi\left(t_1,x_1\right)$, where $t_1 > 2\eta$. Then $\lim_{t\to\infty} d\left[\phi\left(t,x_0\right),\phi\left(t+h_i,x_i\right)\right] = 0,\ i = 1,2$, implies $\lim_{t\to\infty} d\left[\phi\left(t+h_1,x_1\right),\phi\left(t+h_2,x_2\right)\right] =$

0 so that

$$\begin{aligned}
& \lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(t_1+h_2-h_1,\phi\left(t,x_1\right)\right)\right] \\
= & \lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(t+h_2-h_1,\phi\left(t_1,x_1\right)\right)\right] \\
= & \lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(t+h_2-h_1,x_2\right)\right] \\
= & \lim_{t\to\infty} d\left[\phi\left(t+h_1,x_1\right),\phi\left(t+h_2,x_2\right)\right]=0;
\end{aligned}$$

hence

$$\lim_{t\to\infty} d\left[\phi\left(t,x_1\right),\phi\left(\omega,\phi\left(t,x_1\right)\right)\right]=0, \text{where } \omega=t_1+h_2-h_1\geq t_1-2\eta>0 \quad (3.32)$$

Now $x_0\in\Omega\left(x_*\right), x_1\in C_+\left(x_*\right)$ implies there exists a sequence $t_k\to\infty, \phi\left(t_k,x_1\right)\to x_0$, as $k\to\infty$ so that, from (3.32), $x_0=\phi\left(\omega,x_0\right):$ ϕ is periodic with period $\omega>0$ at x_0. Evidently, $C_+\left(x_0\right)\subset\Omega\left(x_*\right)$ and, since ϕ is phase asymptotic at $C_+\left(x_0\right)$, this orbit attracts all nearby orbits and $C_+\left(x_0\right)\supset\Omega\left(x_*\right):$ therefore $C_+\left(x_0\right)=\Omega\left(x_*\right)$. Conversely, if ϕ is phase asymptotic at a omega limit set Ω, it is phase asymptotic at every orbit attracted to Ω. The parenthetic statement on Lyapunov stability may be included throughout this proof. ■

Exercise 3.5.1. *Prove that a omega limit set Ω is a periodic orbit at which ϕ is phase asymptotic if and only if ϕ is phase asymptotic at some $x_0\in\Omega$.*

Exercise 3.5.2. *Formulate and prove analogous results on omega limit sets for discrete dynamical systems $(t,x)\mapsto\phi\left(t,x\right)$ where $t\in\mathbb{Z}_+=\{0,1,2,\cdots\}$.*

Exercise 3.5.3. *A system $\dot{x}=f\left(t,x\right)$, where x, $f\left(t,x\right)\in\mathbb{R}^n$, $f\left(t+\omega,x\right)=f\left(t,x\right)$ and solutions are uniquely determined by initial conditions, is known to have a periodic solution if and only if it has a bounded solution when $n=1$. This result also holds when $n=2$ provided a global existence requirement is satisfied by solutions but does not hold in generality when $n>2$. It is known that this conclusion can in fact be drawn for any n when f has the special form $f\left(t,x\right)=A\left(t\right)x+b\left(t\right)$. These results are due to Massera. Sell(1966) shows that, if $\psi\left(t\right)$ is a bounded solution, then there is a periodic solution of period $k\omega$, $1\leq k\in\mathbb{Z}_+$, if the linear equation $\dot{y}=\frac{\partial f}{\partial x}(t,\psi\left(t\right))y$ is uniformly asymptotically stable. Prove this result. Give, in terms of the linear system, a sufficient condition for the existence of a periodic solution of period ω for the non-linear equation.*

Exercise 3.5.4. *Suppose that $\frac{\partial}{\partial x}f\left(t,x\right)<0$ for all $(t,x)\in[0,\omega]\times\mathbb{R}^n$. Show that the system in Exercise 3.5.3 has at most one periodic solution.*

3.6. Convergence of Solutions

Recall Bendixson's condition for the non-existence of periodic orbits of (3.1) in a simply connected set $D \subset \mathbb{R}^2$: $\operatorname{div} f \neq 0$ on D. Recall also the Poincaré stability condition for the orbital asymptotic stability of a periodic solution $\phi(t)$ in $\mathbb{R}^2$: $\dot{z} = \operatorname{div} f(\phi(t)) z$ is uniformly asymptotically stable. For a general Lagrange stable solution $\phi(t)$, this condition is necessary and sufficient that the omega limit set of the solution is orbitally asymptotically stable hyperbolic periodic as long as the omega limit set does not contain an equilibrium. When $D \subset \mathbb{R}^n$, one generalization of Bendixson's condition, Theorem 19 is that the linear equations $\dot{z} = \frac{\partial f}{\partial x}^{[2]}(\phi(t, x_0)) z, x_0 \in S$, are equi-asymptotically stable for every compact subset $S \subset D$. The corresponding Poincaré necessary and sufficient condition for the equilibrium-free omega limit set of a Lagrange stable solution $\phi(t)$ to be a periodic orbit is that $\dot{z} = \frac{\partial f}{\partial x}^{[2]}(\phi(t)) z$ be uniformly asymptotically stable. We see that this condition, when restricted to a particular orbit, ensures that the omega limit set is a periodic orbit if it does not contain an equilibrium. However, when it is satisfied on all orbits with initial points in a simply connected set, it precludes the existence of non-constant periodic orbits and we conclude that every omega limit set contains an equilibrium. In fact, in a 2-dimensional system which satisfies Bendixson's condition, each non-empty omega limit set is a single equilibrium. It is an interesting exercise to prove this; an elementary proof is given in a classroom note by *McCluskey & Muldowney*. This is a useful observation. For example, if Bendixson's condition holds and there is a single equilibrium, then it attracts all bounded orbits. Also in such a system, if a lone equilibrium is asymptotically stable locally, then it is globally asymptotically stable provided all solutions are bounded. This observation may be extended to higher dimensional systems by the following argument which is based on a use of the Pugh Closing Lemma introduced by *R.A.Smith* and on the centre manifold theorem.

> A point $x_0 \in \mathbb{R}^n$ is *wandering* with respect to (3.1) if there is a neighbourhood U of x_0 and $T > 0$ such that $U \cap \phi(t, U)$ is empty if $t \geq T$. Thus, for example any equilibrium, periodic point or, more generally, an omega limit point is *non-wandering*. The Pugh Closing Lemma shows that, if x_0 is a non-wandering point of (3.1) and $f(x_0) \neq 0$, then for each neighbourhood U of x_0 there exists a C^1 function $x \mapsto g(x)$ arbitrarily C^1−close to $x \mapsto f(x)$ with $g(x) = f(x)$, if $x \in D \backslash U$, and such that the equation $\dot{x} = g(x)$ has a non-constant periodic orbit

through x_0.

Let $\mathcal{M}$ be a set of $n \times n$ matrix-valued functions A. The equations $\dot{x} = A(t)\,x$, $A \in \mathcal{M}$, are *uniformly equi-asymptotically stable* if there exist constants $K, \alpha > 0$ such that each fundamental matrix satisfies $\left|X(t)\,X^{-1}(s)\right| \leq Ke^{-\alpha(t-s)}$, $0 \leq s \leq t < \infty$.

Theorem 29. *Suppose that*

(a) D is an open simply connected subset of $\mathbb{R}^n$.

(b) If $x_0 \in D$, then $\phi(t, x_0)$ exists for all $t \geq 0$.

(c) The equations $\dot{z} = \frac{\partial f}{\partial x}^{[2]}(\phi(t, x_0))\,z$, $x_0 \in S$, are uniformly equi-asymptotically stable if S is a compact subset of D.

Then

(e) Every non-wandering point of (3.1) is an equilibrium.

(f) Every non-empty alpha or omega limit set is a single equilibrium.

(g) Every equilibrium in D is the alpha limit set of at most two distinct non-equilibrium trajectories.

Proof. Recall that $Z(t) = \frac{\partial \phi}{\partial x_0}^{(2)}(t, x_0)$ is a fundamental matrix for the equation in (c). The uniform equi-asymptotic stability condition is equivalent to the existence of constants $K, \alpha > 0$ such that $\left|\frac{\partial \phi}{\partial x_{0.}}^{(2)}(t, x_0)\right| \leq Ke^{-\alpha t}$, $0 \leq t < \infty$, if $x_0 \in S$. It follows that all systems which are C^1-close to (3.1) in the sense of Pugh's Lemma also satisfy (b) and (c); the assertion about (c) can be established with a use of the group property of the flow similar to that in the proof of Theorem 23. From Theorem 19, none of these systems have nontrivial periodic solutions. Therefore the Closing Lemma implies that every non-wandering point of (3.1) is an equilibrium, which is the assertion (e) of the theorem. From this, we see that every alpha or omega limit point is an equilibrium. To see that every non-empty omega limit set is a single equilibrium, let $x_* \in \Omega(x_0)$. Now the uniform asymptotic stability of the constant coefficient equation $\dot{z} = \frac{\partial f}{\partial x}^{[2]}(x_*)\,z$ follows from (c) and is equivalent to $\mathrm{Re}\,(\lambda_i + \lambda_j) < 0$, if $i \neq j$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\frac{\partial f}{\partial x}(x_*)$. Thus, with at most one exception, all the eigenvalues satisfy $\mathrm{Re}\,\lambda < 0$; there is a stable manifold associated with x_* whose dimension is at least $n-1$ and an unstable or centre manifold of dimension at most 1. If the stable manifold has dimension n, then clearly x_* is the unique omega limit point . Suppose now that the stable manifold has dimension $n-1$ and that the omega limit set is not a single point. Then, since all of the limit points are equilibria, the component of the omega limit set containing x_* is a continuum of equilibria. Thus, exactly one eigenvalue satisfies $\operatorname{Re} \lambda = 0$ and there is a non-trivial centre manifold of dimension 1 which contains all nearby equilibria and therefore all nearby points of the omega limit set. The centre manifold theorem now implies $\lim_{t \to \infty} \phi(t, x_0) = x_*$ and $x_* = \Omega(x_0)$; this is assertion (f). When x_* is an alpha limit point of a non-constant solution $\phi(t, x_0)$, a similar argument shows that $\lim_{t \to -\infty} \phi(t, x_0) = x_*$ and that the orbit lies in a 1-dimensional unstable or centre manifold. The unstable manifold is always unique and, in this case, the centre manifold theorem implies that the 1-dimensional centre manifold is always unique if it exists; this gives assertion (g). ■

A more elementary proof of Theorem 29 which uses neither the Pugh lemma nor the centre manifold theorem is also possible but longer than that given here.

A similar result may be based on the Bendixson criterion given in Theorem 18 rather than that in Theorem 19.

Theorem 30. *Suppose that*

(a) D is an open set in $\mathbb{R}^n$ which has the minimum property with respect to the norm $|\cdot|$.

(b) $\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0$ in D [or $\mu\left(-\frac{\partial f}{\partial x}^{[2]}\right) < 0$ in D], where μ is the Lozinskiĭ measure corresponding to the norm $|\cdot|$.

Then the conclusions (e), (f) and (g)[with 'alpha' replaced by 'omega'] of Theorem 29 are satisfied.

3.7. Appendix

Exterior product approach

- $\bigwedge^k \mathbb{R}^n = span\left\{v^1 \wedge \cdots \wedge v^k : v^i \in \mathbb{R}^n\right\} \cong \mathbb{R}^{\binom{n}{k}}$
- $A : \mathbb{R}^n \to \mathbb{R}^m$ linear, $1 \leq k \leq m, n$
- $A^{(k)} : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^m$ is defined by

  $$A^{(k)}\left(v^1 \wedge \cdots \wedge v^k\right) \stackrel{def}{=} Av^1 \wedge \cdots \wedge Av^k,\ v^i \in \mathbb{R}^n,$$

  extended by linearity to $\bigwedge^k \mathbb{R}^n$
- $A^{(k)}$: $k-$th *multiplicative compound* (exterior power) of A. Binet-Cauchy Theorem: $(AB)^{(k)} = A^{(k)} B^{(k)}$

- $A^{[k]} : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$ is defined by $(m = n)$

$$A^{[k]}\left(v^1 \wedge \cdots \wedge v^k\right) \overset{def}{=} \sum_{j=1}^{k} v^1 \wedge \cdots \wedge Av^j \wedge \cdots \wedge v^k, v^i \in \mathbb{R}^n,$$

  extended by linearity to $\bigwedge^k \mathbb{R}^n$

- $A^{[k]}$: k −th *additive compound* of A.
  linearity $\Rightarrow (A+B)^{[k]} = A^{[k]} + B^{[k]}$

- matrix: $m \times n$
  $A = \left[a_i^j\right]$, $1 \le i \le m$, $1 \le j \le n$

- k-th multiplicative compound: $\binom{m}{k} \times \binom{n}{k}$
  $B = A^{(k)}$, $1 \le k \le m, n$
  $b_r^s = a_{r_1 \cdots r_k}^{s_1 \cdots s_k}$
  where $(r) = (r_1, \cdots, r_k)$, $(s) = (s_1, \cdots, s_k)$, $1 \le r \le \binom{m}{k}$, $1 \le s \le \binom{n}{k}$

- e.g. 3×3 :
  $A^{(1)} = A = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix}$, $A^{(2)} = \begin{bmatrix} a_{12}^{12} & a_{12}^{13} & a_{12}^{23} \\ a_{13}^{12} & a_{13}^{13} & a_{13}^{23} \\ a_{23}^{12} & a_{23}^{13} & a_{23}^{23} \end{bmatrix}$, $A^{(3)} = a_{123}^{123}$

- k-th additive compound: $\binom{m}{k} \times \binom{n}{k}$
  $C = A^{[k]}, 1 \le k \le m = n,$

- $c_r^s = a_{r_1}^{r_1} + \cdots + a_{r_k}^{r_k}$, if $(r) = (s)$

- $c_r^s = (-1)^{i+j} a_{r_i}^{s_j}$,if exactly one entry r_i in (r) does not occur in (s) and s_j does not occur in (r)

- $c_r^s = 0$ if (r) differs from (s) in two or more entries

- $n = 2$:

$$\begin{aligned} A^{[1]} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A \\ A^{[2]} &= a_{11} + a_{22} = \mathrm{tr} A \end{aligned}$$

$n = 3:$

$$
\begin{aligned}
A^{[1]} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A \\
A^{[2]} &= \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{bmatrix} \\
A^{[3]} &= a_{11}+a_{22}+a_{33} = \mathrm{tr}A
\end{aligned}
$$

$n = 4:$

$$
A^{[1]} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = A
$$

$$
A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}
$$

$$
A^{[3]} = \begin{bmatrix} a_{11}+a_{22}+a_{33} & a_{34} & -a_{24} & a_{14} \\ a_{43} & a_{11}+a_{22}+a_{44} & a_{23} & -a_{13} \\ -a_{42} & a_{32} & a_{11}+a_{33}+a_{44} & a_{12} \\ a_{41} & -a_{31} & a_{21} & a_{22}+a_{33}+a_{44} \end{bmatrix}
$$

$$
A^{[4]} = a_{11}+a_{22}+a_{33}+a_{44} = trA
$$

Properties

1. $(AB)^{(k)} = A^{(k)}B^{(k)}$ Binet-Cauchy Theorem
2. $(A+B)^{[k]} = A^{[k]} + B^{[k]}$

3. $A^{[k]} = \frac{d}{dt}\left(I + tA\right)^{(k)}|_{t=0} = \lim_{h\to o}\frac{1}{h}\left[\left(I + hA\right)^{(k)} - I^{(k)}\right]$

4. $\left(\exp A\right)^{(k)} = \exp\left(A^{[k]}\right)$, $\log\left(\exp A\right)^{(k)} = A^{[k]}$

5. $A^{(1)} = A^{[1]} = A$

6. $A^{(n)} = \det A,\ A^{[n]} = \operatorname{tr} A$

7. $X\left(t\right)$: matrix solution of $\dot{x} = A\left(t\right)x \Rightarrow$
   $X^{(k)}\left(t\right)$: matrix solution of $\dot{z} = A^{[k]}\left(t\right)z$

8. If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of $A^{(k)}$ and $A^{[k]}$ are $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$ and $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, respectively, $1 \le i_1 < \cdots < i_k \le n$.

9. The corresponding eigenvectors of both $A^{(k)}$ and $A^{[k]}$ are $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ if $v_{i_1}, v_{i_2}, \cdots, v_{i_k}$ are independent eigenvectors of A corresponding to $\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}$, respectively

10. A symmetric, eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$

- $$\lambda_1 = \max \frac{x^* A x}{x^* x},\ x \in \mathbb{R}^n,\ x \ne 0$$

  $$\lambda_1 \lambda_2 \cdots \lambda_k = \max \frac{y^* A^{(k)} y}{y^* y},\ y = x^1 \wedge x^2 \wedge \cdots \wedge x^k,\ y \ne 0$$

- $$\lambda_1 + \lambda_2 + \cdots + \lambda_k = \max \frac{y^* A^{[k]} y}{y^* y},\ y = x^1 \wedge x^2 \wedge \cdots \wedge x^k,\ y \ne 0$$

11. Geršgorin disks for A :

$$\left|z - a_i^i\right| \le \sum_{j \ne i} \left|a_i^j\right|$$

contain the eigenvalues λ_i of A.

12. Geršgorin disks for $A^{(k)}$:

$$\left|z - a_{i_1\cdots i_k}^{i_1\cdots i_k}\right| \le \sum_{(j)\ne(i)} \left|a_{i_1\cdots i_k}^{j_1\cdots j_k}\right|$$

contain the eigenvalues $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$ of $A^{(k)}$.

13. Geršgorin disks for $A^{[k]}$:

$$\left|z-\left(a_{i_1}^{i_1}+\cdots+a_{i_k}^{i_k}\right)\right| \leq \sum_{j \notin (i)} \left(\left|a_{i_1}^{j}\right|+\cdots+\left|a_{i_k}^{j}\right|\right)$$

contain the eigenvalues $\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_k}$ of $A^{[k]}$.

Ordinary Differential Equations

- If $Y(t,s)$ is the evolution matrix of $\dot{y}=A(t)\,y$, then $Z(t,s)=Y^{(k)}(t,s)$ is the evolution matrix of $\dot{z}=A^{[k]}(t)\,z$.

- In particular, if $y^1(t),\cdots,y^k(t)$ are solutions of $\dot{y}=A(t)\,y$, then $z(t)=y^1(t)\wedge\cdots\wedge y^k(t)$ is a solution of $\dot{z}=A^{[k]}(t)\,z$

$$\begin{aligned} y^1(t)\wedge\cdots\wedge y^k(t) &= Y(t,s)\,y^1(t)\wedge\cdots\wedge Y(t,s)\,y^k(t) \\ &= Y^{(k)}(t,s)\,y^1(s)\wedge\cdots\wedge y^k(s) \end{aligned}$$

- If $y^1(t),\cdots,y^k(t)$ are considered as an ordered set of oriented line segments in $\mathbb{R}^n$ changing with time, then $z(t)$ may be interpreted as the corresponding $k-$dimensional oriented parallelopiped.The $\binom{n}{k}$components of $z(t)$ are the determinants $z_{i_1\cdots i_k}^{1\cdots k}(t)$ which are the projections of $z(t)$ onto the $k-$dimensional coordinate subspace spanned by $e_{i_1},\cdots,e_{i_k}$ and $|z(t)|$ is a measure of the $k-$dimensional volume of $z(t)$ if $|\cdot|$ is any norm on $\bigwedge^k\mathbb{R}^n$.

- The nonlinear autonomous differential equation in $\mathbb{R}^n$, $\dot{x}=f(x)$, generates a semigroup $\phi(t,x)$.

- Under the map $x_0\mapsto x_t=\phi(t,x_0)$, an infinitesimal oriented line segment dx_0 evolves in time as $dx_t=\frac{\partial\phi}{\partial x_0}(t,x_0)\,dx_0$, a solution of $\dot{y}=\frac{\partial f}{\partial x}(\phi(t,x_0))\,y$

- An oriented infinitesimal $k-$dimensional volume $dx_0^1\wedge\cdots\wedge dx_0^k$ evolves as $dx_t^1\wedge\cdots\wedge dx_t^k=[\frac{\partial\phi}{\partial x_0}(t,x_0)\,dx_0^1]\wedge\cdots\wedge[\frac{\partial\phi}{\partial x_0}(t,x_0)\,dx_0^k]$
$=\frac{\partial\phi}{\partial x_0}^{(k)}(t,x_0)\,dx_0^1\wedge\cdots\wedge dx_0^k$, which is a solution of $\dot{z}=\frac{\partial f}{\partial x}^{[k]}(\phi(t,x_0))\,z$.

- In particular, the evolution of infinitesimal $n-$dimensional volumes is governed by the Liouville equation, $\dot{z}=\operatorname{div} f(\phi(t,x_0))\,z$