

A Research Story: Compound Equations and Dynamics. Part 2

James Muldowney, University of Alberta

July 19, 2021

Exterior Products

Vector Space: $u \in \mathbb{X}$

Dual Space: $v \in \mathbb{X}^* = \mathcal{L}(\mathbb{X} \rightarrow \mathbb{R})$, the space of real linear functionals $v(u)$ on \mathbb{X}

Bilinear Map $\langle \cdot, \cdot \rangle : \mathbb{X}^* \times \mathbb{X} \rightarrow \mathbb{R}$

$$\langle v, u \rangle = v(u)$$

Multilinear Maps: $u^1, \dots, u^k \in \mathbb{X}$. Define $u^1 \wedge u^2 \wedge \dots \wedge u^k : (\mathbb{X}^*)^k \rightarrow \mathbb{R}$ by

$$u^1 \wedge u^2 \wedge \dots \wedge u^k (v_1, v_2, \dots, v_k) \stackrel{\text{def}}{=} \det [v_i(u^j)] = \det [\langle v_i, u^j \rangle]$$

if $v_i \in \mathbb{X}^*$ $i = 1, 2, \dots, k$.

$\bigwedge^k \mathbb{X}$ is the set of all finite linear combinations of the maps $u^1 \wedge u^2 \wedge \dots \wedge u^k$.

Similarly $v_1 \wedge v_2 \wedge \cdots \wedge v_k : \mathbb{X}^k \rightarrow \mathbb{R}$ is defined by

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k (u^1, u^2, \dots, u^k) \stackrel{\text{def}}{=} \det [\langle v_i, u^j \rangle].$$

and $\wedge^k \mathbb{X}^*$ is the set of all finite linear combinations of these.

The expression

$$\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k, u^1 \wedge u^2 \wedge \cdots \wedge u^k \rangle \stackrel{\text{def}}{=} \det [\langle v_i, u^j \rangle]$$

extended by linearity gives a bilinear map $\langle \cdot, \cdot \rangle : \wedge^k \mathbb{X}^* \times \wedge^k \mathbb{X} \rightarrow \mathbb{R}$.

Let $\{e^j\}$ be a basis for \mathbb{X} . Define real functionals e_i on the basis by $\langle e_i, e^j \rangle = \delta_i^j$. Extend e_i to \mathbb{X} by linearity. if $u \in \mathbb{X}$, then

$$v = \sum_j c_j e^j, \quad c_i = \langle e_i, v \rangle$$

Moreover $\{e^{j_1} \wedge \dots \wedge e^{j_k}\}$ is a basis for $\wedge^k \mathbb{X}$

Note that

$$\langle e_{i_1} \wedge \dots \wedge e_{i_k}, e^{j_1} \wedge \dots \wedge e^{j_k} \rangle = \delta_{(i)}^{(j)}$$

If $v^1, \dots, v^k \in \mathbb{X}$, then $v^s = \sum_j c_j^s e^j$, $c_i^s = \langle e_i, v^s \rangle$ and

$$\begin{aligned} v^1 \wedge \dots \wedge v^k &= \sum_{(j)} c_{(j)} e^{j_1} \wedge \dots \wedge e^{j_k}, \\ c_{(i)} &= \left\langle e_{i_1} \wedge \dots \wedge e_{i_k}, v^1 \wedge \dots \wedge v^k \right\rangle \\ &= \det [\langle e_{i_r}, v^s \rangle], \quad r, s = 1, \dots, k \\ &= \det [c_{i_r}^s] \stackrel{\text{def}}{=} c_{i_1 i_2 \dots i_k}^{12 \dots k} \end{aligned} \tag{1}$$

Example:

$$v^1 = \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \\ v_4^1 \end{bmatrix}, v^2 = \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \\ v_4^2 \end{bmatrix} \in \mathbb{R}^4$$

$$v^1 \wedge v^2 = \begin{bmatrix} v_{12}^{12} \\ v_{13}^{12} \\ v_{14}^{12} \\ v_{23}^{12} \\ v_{24}^{12} \\ v_{34}^{12} \end{bmatrix} \in \bigwedge^2 \mathbb{R}^4 \cong \mathbb{R}^6$$

Compound Operators

\mathbb{X}, \mathbb{Y} vector spaces, $A \in \mathcal{L}(\mathbb{X} \rightarrow \mathbb{Y})$

Multiplicative compound $A^{(k)} \in \mathcal{L}(\wedge^k \mathbb{X} \rightarrow \wedge^k \mathbb{Y})$

$$A^{(k)}(v^1 \wedge \dots \wedge v^k) \stackrel{\text{def}}{=} Av^1 \wedge \dots \wedge Av^k, \quad v^j \in \mathbb{X}$$

extended by linearity to $\wedge^k \mathbb{X}$.

$\mathbb{X} \subset \mathbb{Y}$: *Additive compound* $A^{[k]} \in \mathcal{L}(\wedge^k \mathbb{X} \rightarrow \wedge^k \mathbb{Y})$.

$$A^{[k]}(v^1 \wedge \dots \wedge v^k) \stackrel{\text{def}}{=} \sum_{j=1}^k v^1 \wedge \dots \wedge Av^j \wedge \dots \wedge v^k$$

extended by linearity to $\wedge^k \mathbb{X}$.

Matrices

$$C = A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$[c_r^s] = \langle e_r, Ae^s \rangle = [a_r^s]$$

$$C = A^{(k)} : \wedge^k \mathbb{R}^n \rightarrow \wedge^k \mathbb{R}^m$$

$$\begin{aligned} c_{(r)}^{(s)} &= \left\langle e_{r_1} \wedge \cdots \wedge e_{r_k}, A^{(k)} (e^{s_1} \wedge \cdots \wedge e^{s_k}) \right\rangle \\ &= \langle e_{r_1} \wedge \cdots \wedge e_{r_k}, Ae^{s_1} \wedge \cdots \wedge Ae^{s_k} \rangle \\ &= \det \langle e_{r_i}, Ae^{s_j} \rangle = \det [a_{r_i}^{s_j}] \quad i = 1, \dots, k, j = 1, \dots, k \\ &= a_{r_1 \dots r_k}^{s_1 \dots s_k} \end{aligned}$$

$C = A^{[k]} : \wedge^k \mathbb{R}^n \rightarrow \wedge^k \mathbb{R}^n$. Replace $A^{(k)}$ by $A^{[k]}$ above

$$c_{(r)}^{(s)} = \left\langle e_{r_1} \wedge \cdots \wedge e_{r_k}, \sum_{j=1}^k e^{s_1} \wedge \cdots \wedge Ae^{s_j} \wedge \cdots \wedge e^{s_k} \right\rangle$$

so that $c_{(r)}^{(s)} = a_{r_1}^{s_1} + \cdots + a_{r_k}^{s_k}$, if $(r) = (s)$. $c_{(r)}^{(s)} = (-1)^{i+j} a_{r_i}^{s_j}$, if exactly one entry r_i in (r) does not occur in (s) and s_j does not occur in (r) ,

$c_{(r)}^{(s)} = 0$ if (r) differs from (s) in two or more entries.

Properties

1. The Binet-Cauchy identity:

$$(AB)^{(k)} = A^{(k)} B^{(k)}$$

$$\begin{aligned}(AB)^{(k)} v^1 \wedge \dots \wedge v^k &= (AB) v^1 \wedge \dots \wedge (AB) v^k \\ &= A (Bv^1) \wedge \dots \wedge A (B) v^k = A^{(k)} [(Bv^1) \wedge \dots \wedge (B) v^k] \\ &= A^{(k)} [B^{(k)} v^1 \wedge \dots \wedge v^k] = A^{(k)} B^{(k)} v^1 \wedge \dots \wedge v^k\end{aligned}$$

2. Additive property:

$$(A + B)^{[k]} = A^{[k]} + B^{[k]}$$

$$\begin{aligned}(A + B)^{[k]} (v^1 \wedge \dots \wedge v^k) &= \sum_{j=1}^k v^1 \wedge \dots \wedge (A + B) v^j \wedge \dots \wedge v^k \\ &= \sum_{j=1}^k v^1 \wedge \dots \wedge A v^j \wedge \dots \wedge v^k + \sum_{j=1}^k v^1 \wedge \dots \wedge B v^j \wedge \dots \wedge v^k \\ &= A^{[k]} v^1 \wedge \dots \wedge v^k + B^{[k]} v^1 \wedge \dots \wedge v^k = (A^{[k]} + B^{[k]}) v^1 \wedge \dots \wedge v^k\end{aligned}$$

3. Multiplicative-additive relationship

$$\left. \frac{d}{dh} (I + hA)^{(k)} \right|_{h=0} = A^{[k]}$$

Proof:

$$(I + hA)^{(k)} v^1 \wedge \dots \wedge v^k = (I + hA) v^1 \wedge \dots \wedge (I + hA) v^k$$

Differentiate using the product formula and set $h = 0$

$$\left. \frac{d}{dh} (I + hA)^{(k)} \right|_{h=0} v^1 \wedge \dots \wedge v^k = \sum_{j=1}^k v^1 \wedge \dots \wedge Av^j \wedge \dots \wedge v^k = A^{[k]} v^1 \wedge \dots \wedge v^k$$

4. Differential Equations: $L: \dot{x} = A(t)x$, $L_k: \dot{z} = A^{[k]}(t)z$

(i) If $x^1(t), \dots, x^k(t)$ are solutions of L , then

$z(t) = x^1(t) \wedge \dots \wedge x^k(t)$ is a solution of L_k since

$$\dot{z}(t) = \sum_{j=1}^k x^1(t) \wedge \dots \wedge \dot{x}^j(t) \wedge \dots \wedge x^k(t) =$$

$$\sum_{j=1}^k x^1(t) \wedge \dots \wedge A(t)x^j(t) \wedge \dots \wedge x^k(t) = A^{[k]}(t)x^1(t) \wedge \dots \wedge x^k(t)$$

(ii) If $X(t)$ is a fundamental matrix for L , then $X^{(k)}(t)$ is a fundamental matrix of L_k since from (i), if $x^j(t) = X(t)c^j$ are solutions of L , $X^{(k)}(t)c^1 \wedge \dots \wedge c^k = X(t)c^1 \wedge \dots \wedge X(t)c^k$ is a solution of L_k

(iii) For any square matrix A

$$A^{(1)} = A^{[1]}, \quad A^{(n)} = \det A, \quad A^{[n]} = \operatorname{tr} A,$$

$$(\exp A)^{(k)} = \exp\left(A^{[k]}\right), \quad \log(\exp A)^{(k)} = A^{[k]}$$

$\dot{x} = Ax$. A fundamental matrix $X(t) = \exp(At)$ (A is constant) and $X(0) = I$. The fundamental matrix $Z(t)$ of $\dot{z} = A^{[k]}z$, $Z(0) = I$, is $Z(t) = \exp\left(A^{[k]}t\right)$ and also $Z(t) = X^{(k)}(t) = \exp(At)^{(k)}$ so that $\exp(At)^{(k)} = \exp\left(A^{[k]}t\right)$; let $t = 1$.

5. Eigenvalues $n \times n$ matrix A

(i) If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A , then the eigenvalues of $A^{(k)}$ and $A^{[k]}$ are $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$ and $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, respectively, $1 \leq i_1 < \cdots < i_k \leq n$.

(ii) The corresponding eigenvectors of both $A^{(k)}$ and $A^{[k]}$ are $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ if $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ are independent eigenvectors of A corresponding to $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_k}$, respectively

(iii) Geršgorin disks for A :

$$|z - a_i^i| \leq \sum_{j \neq i} |a_i^j|$$

contain the eigenvalues λ_i of A

(iv) Geršgorin disks for $A^{(k)}$:

$$\left| z - a_{i_1 \dots i_k}^{i_1 \dots i_k} \right| \leq \sum_{(j) \neq (i)} \left| a_{i_1 \dots i_k}^{j_1 \dots j_k} \right|$$

contain the products $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$

(v) Geršgorin disks for $A^{[k]}$:

$$\left| z - \left(a_{i_1}^{i_1} + \cdots + a_{i_k}^{i_k} \right) \right| \leq \sum_{j \notin (i)} \left(\left| a_{i_1}^j \right| + \cdots + \left| a_{i_k}^j \right| \right)$$

contain the sums $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$

Courant-Fischer

(vi) A a real positive definite symmetric matrix, eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$$\lambda_1 = \max \frac{x^* A x}{x^* x}, \quad x \in \mathbb{R}^n, \quad x \neq 0$$

$$\lambda_1 \lambda_2 \cdots \lambda_k = \max \frac{y^* A^{(k)} y}{y^* y}, \quad y = x^1 \wedge x^2 \wedge \cdots \wedge x^k, \quad y \neq 0$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = \max \frac{y^* A^{[k]} y}{y^* y}, \quad y = x^1 \wedge x^2 \wedge \cdots \wedge x^k, \quad y \neq 0$$

C any positive definite symmetric, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n(C)$, are the eigenvalues of C . Courant-Fischer implies

$$\lambda_1(AB) \leq \lambda_1(A) \lambda_1(B)$$

$$\lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B)$$

It follows from the compound matrices that

$$\lambda_1 \lambda_2 \cdots \lambda_k(AB) \leq \lambda_1 \lambda_2 \cdots \lambda_k(A) \lambda_1 \lambda_2 \cdots \lambda_k(B)$$

$$\begin{aligned} & [\lambda_1 + \lambda_2 + \cdots + \lambda_k](A+B) \\ \leq & [\lambda_1 + \lambda_2 + \cdots + \lambda_k](A) + [\lambda_1 + \lambda_2 + \cdots + \lambda_k](B) \end{aligned}$$

Dimension Problems

If the differential equation

$$\dot{x} = A(t)x$$

satisfies *certain technical conditions**

$$\dim \left\{ x(t) : \lim_{t \rightarrow \infty} x(t) = 0 \right\} \geq n - k + 1$$

if and only if *all solutions* of $\dot{z} = A^{[k]}(t)z$ satisfy

$$\lim_{t \rightarrow \infty} z(t) = 0.$$

So for example, with $k = n$ and the technical conditions*, there exists at least one non-zero solution $x(t)$ of $\dot{x} = A(t)x$ such that $\lim_{t \rightarrow \infty} x(t) = 0$ if and only if $\int_0^\infty \operatorname{tr} A(s) ds = -\infty$. In this case

$$A^{[n]}(t) = \operatorname{tr} A(t)$$

and the solutions of the scalar equation $\dot{z} = \operatorname{tr} A(t)z$ are

$$z(t) = z(0) \exp \left(\int_0^t \operatorname{tr} A(s) ds \right).$$