A Research Story: Compound Equations and Dynamics. Part I

James Muldowney, University of Alberta

July 16, 2021

Compound Matrices

$m \times n$ matrix:

$$A = \left[a_i^j\right], \; 1 \le i \le m, \; 1 \le j \le n$$

$p \times p$ minor:

$$a_{r_1 \dots r_p}^{s_1 \dots s_p} = \det \left[a_{r_i}^{s_j}\right], \; 1 \le i, j \le p,$$

minor of A determined by the rows $r_1, \dots, r_p$ and the columns $s_1, \dots, s_p$

examples:

$$\begin{aligned} a_{11}^{12} &= \begin{vmatrix} a_1^1 & a_1^2 \\ a_1^1 & a_1^2 \end{vmatrix} = 0 \\ a_{12}^{12} &= \begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix}, \; a_{13}^{12} = \begin{vmatrix} a_1^1 & a_1^2 \\ a_3^1 & a_3^2 \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
a^{123}_{122} &= \begin{vmatrix} a^1_1 & a^2_1 & a^3_1 \\ a^1_2 & a^2_2 & a^3_2 \\ a^1_2 & a^2_2 & a^3_2 \end{vmatrix} = 0 \\
a^{123}_{123} &= \begin{vmatrix} a^1_1 & a^2_1 & a^3_1 \\ a^1_2 & a^2_2 & a^3_2 \\ a^1_3 & a^2_3 & a^3_3 \end{vmatrix}, \; a^{124}_{123} = \begin{vmatrix} a^1_1 & a^2_1 & a^4_1 \\ a^1_2 & a^2_2 & a^4_2 \\ a^1_3 & a^2_3 & a^4_3 \end{vmatrix}
\end{aligned}
$$

$m = n > p$, cofactor matrix:

$$A^{s_1 \dots s_p}_{r_1 \dots r_p} \text{ is the } \textit{cofactor} \text{ of } a^{s_1 \dots s_p}_{r_1 \dots r_p}\,,$$

i.e. it is the *signed minor* determined by the rows complementary to rows $r_1, \dots, r_p$ and by the columns complementary to columns $s_1, \dots, s_p$ multiplied by $(-1)^{r_1+s_1+\dots+r_p+s_p}$.

If $p = n$, define

$$A^{12\dots n}_{12\dots n} = 1.$$

3×3 matrix $A = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix}$:

$$A_1^1 = a_{23}^{23} = \begin{vmatrix} a_2^2 & a_2^3 \\ a_3^2 & a_3^3 \end{vmatrix}, \; A_2^3 = -a_{13}^{12} = -\begin{vmatrix} a_1^1 & a_1^2 \\ a_3^1 & a_3^2 \end{vmatrix}$$

$$A_{12}^{12} = a_3^3, \; A_{13}^{12} = -a_2^3 \text{ and } A_{123}^{123} = 1.$$

Note that in this case we have

$$\det A = a_i^1 A_i^1 + a_i^2 A_i^2 + a_i^3 A_i^3, \; i = 1, 2, 3$$

and

$$0 = a_i^1 A_r^1 + a_i^2 A_r^2 + a_i^3 A_r^3, \text{ if } i \neq r.$$

$n \times n$ matrix A: *Laplace* expansion

$$\begin{aligned} \text{by rows } \det A &= \sum_{j=1}^{n} a_i^j A_i^j, \qquad i = 1, 2, \cdots, n \\ \text{by columns } \det A &= \sum_{i=1}^{n} a_i^j A_i^j, \qquad j = 1, 2, \cdots, n \end{aligned}$$

If A is a $n \times n$ matrix and $1 \leq k \leq n$, then the *Laplace* expansions by minors are:

$$\det A = \sum_{1 \leq s_1 < \ldots < s_k \leq n} a^{s_1 \ldots s_k}_{r_1 \ldots r_k} A^{s_1 \ldots s_k}_{r_1 \ldots r_k}, \text{ if } 1 \leq r_1 < \ldots < r_k \leq n$$

$$\det A = \sum_{1 \leq r_1 < \ldots < r_k \leq n} a^{s_1 \ldots s_k}_{r_1 \ldots r_k} A^{s_1 \ldots s_k}_{r_1 \ldots r_k}, \text{ if } 1 \leq s_1 < \ldots < s_k \leq n$$

Note: $0 = \sum_{(s)} a^{s_1 \ldots s_k}_{r_1 \ldots r_k} A^{s_1 \ldots s_k}_{t_1 \ldots t_k}$, if $(r) \neq (t)$, and $0 = \sum_{(r)} a^{s_1 \ldots s_k}_{r_1 \ldots r_k} A^{t_1 \ldots t_k}_{r_1 \ldots r_k}$, if $(s) \neq (t)$.

$n \times n$ matrix A: *Cofactor* matrix of A:

$$\mathrm{cof}A = \left[A_i^j\right],\; i, j = 1, ..., n$$

*adjugate (*or *classical adjoint)* matrix of A:

$$\mathrm{adj}A = (\mathrm{cof}A)^T .$$

Properties:

$$A\,(\mathrm{adj}A) = (\mathrm{adj}A)\,A = (\det A)\,I$$

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}A$$

$$\det(\mathrm{cof}A) = \det(\mathrm{adj}A) = (\det A)^{n-1}$$

Multiplicative Compounds

$n \times m$ matrix A, $1 \le k \le \min\{n, m\}$
$k-$th multiplicative compound is the $\binom{n}{k} \times \binom{m}{k}$ matrix

$$A^{(k)} = \left[a^{s_1 \dots s_k}_{r_1 \dots r_k}\right] = \left[a^{(s)}_{(r)}\right]$$

The entry in the $r-$th row and the $s-$th column of $A^{(k)}$ is $a^{s_1 \dots s_k}_{r_1 \dots r_k} = a^{(s)}_{(r)}$, where $(r) = (r_1, ..., r_k)$ is the $r-$th member of the lexicographic ordering of the integers $1 \le r_1 < r_2 < ... < r_k \le m$ and $(s) = (s_1, ...s_k)$ is the s-th member in the lexicographic (dictionary) ordering of all k-tuples of the integers $1 \le s_1 < s_2 < ... < s_k \le n$:

$$\begin{aligned} 1 &\le r_1 < r_2 < r_3 \le 5 \\ (1) &= (123)\,,\ (2) = (124)\,,\ (3) = (125)\,,\ (4) = (134)\,,\ (5) = (135)\,, \\ (6) &= (145)\,,\ (7) = (234)\,,\ (8) = (235)\,,\ (9) = (245)\,, (10) = (345)\,. \end{aligned}$$

Example:

$$A = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ \vdots & \vdots \\ a_m^1 & a_m^2 \end{bmatrix}_{m\times 2}, \quad A^{(2)} = \begin{bmatrix} a_{12}^{12} \\ a_{13}^{12} \\ \vdots \\ a_{m-1,m}^{12} \end{bmatrix}_{\binom{m}{2}\times 1}$$

Binet-Cauchy Theorem:

$$AB = C \;\Rightarrow\; A^{(k)}B^{(k)} = C^{(k)}$$

Sylvester's Theorem:

$$\det A^{(k)} = (\det A)^{\binom{n-1}{k-1}}$$

Linear Differential Equations

$$\dot{x} = A(t)\,x \tag{L}$$

$t \in [0, \infty)$, $x \in \mathbb{R}^n$, $t \longrightarrow A(t)_{n \times n}$ continuous.

A solution $x(t)$ of (L) is uniquely determined by its value $x(t_0)$ at any point $t_0 \in [0, \infty)$.

$X(t)_{n \times m}$ is a *solution matrix* of (L) if $\dot{X}(t) = A(t)\,X(t)$

$X(t)$ is a *fundamental matrix* of (L) if it is $n \times n$, non-singular and

$$\dot{X}(t) = A(t)\,X(t)$$

The columns of a fundamental matrix span the solution space of (L) :
$x(t)$ is a solution of $(L) \iff$ there exists $c \in \mathbb{R}^n$ such that

$$x(t) = X(t)\,c.$$

Equivalently, the columns of $X(t)$ are solutions of (L) which span the solution space of (L).

In particular, each column of $X(t)$ is a solution of (L).

Suppose that $X(t)$ is a fundamental matrix of (L), then a $n \times n$ matrix $Y(t)$ is a fundamental matrix of (L) if and only if there is a constant non-singular matrix C such that $Y(t) = X(t)\,C$.

Any continuously differentiable $n \times n$ matrix $X(t)$ is a fundamental matrix for *some* linear differential equation $(L) \iff X(t)$ is non-singular:

$$
\begin{aligned}
A(t) &= \dot{X}(t)X^{-1}(t) \\
\dot{X}(t) &= A(t)\,X(t)
\end{aligned}
$$

Compound Differential Equations

Recall, from Sylvester's Theorem, $\det X(t)^{(k)} = (\det X(t))^{\binom{n-1}{k-1}}$ so that $\det X(t) \neq 0 \Rightarrow \det X^{(k)}(t) \neq 0$. So $Y(t) = X^{(k)}(t) = \left[x_{r_1\ldots r_k}^{s_1\ldots s_k}(t)\right]$ is a fundamental matrix for a $\binom{n}{k}$-dimensional equation. The coefficient matrix in this equation is denoted $A^{[k]}$

$$\dot{y} = A^{[k]}(t)\, y \tag{k}$$

the k-th *compound equation* of (L). Note that $A^{[1]} = A,\ A^{[n]} = \operatorname{tr} A$

$$\dot{y} = A(t)\, y \tag{1}$$

$$\dot{y} = \operatorname{tr} A(t)\, y \tag{n}$$

In the case $k = n$, $X^{(n)}(t) = \det X(t)$, and (n) is the famous *Abel-Jacobi* scalar equation which gives

$$\det X(t) = \det X(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right)$$

If $X(t)$ is a $n \times m$ solution matrix of (L), then $Y(t) = X^{(k)}(t)$ is a $\binom{n}{k} \times \binom{m}{k}$ solution matrix of (k)

Example:

$$X = \begin{bmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \\ \vdots & \vdots \\ x_m^1 & x_m^2 \end{bmatrix}_{m\times 2}, \quad x^{(2)} = \begin{bmatrix} x_{12}^{12} \\ x_{13}^{12} \\ \vdots \\ x_{m-1,m}^{12} \end{bmatrix}_{\binom{m}{2}\times 1}$$

Additive Compounds

$A = \left[a_i^j \right], 1 \le i, j \le m = n$

$C = A^{[k]}, 1 \le k \le m = n$ is called the $k-$th *additive compound A*

$$c_r^s = \begin{cases} a_{r_1}^{r_1} + \cdots + a_{r_k}^{r_k}, & \{ \text{if } (r) = (s) \\ (-1)^{i+j} a_{r_i}^{s_j}, & \begin{cases} \text{if exactly one entry } r_i \text{ in } (r) \\ \text{does not occur in } (s) \text{ and } s_j \\ \text{does not occur in } (r) \end{cases} \\ 0, & \begin{cases} \text{if } (r) \text{ differs from } (s) \text{ in two} \\ \text{or more entries} \end{cases} \end{cases}$$

Additivity:

$$(A + B)^{[k]} = A^{[k]} + B^{[k]}$$

Examples:

$n = 2:$

$$A^{[1]} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$$

$$A^{[2]} = a_{11} + a_{22} = \text{tr}A$$

$n = 3:$

$$A^{[1]} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

$$A^{[2]} = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}$$

$$A^{[3]} = a_{11} + a_{22} + a_{33} = \text{tr}A$$

$n = 4:$

$$A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}$$

$$A^{[3]} = \begin{bmatrix} a_{11}+a_{22}+a_{33} & a_{34} & -a_{24} & a_{14} \\ a_{43} & a_{11}+a_{22}+a_{44} & a_{23} & -a_{13} \\ -a_{42} & a_{32} & a_{11}+a_{33}+a_{44} & a_{12} \\ a_{41} & -a_{31} & a_{21} & a_{22}+a_{33}+a_{44} \end{bmatrix}$$

$$A^{[4]} = a_{11}+a_{22}+a_{33}+a_{44} = trA$$

Geometrical Interpretation

Solutions $x^1(t)$, $x^2(t)$ of (L) with $n = 3$ may be interpreted as oriented line segments in $\mathbb{R}^3$ whose projections on a basis e^1, e^2, e^3 are $\begin{bmatrix} x_1^1(t) \\ x_2^1(t) \\ x_3^1(t) \end{bmatrix}$, $\begin{bmatrix} x_1^2(t) \\ x_2^2(t) \\ x_3^2(t) \end{bmatrix}$ and whose evolution in time is governed by (L). If

$X(t) = \begin{bmatrix} x_1^1(t) & x_1^2(t) \\ x_2^1(t) & x_2^2(t) \\ x_3^1(t) & x_3^2(t) \end{bmatrix}$, then $X^{(2)}(t) = \begin{bmatrix} x_{12}^{12} \\ x_{13}^{12} \\ x_{23}^{12} \end{bmatrix}$ satisfies (2) and may be considered as an oriented 2-dimensional parallelogram in $\mathbb{R}^3$ whose projection onto the $\left(e^i, e^j\right)$ coordinate plane, $i < j$, is a parallelogram with area x_{ij}^{12}.

If $x^1(t), \cdots, x^k(t)$ are considered as an ordered set of oriented line segments in $\mathbb{R}^n$ changing with time, then $y(t) = x^{12\cdots k}_{r_1 r_2 \cdots r_k}(t)$ may be interpreted as the projection of the corresponding $k-$dimensional oriented parallelopiped in $\mathbb{R}^n$ onto the k-dimensional coordinate subspace spanned by $e_{r_1}, \cdots, e_{r_k}$.

$$(\exp A)^{(k)} = \exp\left(A^{[k]}\right)$$

$$\frac{d}{dt}\left.(I + tA)^{(k)}\right|_{t=0} = A^{[k]}$$

The last expression is sometimes taken as the definition of $A^{[k]}$