Surfaces in $\mathbb{R}^3$

Note Title 18/07/2018

Quadric Surfaces

$$0 = ax^2 + by^2 + cz^2 + d\,xy + e\,xz + f\,yz + gx + hy + kz + l$$

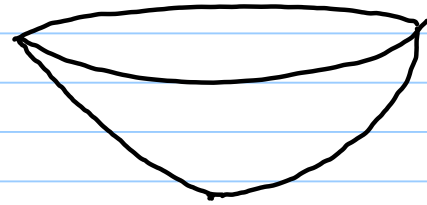
for constants $a, \ldots, l$.

Can eliminate the "cross-terms" by simple changes of variable

$$\Rightarrow 0 = ax^2 + by^2 + cz^2 + gx + hy + kz + l$$

(where the meanings of $a, \ldots, l$, x, y, z may have changed.)

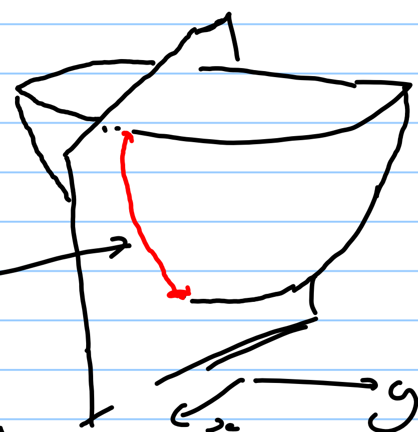
e.g. Paraboloid $z = ax^2 + by^2$



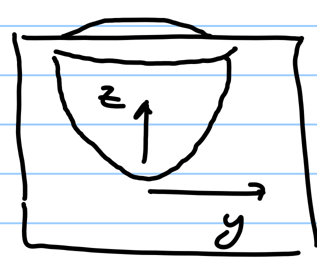
Paraboloid intersected by vertical plane

$$\alpha x + \beta y = 0$$

Curve of intersection obeys both the above equations.



e.g. Plane $x = 0$:



Curve of intersection is

$$\left.\begin{aligned} x(t) &= 0 \\ y(t) &= t \\ z(t) &= bt^2 \end{aligned}\right\}$$

Curvature at $(0,0,0)$ is

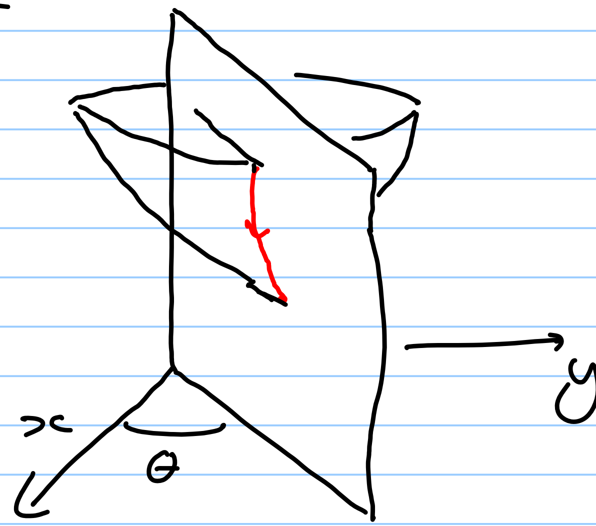
$$\kappa = \left.\frac{2b}{(1+4b^2t^2)^{3/2}}\right|_{t=0} = 2b$$

Plane $y=0$:

$x(t)=t$
$y(t)=0$
$z(t)=at^2$

$K=2a$

Vertical Plane making angle θ with x-axis



$K=2\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$

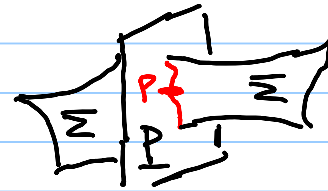
Extreme Values are $K=2a$ and $K=2b$

They <u>always</u> occur as the inverse radii of osculating circles in two <u>orthogonal planes</u>.

True for any surface.

<u>Definition</u>: Consider a point p on a surface $\Sigma \subseteq \mathbb{R}^3$, and consider all planes through p that contain the vector normal to Σ at p.

Each such plane $\underline{P}$ intersects Σ in a curve through p. Let $K_{\underline{P}}$ be the curvature of this curve in $\underline{P}$ at p.



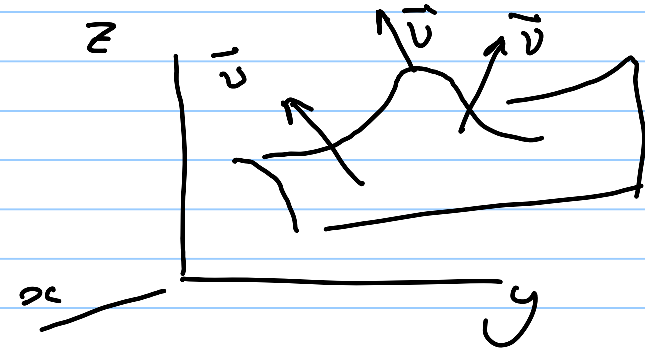
Find the maximum K_1 over all such planes and find the minimum K_2 over all such planes (the plane yielding K_2 will always be orthogonal to the plane yielding K_1).

Then κ_1 and κ_2 are called the <u>principal curvatures</u> of the surface Σ at p.

The <u>mean curvature</u> of Σ at p is $H := \frac{1}{2}(\kappa_1 + \kappa_2)$.

The <u>Gauss curvature</u> of Σ at p is $K_G := \kappa_1 \kappa_2$.

Minimal Surfaces:



$z = f(x,y)$

A formula for the principal curvatures:

1. Define $S(x,y,z) = z - f(x,y)$

2. Surfaces $S(x,y,z) = \text{constant}$ are called <u>level sets</u>.

   So graph $z = f(x,y)$ is the <u>zero level set</u> of S.

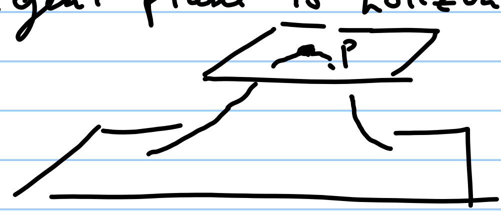
3. The gradient
$$\text{grad}(S) = \vec{\nabla} S = \left\langle \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z} \right\rangle = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

   is normal (i.e., perpendicular) to level sets.

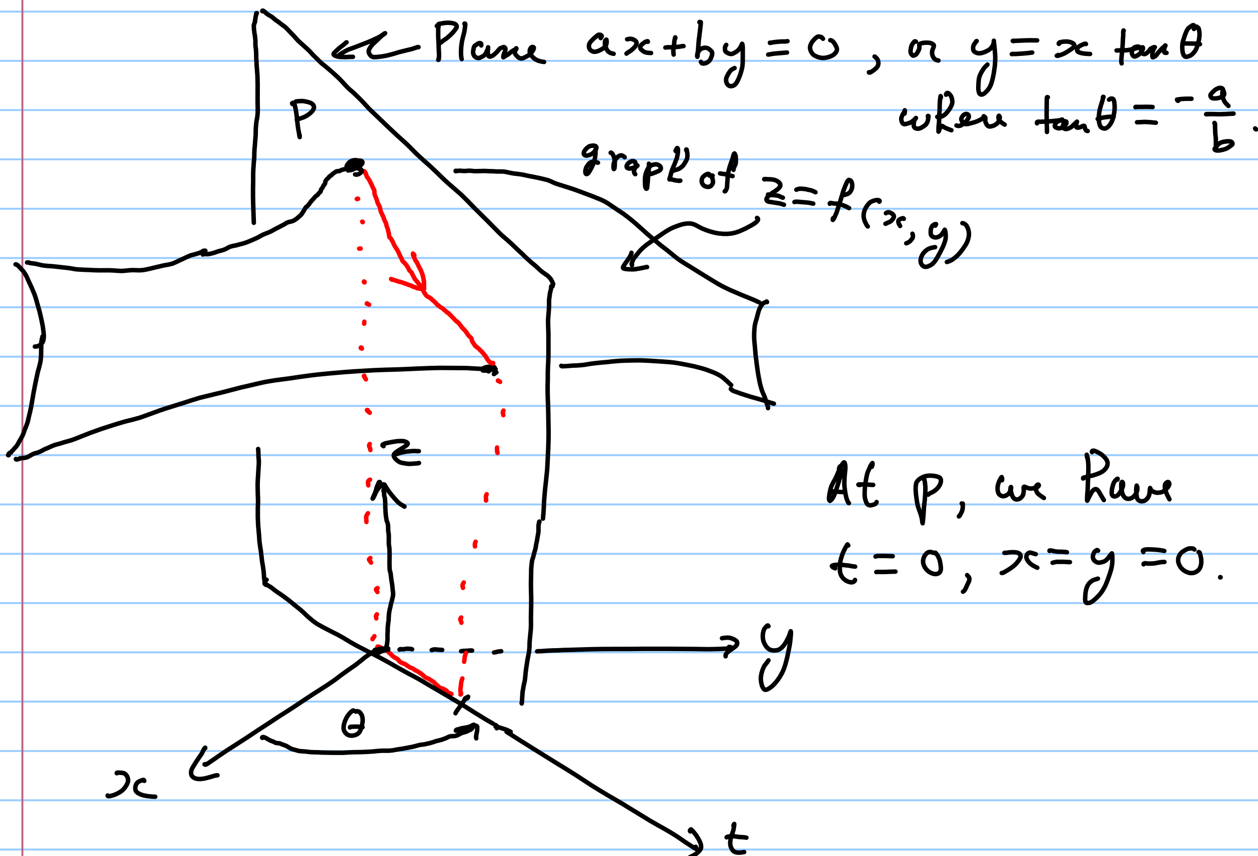
4. $\vec{\nu} = \vec{\nabla} S / |\vec{\nabla} S| =$ unit normal.

Consider a <u>critical point</u> p: $\vec{\nabla} f \big|_p = 0$

Then $\vec{\nu}$ is vertical (parallel to z-direction) and the tangent plane is horizontal ($\perp$ to z-direction)

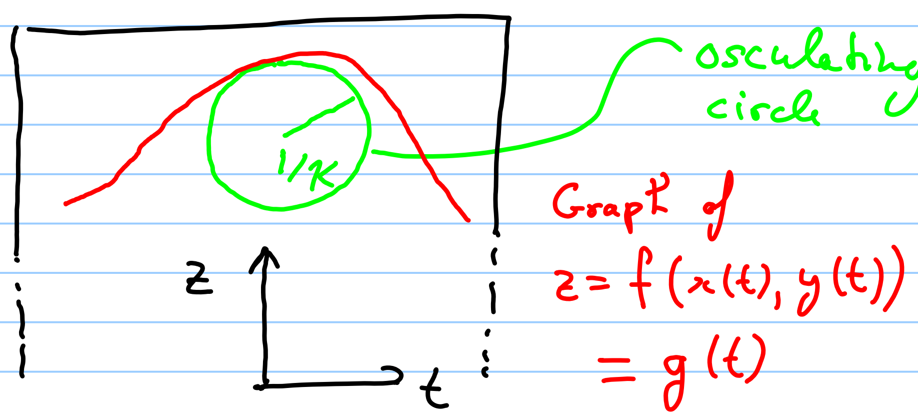


Consider all vertical planes through p:



At p, we have $t=0$, $x=y=0$.

In the (t,z) plane:



$$\text{Curvature} = \kappa = \frac{g''(0)}{\left[1+(g'(0))^2\right]^{3/2}} = g''(0)$$

because p $(t=0)$ is a critical point so $g'(0)=0$.

$g(t) = f(x(t), y(t))$

$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ --- Chain Rule

Because the curve lies on the plane $ax+by=0$, if we choose the parameter t along the curve so that $x(t)=t$, then necessarily $y(t) = -bt = (a\tan\theta)t$.

$\Rightarrow g'(t) = a\frac{\partial f}{\partial x} - b\frac{\partial f}{\partial y} = 0$ at critical point

$$\left.\begin{aligned} g''(t) &= \frac{d}{dt}\left[a\frac{\partial f}{\partial x} - b\frac{\partial f}{\partial y}\right] \\ &= a^2\frac{\partial^2 f}{\partial x^2} - 2ab\frac{\partial^2 f}{\partial x\partial y} + b^2\frac{\partial^2 f}{\partial y^2} \\ &= \begin{bmatrix} a & b \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx} & -f_{xy} \\ -f_{xy} & f_{yy} \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}\right\} \quad (*)$$

The matrix A is called the <u>second fundamental form (2FF)</u> of the surface $z=f(x,y)$ at the critical point p.

At other points, A is slightly more complicated.

Eigenvalues: $\det(A - \lambda I) = 0$

$$0 = \begin{vmatrix} f_{xx}-\lambda & -f_{xy} \\ -f_{xy} & f_{yy}-\lambda \end{vmatrix} = (f_{xx}-\lambda)(f_{yy}-\lambda) - f_{xy}^2$$

$$= \lambda^2 - (f_{xx}+f_{yy})\lambda + f_{xx}f_{yy} - f_{xy}^2$$

$\Rightarrow 0 = \lambda^2 - 2H\lambda + K_G^2$

when $H = \frac{1}{2} tr(A) =$ "mean curvature"

$K_G = \det(A) =$ "Gauss curvature"

$$\lambda = H \pm \sqrt{H^2 - K_G}$$
$$= \frac{1}{2}(f_{xx} + f_{yy}) \pm \sqrt{\frac{1}{4}(f_{xx}+f_{yy})^2 - f_{xx}f_{yy} + f_{xy}^2}$$
$$= \frac{1}{2}(f_{xx} + f_{yy}) \pm \sqrt{\frac{1}{4}(f_{xx}-f_{yy})^2 + f_{xy}^2}$$

Call these λ_i, $i = 1, 2$. Pick $\lambda_1 = \frac{1}{2}(\dots) - \sqrt{\dots}$, $\lambda_2 = \frac{1}{2}(\dots) + \sqrt{\dots}$.

Then we can find eigenvectors $\vec{e}_i$ such that

(i) $A\vec{e}_i = \lambda_i \vec{e}_i$ (ii) $|\vec{e}_i| = 1$, (iii) $\vec{e}_1 \perp \vec{e}_2$.

Now let $c_1 = a$, $c_2 = b$, so that $\sum_{i=1}^{2} c_i^2 = a^2 + b^2 = 1$.

$$g''(t) = [a \; b] A \begin{bmatrix} a \\ b \end{bmatrix} = \left(\sum_{i=1}^{2} c_i \vec{e}_i\right)^T A \left(\sum_{j=1}^{2} c_j \vec{e}_j\right)$$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} c_i c_j \, e_i^T A e_j, \quad \text{and } Ae_j = \lambda_j e_j$$

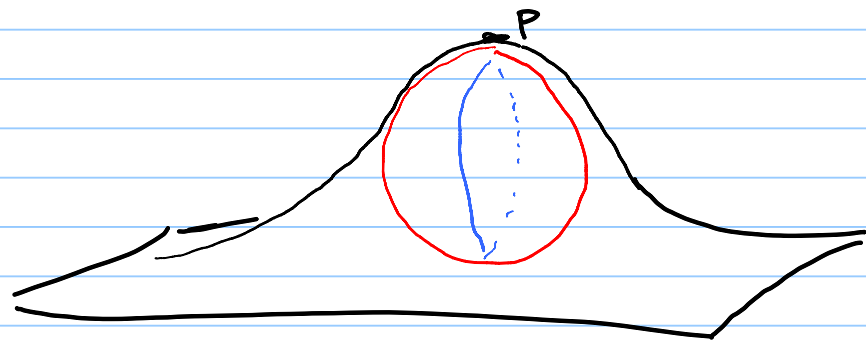
$$= \sum_{i=1}^{2} \sum_{j=1}^{2} c_i c_j \lambda_j (e_i \cdot e_j), \quad \text{and } e_i \cdot e_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$= \sum_{i=1}^{2} c_i^2 \lambda_i \quad \text{when } \sum_{i=1}^{2} c_i^2 = 1.$$

$$= c_1^2 \lambda_1 + c_2^2 \lambda_2 = a^2 \lambda_1 + b^2 \lambda_2, \quad a^2 + b^2 = 1$$

Now $\lambda_2 \geq \lambda$. Thus $g''(t)$ is maximized when $a = 0, b = 1$, and the max. value is $\lambda_2 = \lambda_+$. It's minimized when $a = 1, b = 0$, and the min. value is $\lambda_1 = \lambda_-$

$\Rightarrow \lambda_{\pm}$ are our old friends, the principal curvatures at a critical point p.



Mean Curvature $= H = \frac{1}{2}(K_1 + K_2) = \frac{1}{2}(f_{xx} + f_{yy})$

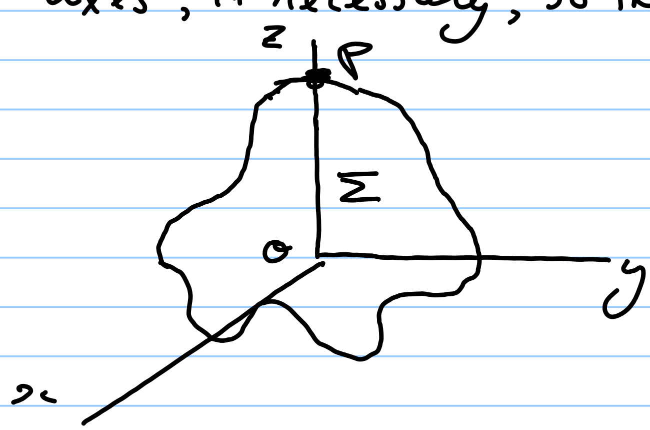
Gauss Curvature $= K_G = K_1 K_2$

The formulas are more complicated at generic points (i.e., at points that are not critical points).

A surface in $\mathbb{R}^3$ with $H = 0$ everywhere is called a minimal surface.

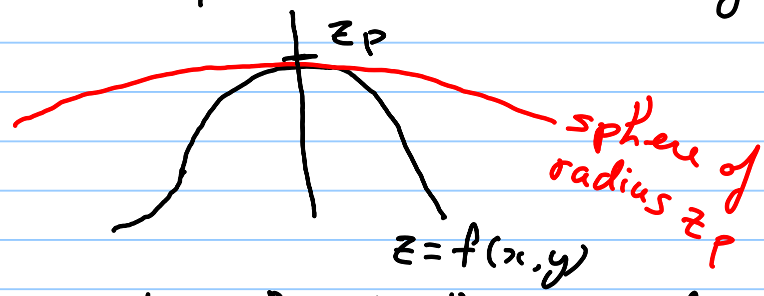
Bernstein Theorem: There are no compact minimal surfaces in $\mathbb{R}^3$.

Proof: On a compact surface Σ we can always find a point p farthest from the origin. Rotate the axes, if necessary, so that p lies on z-axis, $z>0$.



Near p, Σ is the graph of a function $z = f(x,y)$, and p is a maximum, so it's a critical point. Denote the value of z at p by z_p.

Draw the sphere centred at the origin, with radius z_p.



This sphere has both principal curvatures $= 1/z_p$.

But we have (see diagram) $\frac{1}{z_p} \le \kappa_1 \le \kappa_2$

when κ_1, κ_2 are the principal curvatures of the graph.

$$\Rightarrow \ 0 < \frac{1}{z_p} = \frac{1}{2}\left(\frac{1}{z_p} + \frac{1}{z_p}\right) \le \frac{1}{2}(\kappa_1 + \kappa_2) = H \text{ at } p.$$

$\Rightarrow$ $H > 0$ at p, so Σ cannot be a minimal surface.

$\square$

<u>Minimal Surface Equation</u>

If p is not a critical point, the principal curvatures and the mean curvature have some extra factors.

$$H = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right)\right.$$

where $z = f(x,y)$, $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$.

The minimal surface equation is still $\underbrace{H = 0}$.

$\boxed{\text{* Every plane in } \mathbb{R}^3 \text{ is a minimal surface.}}$

Some Nontrivial Examples

Scherk's doubly periodic surface

For $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ this has equation

$$z = f(x,y) = \log \frac{\cos y}{\cos x} = \log(\cos y) - \log(\cos x)$$

Then take the π-periodic even extension in x and y directions.

Check: $f_x = \tan x$, $f_y = -\tan y$

$$\Rightarrow \frac{\partial}{\partial x} \frac{f_x}{\sqrt{1+f_x^2+f_y^2}} = \frac{\partial}{\partial x} \frac{\tan x}{\sqrt{1+\tan^2 x+\tan^2 y}}$$ Use quotient rule

$$= \frac{\sec^2 x(1+\tan^2 x+\tan^2 y) - \tan^2 x \sec^2 x}{(1+\tan^2 x+\tan^2 y)^{3/2}}$$

$$= \frac{\sec^2 x \sec^2 y}{(1+\tan^2 x+\tan^2 y)^{3/2}}$$ where we have skipped a few steps

Likewise $$\frac{\partial}{\partial y} \frac{f_y}{\sqrt{1+f_x^2+f_y^2}} = -\frac{\sec^2 x \sec^2 y}{(1+\tan^2 x+\tan^2 y)^{3/2}}$$

$\Rightarrow H = 0$. ✓

See www.ugr.es/~fmartin/scherk-4.jpg
or https://en.wikipedia.org/wiki/Scherk_surface

Enneper's surface:

$$\left.\begin{aligned} x(t,s) &= s - s^3/3 + st^2 \\ y(t,s) &= -t - s^2 t + t^3/3 \\ z(t,s) &= s^2 - t^2 \end{aligned}\right\}$$ Defined parametrically, with parameters s, t.

https://www.math.hmc.edu/~gu/curves_and_surfaces/surfaces/enneper.html

* Compact minimal surfaces exist, but not in $\mathbb{R}^n$.

3-sphere $S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2=1\}$

- The equatn $w=0$ is a minimal 2-sphere in S^3
- The "Clifford torus" is also a minimal surface in S^3.

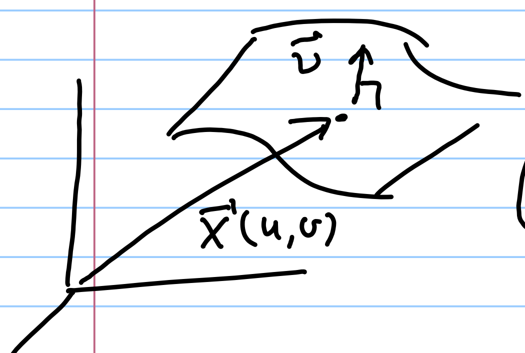
* Static Black Hole Horizons in General Relativity are compact minimal surfaces

<u>CMC surfaces:</u> If a surface has $H=$ const, the surface is <u>CMC</u> (Constant Mean Curvature)

* Compact CMC surfaces <u>do</u> exist in $\mathbb{R}^3$: soap bubbles!

Mean Curvature Flow (MCF)

The curve shortening flow for curves in $\mathbb{R}^2$ generalizes to become the Mean Curvature Flow for surfaces in $\mathbb{R}^3$



$\vec{\nu}$ = unit normal to surface

$$\boxed{\text{MCF: } \frac{\partial \vec{X}}{\partial t} = H\vec{\nu}}$$