A Research Story: Compound Equations and Dynamics. Part 3

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Curves and Surfaces

A smooth *curve* γ in $\mathbb{R}^n$ is a C^1 function $s \to x(s)$, $s \in I \subset \mathbb{R}$, $x(s) \in \mathbb{R}^n$.

A measure of the *length* of γ is

$$l(\gamma) = \int_\gamma dl \stackrel{def}{=} \int_I \left\| \frac{dx}{ds}(s) \right\| ds$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. For example, the euclidean norm $\|x\| = \sqrt{(x_1)^2 + \cdots + (x_n)^2}$ gives the usual measure of length

$$l(\gamma) = \int_I \sqrt{\frac{dx_1}{ds}^2 + \cdots + \frac{dx_n}{ds}^2} ds$$

A smooth 2-*surface* σ in $\mathbb{R}^n$ is a C^1 function $(s_1, s_2) \to x(s_1, s_2)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$, $x(s_1, s_2) \in \mathbb{R}^n$.
A measure of the *area* of σ is

$$a_2(\sigma) = \int_\sigma da \overset{def}{=} \int_U \left\| x_{s_1} \wedge x_{s_2} \right\| ds_1 ds_2$$

where $x_{s_i} = \frac{\partial}{\partial s_i} x(s_1, s_2)$ and $\|\cdot\|$ is a norm on $\mathbb{R}^{\binom{n}{2}}$. If $\|\cdot\|$ is the Euclidean norm we have

$$a_2(\sigma) = \int_U \sqrt{\sum_{1 \le i < j \le n} \frac{\partial(x_i, x_j)}{\partial(s_1, s_2)}^2} ds_1 ds_2$$

where

$$\frac{\partial(x_i, x_j)}{\partial(s_1, s_2)} = \det \begin{bmatrix} \frac{\partial x_i}{\partial s_1} & \frac{\partial x_i}{\partial s_2} \\ \frac{\partial x_j}{\partial s_1} & \frac{\partial x_j}{\partial s_2} \end{bmatrix}.$$

A smooth *k-surface* σ in $\mathbb{R}^n$ is a C^1 function $(s_1, \cdots, s_k) \to x(s_1, \cdots, s_k)$, $s_1, \cdots, s_k \in U \subset \mathbb{R}^k$, $x(s_1, \cdots, s_k) \in \mathbb{R}^n$. A measure of the *k-area* of σ is

$$a_k(\sigma) = \int_\sigma da_k \overset{def}{=} \int_U \left\| x_{s_1} \wedge \cdots \wedge x_{s_k} \right\| ds_1 \cdots ds_k$$

where $x_{s_i} = \frac{\partial}{\partial s_i} x(s_1, \cdots, s_k)$ and $\|\cdot\|$ is a norm on $\mathbb{R}^{\binom{n}{k}}$. If $\|\cdot\|$ is the Euclidean norm we have

$$a_k(\sigma) = \int_U \sqrt{\sum_{1 \le i_1 < \cdots < i_k \le n} \frac{\partial(x_{i_1}, \cdots, x_{i_k})}{\partial(s_1, \cdots, s_k)}^2} ds_1 ds_2$$

where

$$\frac{\partial(x_{i_1}, \cdots, x_{i_k})}{\partial(s_1, \cdots, s_k)} = \det \begin{bmatrix} \frac{\partial x_{i_1}}{\partial s_1} & \frac{\partial x_{i_1}}{\partial s_2} & . & \frac{\partial x_{i_1}}{\partial s_k} \\ \frac{\partial x_{i_2}}{\partial s_1} & \frac{\partial x_{i_2}}{\partial s_2} & . & \frac{\partial x_{i_2}}{\partial s_k} \\ . & . & . & . \\ \frac{\partial x_{i_k}}{\partial s_1} & \frac{\partial x_{i_k}}{\partial s_2} & . & \frac{\partial x_{i_k}}{\partial s_k} \end{bmatrix}.$$

Nonlinear Differential Equations

$f \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$

$$\dot{x} = f(x) \tag{N}$$

Solution : $x(t) = \phi(t) = \phi(t, x_0)$, is uniquely determined by $x(0) = x_0$ and, for simplicity, we will only consider equations for which solutions exist for all $t > 0$

If $\phi(t, x_0) = x_0$ for all t, then x_0 is called an *equilibrium.*

If $\phi(t + \omega) = \phi(t)$, $\omega > 0$,the solution is *periodic* of period ω.

An *orbit* (positive semi-orbit) is a set $\{\phi(t) : 0 \leq t < \infty\}$.

The orbit of an equilibrium is a single point.

The orbit of a periodic solution is a simple closed curve (Jordan curve)

Linearization about a solution $\phi(t)$:

$$\dot{y} = \frac{\partial f}{\partial x}(\phi(t))\, y \qquad \text{(L)}$$

Solution is:

$$\begin{aligned} y(t) &= \frac{\partial \phi}{\partial x_0}(t, x_0)\, y(0), \quad x_0 = \phi(0) \\ Y(t) &= \frac{\partial \phi}{\partial x_0}(t, x_0)\,.\text{fundamental matrix, } Y(0) = I \end{aligned}$$

"Proof": $y = \phi(t, x_0)$ solves $\dot{y} = f(y)$

$$\Rightarrow \frac{\partial \phi}{\partial t}(t, x_0) = f(\phi(t, x_0))$$

Differentiate with respect to x_0

$$\begin{aligned} &\Rightarrow \quad \frac{\partial^2 \phi}{\partial t \partial x_0}(t, x_0) = \frac{\partial^2 \phi}{\partial x_0 \partial t}(t, x_0) = \frac{\partial f}{\partial x}(\phi(t, x_0))\, \frac{\partial \phi}{\partial x_0}(t, x_0) \\ &\Rightarrow \quad \dot{Y} = \frac{\partial f}{\partial x}(\phi(t))\, Y \end{aligned}$$

The $k-$th compound equation of (L) is:

$$\dot{z} = \frac{\partial f}{\partial x}^{[k]} \left(\phi\left(t\right)\right) z \qquad (\mathrm{L}_k)$$

Solution: $z\left(t\right) = \frac{\partial \phi}{\partial x_0}^{(k)} \left(t, x_0\right) z\left(0\right),\ x_0 = \phi\left(0\right)$

The case $k = n$ of (L_k) is the Liouville equation:

$$\dot{z} = \operatorname{div} f\left(\phi\left(t\right)\right) z \qquad (\mathrm{L}_n)$$

Solution: $z\left(t\right) = \det \frac{\partial \phi}{\partial x_0} \left(t, x_0\right) z\left(0\right),\ x_0 = \phi\left(0\right)$

Suppose that $D \subset \mathbb{R}^n$ has finite n-dimensional measure $a_n(D)$, then the measure of $\phi(t, D)$ is

$$a_n(\phi(t, D)) = \int_{x \in \phi(t,D)} dx = \int_{x_0 \in D} \left| \det \frac{\partial \phi}{\partial x_0}(t, x_0) \right| dx_0$$

$(L_n) \Rightarrow \det \frac{\partial \phi}{\partial x_0}(t, x_0) = \exp\left[\int_0^t \operatorname{div} f(\phi(s, x_0))\, ds\right]$. So, for example, if $\operatorname{div} f < 0$ in $\mathbb{R}^n$, then the measure of the set $\phi(t, D)$ decreases with time.

When $n = 2$ this observation implies that no simply connected region where $\operatorname{div} f < 0$ can contain a non-trivial periodic orbit of (L). This is known as *Bendixson's Condition.* Most textbooks prove this as a very nice application of Green's Theorem.

Stability of the linearized equations (L) and its compounds (L_k) have many implications for the dynamics of (N)

If γ_0: $x = x_0(s)$, $0 \leq s \leq 1$ is a curve in $\mathbb{R}^n$, then $\gamma_t : x = \phi(t, x_0(s))$, $0 \leq s \leq 1$ is also a curve in $\mathbb{R}^n$ for each $t \geq 0$.

$$
\begin{aligned}
l\gamma_0 &= \int_0^1 \left\| \frac{d}{ds} x_0(s) \right\| ds \\
l\gamma_t &= \int_0^1 \left\| \frac{d}{ds} \phi(t, x_0(s)) \right\| ds = \int_0^1 \left\| \frac{\partial \phi}{\partial x_0}(t, x_0(s)) \frac{d}{ds}(x_0(s)) \right\| ds \\
&\leq \int_0^1 \left\| \frac{\partial \phi}{\partial x_0}(t, x_0(s)) \right\| \left\| \frac{d}{ds}(x_0(s)) \right\| ds
\end{aligned}
$$

We can conclude for example that, if $\left\| \frac{\partial \phi}{\partial x_0}(t, x_0) \right\| \underset{t \to \infty}{\longrightarrow} 0$ uniformly with respest to $x_0 \in \mathbb{R}^n$, then

- there is at most one equilibrium of (N) and,
- any equilibrium attracts all other orbits

If $\sigma_0 : (s_1, s_2) \to x(s_1, s_2)$ is a 2-surface in $\mathbb{R}^n$ then so also is $\sigma_t : (s_1, s_2) \to \phi(t, x(s_1, s_2))$.

We can use similar ideas to get higher dimensional *Bendixson Conditions* to rule out the existence of periodic orbits. These are conditions on (L_2) that typically imply that some measure of surface area decreases in the dynamics. Another related type of condition would imply that $a_2\sigma_t \underset{t\to\infty}{\longrightarrow} 0$.

The central idea is to observe that a periodic orbit γ is invariant in the dynamics, $\phi(t, \gamma) = \gamma$. So, if Σ_0 is any surface which has γ as its boundary, then $\Sigma_t = \phi(t, \Sigma_0)$ is also a surface with γ as boundary. But if, among all surfaces with boundary γ, Σ_0 is a surface with minimum area and (N) diminishes area we would contradict the minimality of $\Sigma_{0.}$*So* no such invariant closed curve can exist.

The following are Bendixson conditions for various measures of 2-surface area. Each reduces to the classical result when $n = 2$:

$$\lambda_1 + \lambda_2 < 0 \, \text{(RA Smith)}$$

$$\max_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r,s} \left(\left| \frac{\partial f_r}{\partial x_q} \right| + \left| \frac{\partial f_s}{\partial x_q} \right| \right) \right\} < 0$$

$$\max_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r,s} \left(\left| \frac{\partial f_q}{\partial x_r} \right| + \left| \frac{\partial f_q}{\partial x_s} \right| \right) \right\} < 0$$

$$\lambda_{n-1} + \lambda_n > 0$$

$$\min_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r,s} \left(\left| \frac{\partial f_r}{\partial x_q} \right| + \left| \frac{\partial f_s}{\partial x_q} \right| \right) \right\} > 0$$

$$\min_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r,s} \left(\left| \frac{\partial f_q}{\partial x_r} \right| + \left| \frac{\partial f_q}{\partial x_s} \right| \right) \right\} > 0$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\frac{1}{2} \left(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x} \right)$

General Compounds

M Fiedler, Czech Math J 24(1974), pp 392 - 402

$\mathbb{X} \subset \mathbb{Y}$: *General compound* $A^{[k]} \in \mathcal{L}\left(\wedge^k \mathbb{X} \to \wedge^k \mathbb{Y}\right)$. $0 \le m \le k$

$$A^{[k,m]}\left(v^1 \wedge \cdots \wedge v^k\right) \stackrel{def}{=} \sum_{(\varepsilon_1,\cdots,\varepsilon_k)} A^{\varepsilon_1} v^1 \wedge A^{\varepsilon_2} v^2 \wedge \cdots \wedge A^{\varepsilon_k} v^k$$

$$\varepsilon_i \in \{0,1\}, \quad \varepsilon_1 + \cdots + \varepsilon_k = m, \quad A^0 = I$$

$$A^{[k,0]} = I^{(k)}, \quad A^{[k,1]} = A^{[k]}, \quad A^{[k,k]} = A^{(k)}$$

$$D_h^m (I + hA)^{(k)}\Big|_{t=0} = m! A^{[k,m]}$$

$$D_h^m \left(I + hA\right)^{(k)}\Big|_{t=0} = m!A^{[k,m]}$$

$$\begin{aligned}(I + hA)^{(k)} &= \sum_{m=0}^{k} h^m A^{[k,m]} \\ &= hA^{[k,1]} + h^2 A^{[k,2]} + \cdots + h^k A^{[k,k]}\end{aligned}$$

If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A with eigenvectors $v^1, \cdots, v^n$, then the eigenvalues of $(I + hA)^{(k)}$ are

$$h\left(\lambda_{i_1} + \cdots + \lambda_{i_k}\right) + h^2\left(\lambda_{i_1}\lambda_{i_2} + \cdots + \lambda_{i_{k-1}}\lambda_{i_k}\right) + \cdots + h^k\left(\lambda_{i_1}\lambda_{i_2}\cdot\cdot\lambda_{i_k}\right)$$

with eigenvectors $v^{i_1} \wedge v^{i_2} \wedge \cdots \wedge v^{i_k}$.

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