Note Title 17/07/2018

Geometric Analysis

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Geometric Analysis = Geometry + Analysis (Calculus)

- Curves, surfaces, manifolds
- Curvature
- Applications: General relativity
  Topology
  Applied Mathematics
  Information Theory
  Statistics
  - - -

A project: Can you write a computer program to do this?
http://a.carapetis.com/csf/

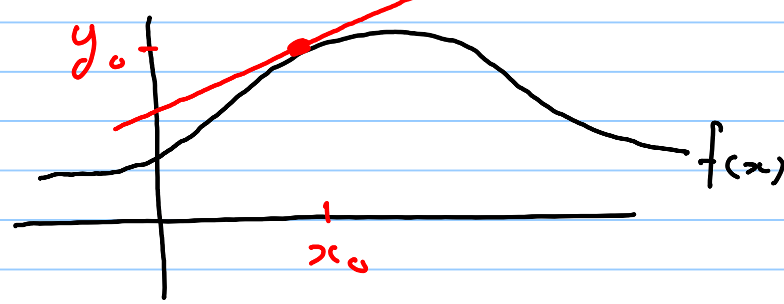
Today's topic: What is this webpage doing?
Answer: "Curve shortening flow"
To explain, first we have to study curves and their curvature.

Curvature

Graph of the function $x \mapsto f(x)$

Tangent line ("linear approximation") at (x_0, y_0)

$$y = f'(x_0)(x - x_0) + y_0$$



The osculating circle — Find a circle that

(i) passes through (x_0, y_0),

(ii) has the same tangent line as $f(x)$ does at (x_0, y_0), and

(iii) has the same second derivative (concavity/convexity) as $f(x)$ does at (x_0, y_0)

Notice: These are 3 conditions for the 3 free parameters a, b, c in the equation of a circle:

$$(x-a)^2 + (y-b)^2 = c^2 \quad --- (*)$$

Solution: Using (i), then $c^2 = (x_0 - a)^2 + (y_0 - b)^2$ --- (i)

Differentiate $(*)$: $\quad x - a + (y-b)\,y' = 0 \quad --- (**)$

But when $(x, y) = (x_0, y_0)$ then $y' = f'(x_0)$.

$$\Rightarrow (x_0 - a)^2 = (f'(x_0))^2 (y_0 - b)^2 \quad --- (ii)$$

Combine (i), (ii) $\Rightarrow c^2 = \left[1 + (f'(x_0))^2\right](y_0 - b)^2$ --- (ii′)

Finally, differentiate (∗∗) to get

$$1+(y')^2+(y-b)y''=0$$

At (x_0, y_0) we have $y'=f'(x_0)$, $y''=f''(x_0)$.

$$\Rightarrow \quad 1+(f'(x_0))^2+(y_0-b)f''(x_0)=0$$

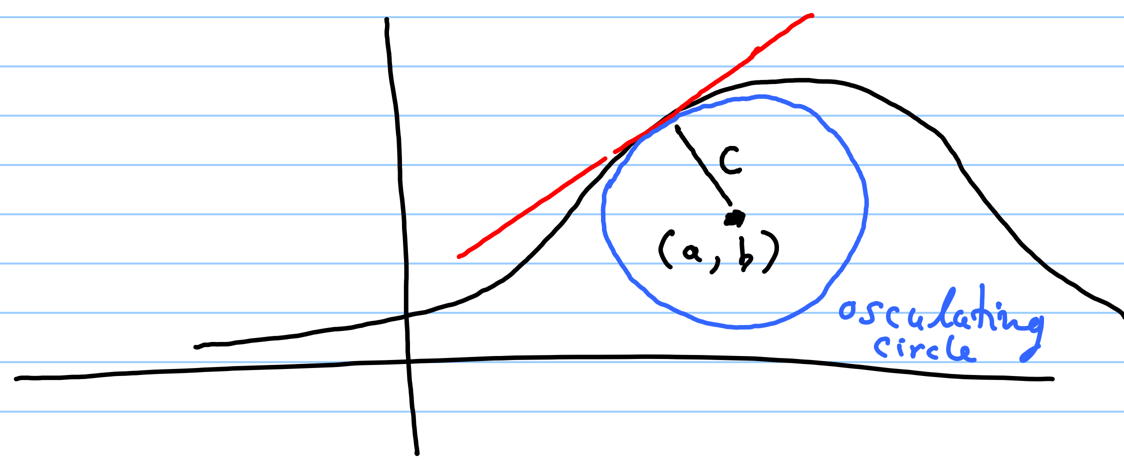
$$\Rightarrow \quad (y_0-b)^2\,(f''(x_0))^2=\left[1+(f'(x_0))^2\right]^2 \quad \text{--- (iii)}$$

Using (ii′), we get:

$$c^2=\left[1+(f'(x_0))^2\right]^3\Big/(f''(x_0))^2 \quad \text{--- (∗∗∗)}$$

provided $f''(x_0)\neq 0$ (this happens, for example, at smooth points of inflection).

∗ Now can find a, b in terms of $f'(x_0)$, $f''(x_0)$.

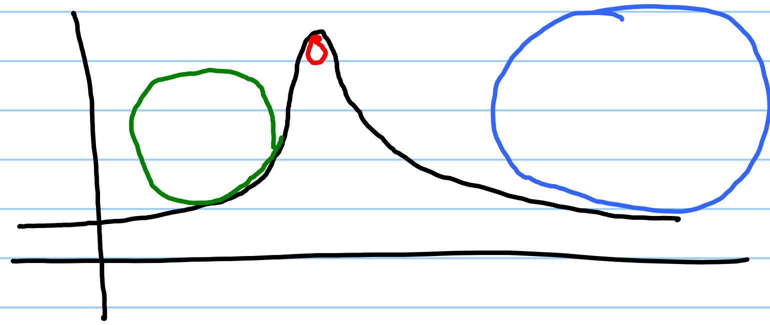


∗ The osculation circle is a "better" approximation to $x \mapsto f(x)$ at (x_0, y_0).

$$f(x)=\underbrace{\underbrace{f(x_0)+f'(x_0)(x-x_0)}_{\text{tangent line approximation}}+\tfrac{1}{2}f''(x_0)(x-x_0)^2}_{\text{quadratic approximation}}+\cdots$$

Small circle $\Rightarrow$ large curvature

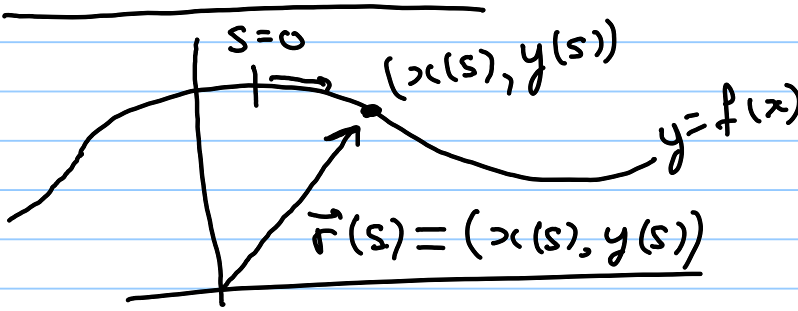
Large circle $\Rightarrow$ small curvature



Define the curvature κ to be the inverse of the radius of the osculating circle:

$$\kappa := \frac{1}{c} = \frac{|f''(x_0)|}{\left[1+(f'(x_0))^2\right]^{3/2}}$$

Parametrized Curves



$$\frac{dy}{ds} = \frac{df}{dx}\frac{dx}{ds} \quad \text{by chain rule.}$$

$$\Rightarrow \frac{df}{dx} = \frac{dy}{ds} \Big/ \frac{dx}{ds} \quad \left(\text{assume } \frac{dx}{ds} \neq 0\right) \text{ --- [1]}$$

Now $$\frac{d^2y}{ds^2} = \frac{d}{ds}\left(\frac{df}{dx}\frac{dx}{ds}\right) = \left(\frac{d}{ds}\frac{df}{dx}\right)\frac{dx}{ds} + \frac{df}{dx}\frac{d^2x}{ds^2}$$

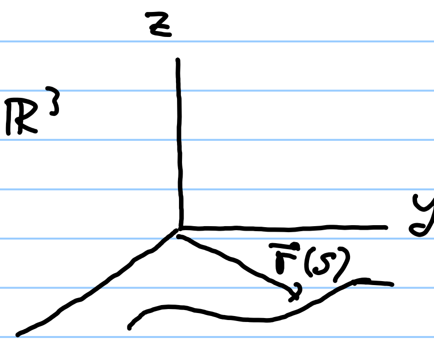
$$= \frac{d^2f}{dx^2}\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy/ds}{dx/ds}\right)\frac{d^2x}{ds^2}$$

$$\Rightarrow \frac{d^2 f}{dx^2} = \frac{\frac{d^2 y}{ds^2} - \frac{dy/d}{dx/ds}\frac{d^2 x}{ds^2}}{(dx/ds)^2} = \frac{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}}{(dx/ds)^3} \quad --[2]$$

$$\Rightarrow K = \frac{\left|\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right|}{\left(\frac{dx}{ds}\right)^3\left[1+\left(\frac{dy/ds}{dx/ds}\right)^2\right]^{3/2}}$$

$$= \frac{\left|\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right|}{\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right]^{3/2}}$$

Think of xy-plane as lying in $\mathbb{R}^3$



$\vec{r}(s) = (x(s), y(s), 0)$

$\vec{r}\,'(s) = (x'(s), y'(s), 0)$

$\vec{r}\,''(s) = (x''(s), y''(s), 0)$

$$\Rightarrow K = \pm \frac{|\vec{r}\,'(s) \times \vec{r}\,''(s)|}{|\vec{r}\,'(s)|^3}$$

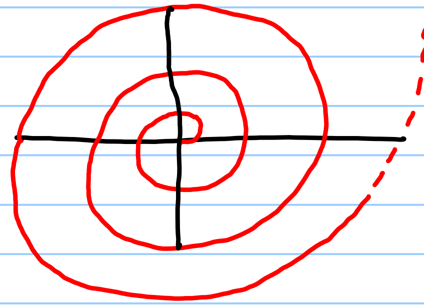
<u>Examples:</u>

1. Circle centred at origin, radius a.

$x(\theta) = a\cos\theta \Rightarrow x'(\theta) = -a\sin\theta \Rightarrow x''(\theta) = -a\cos\theta$
$y(\theta) = a\sin\theta \quad\;\; y'(\theta) = a\cos\theta \quad\;\; y''(\theta) = -a\sin\theta$

$$\Rightarrow K = \frac{a^2\sin^2\theta + a^2\cos^2\theta}{(a^2\sin^2\theta + a^2\cos^2\theta)^{3/2}} = \frac{1}{a} \checkmark.$$

2. Spiral curve $x(\theta) = \theta\cos\theta$
$y(\theta) = \theta\sin\theta$
$\theta \in [0, \infty)$



$$\Rightarrow x'(\theta) = -\theta\sin\theta + \cos\theta$$
$$y'(\theta) = \theta\cos\theta + \sin\theta$$

$$\Rightarrow x''(\theta) = -\theta\cos\theta - 2\sin\theta$$
$$y''(\theta) = -\theta\sin\theta + 2\cos\theta$$

$$\Rightarrow x'(\theta)y''(\theta) - y'(\theta)x''(\theta) = (-\theta\sin\theta + \cos\theta)(-\theta\sin\theta + 2\cos\theta) - (\theta\cos\theta + \sin\theta)(-\theta\cos\theta - 2\sin\theta)$$

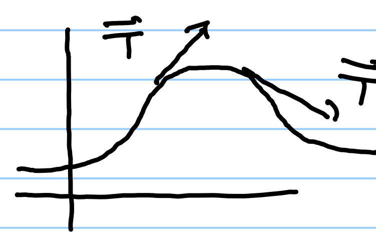
$$= \theta^2\sin^2\theta + 2\cos^2\theta + \theta^2\cos^2\theta + 2\sin^2\theta$$

$$= 2 + \theta^2$$

$$\left[\left(x'(\theta)\right)^2 + \left(y'(\theta)\right)^2\right]^{3/2} = \left[\theta^2\sin^2\theta + \cos^2\theta + \theta^2\cos^2\theta + \sin^2\theta\right]^{3/2}$$
$$= (1+\theta^2)^{3/2}$$

$$\Rightarrow K = \frac{2+\theta^2}{(1+\theta^2)^{3/2}} \sim \frac{1}{\theta} \text{ for large } \theta. \checkmark$$

Unit tangent vector $\vec{T}(s) = \dfrac{\vec{r}'(s)}{|\vec{r}'(s)|}$ so that $|\vec{T}(s)| = 1$.



Now $\dfrac{d}{ds}|\vec{r}'(s)|$

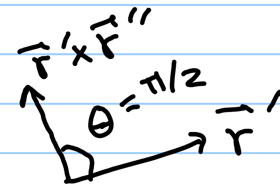
$$= \frac{1}{2|\vec{r}'(s)|}\frac{d}{ds}(\vec{r}'\cdot\vec{r}')$$

$$= \frac{\vec{r}'}{|\vec{r}'|}\cdot\frac{d}{ds}(\vec{r}') = \frac{\vec{r}'\cdot\vec{r}''}{|\vec{r}'|}$$

$$\Rightarrow \frac{d}{ds}\vec{T} = \frac{d}{ds}\left(\frac{\vec{r}'}{|\vec{r}'|}\right) = \frac{\vec{r}''|\vec{r}'| - \vec{r}'\frac{d}{ds}(|\vec{r}'|)}{|\vec{r}'|^2}$$

$$= \frac{\vec{r}''}{|\vec{r}'|} - \frac{(\vec{r}'\cdot\vec{r}'')\vec{r}'}{|\vec{r}'|^3}$$

$$= \frac{(\vec{r}'\cdot\vec{r}')\vec{r}'' - (\vec{r}'\cdot\vec{r}'')\vec{r}'}{|\vec{r}'|^3}$$



$$= \frac{\vec{r}'\times(\vec{r}''\times\vec{r}')}{|\vec{r}'|^3}$$

$$\Rightarrow \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{r}'\times(\vec{r}''\times\vec{r}')|}{|\vec{r}'|^3} = \frac{|\vec{r}'|\,|\vec{r}''\times\vec{r}'|}{|\vec{r}'|^3} \quad \sin\theta = 1$$

$$= \frac{|\vec{r}''\times\vec{r}'|}{|\vec{r}'|^2} = \pm K\,|\vec{r}'|$$

$$\Rightarrow K = \pm\left|\frac{d\vec{T}}{ds}\right| \Big/ \left|\frac{d\vec{r}}{ds}\right|$$

$\Rightarrow$ K measures

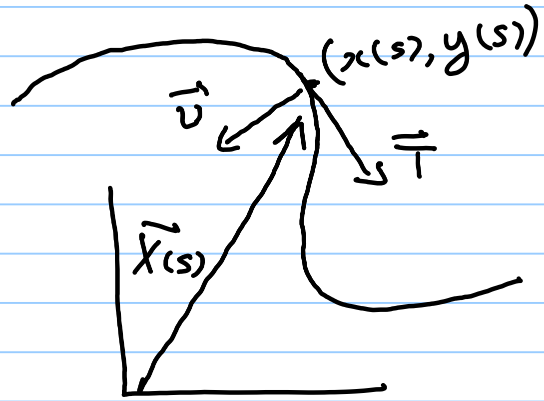
(i) the (inverse of the) radius of the osculating circle.

(ii) the rate of change of the unit tangent vector along the curve

<u>Remark:</u> Often we choose the parameter s so that $\left|\frac{d\vec{r}}{ds}\right| = 1$. Then s is called an <u>arc length parameter</u>.

Curves parametrized by an arclength parameter are called <u>unit speed curves</u>.

A curvature flow:



$\vec{\nu}$ = unit normal vector

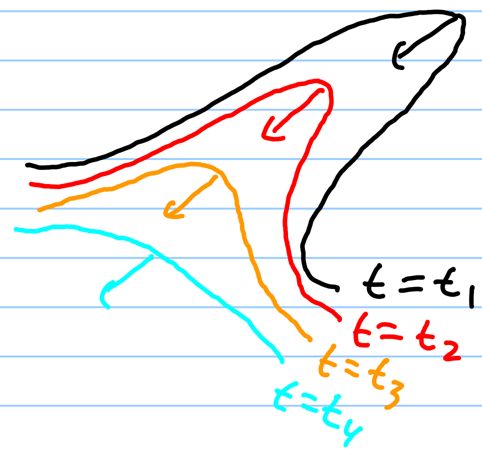
(i) $|\vec{\nu}| = 1$

(ii) $\vec{\nu} \perp \vec{T}$

(iii) $\vec{\nu}$ points toward centre of osculating circle.

The curve-shortening flow is $\vec{X}(s,t)$ where

$$\frac{\partial \vec{X}}{\partial t} = |\kappa| \, \vec{\nu}$$



$t_1 < t_2 < t_3 < t_4$

See the demonstration at a.carapetis.com/csf/

Space Curves: Curves in $\mathbb{R}^3$

Parametric form: $t \mapsto \vec{r}(t) = (x(t), y(t), z(t))$

t = parameter (not the t of the curve shortening flow)

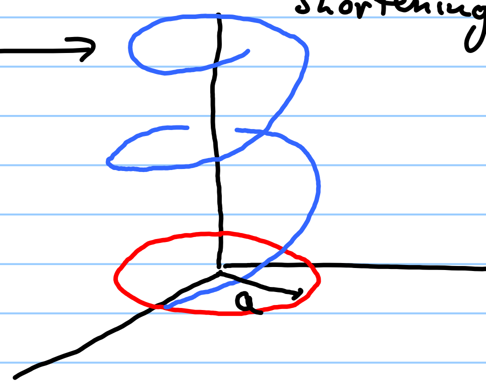
e.g. Circular helix: →

$x(t) = a\cos t$

$y(t) = a\sin t$

$z(t) = t$

$t \in [0, \infty)$



arclength of a space curve: $s(t) = \int_0^t |\vec{r}\,'|\,dt$

For helix (above), $\vec{r}\,' = (-a\sin t, a\cos t, 1)$

$$|\vec{r}\,'| = \sqrt{a^2+1}$$

$$s = \sqrt{a^2+1}\; t.$$

Arclength parametrization: a parameter u is an arclength parameter if $|\vec{r}\,'(u)| = 1$.

e.g. In above example, use $t = s/\sqrt{a^2+1}$ to write

$$\vec{r}(s) = \left(a\cos\frac{s}{\sqrt{a^2+1}},\ a\sin\frac{s}{\sqrt{a^2+1}},\ \frac{s}{\sqrt{a^2+1}}\right).$$

Then

$$\vec{r}\,'(s) = \left(\frac{-a}{\sqrt{a^2+1}}\sin\frac{s}{\sqrt{a^2+1}},\ \frac{a}{\sqrt{a^2+1}}\cos\frac{s}{\sqrt{a^2+1}},\ \frac{1}{\sqrt{a^2+1}}\right)$$

and so $|\vec{r}\,'(s)| = \dfrac{a^2+1}{a^2+1} = 1.$ $\Rightarrow$ s is an arclength parameter and $\vec{r}(s)$ is <u>unit speed.</u>

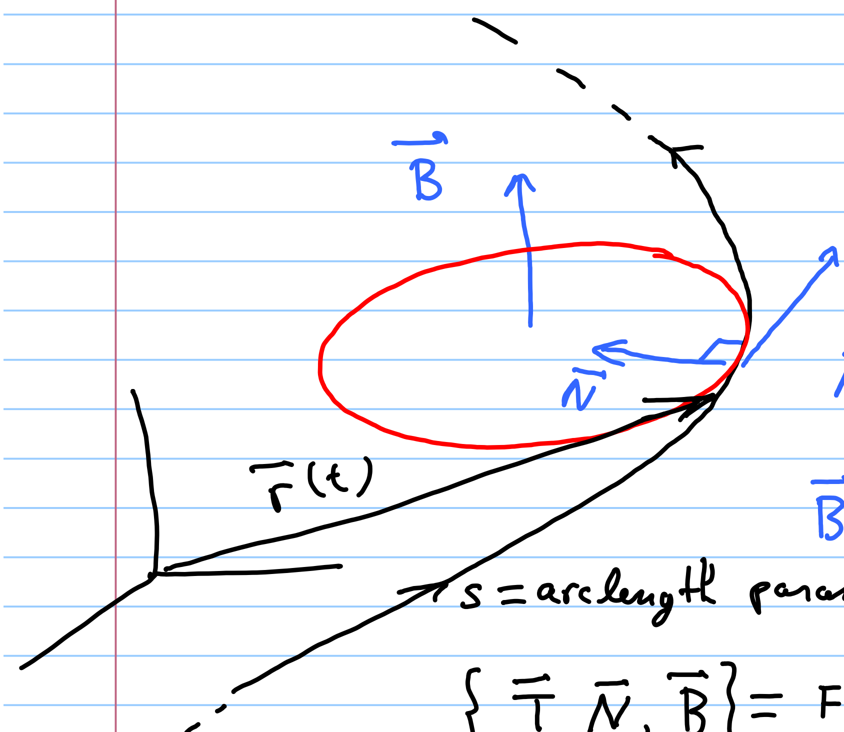
Curvature of a space curve: $\vec{T} = \vec{r}\,'(s) =$ unit tangent vector.

Then the curvature is

$$\kappa = \left| d\vec{T}/ds \right|$$

Helix with osculating circles with radii $1/\kappa$

Torsion: This measures how much the osculating circles "tip" or "tilt" as they move along the curve.



$$\vec{T} = \vec{r}\,'/|\vec{r}\,'|$$

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} \Rightarrow \begin{cases} |\vec{N}| = 1 \\ \vec{N} \perp \vec{T} \end{cases}$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{cases} |\vec{B}| = 1 \\ \vec{B} \perp \vec{T} \\ \vec{B} \perp \vec{N} \end{cases}$$

$s =$ arclength parameter

$\{\vec{T}, \vec{N}, \vec{B}\} =$ Fresnet frame

$=$ orthonormal basis along curve

$$\frac{d}{ds}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix},$$

τ is called the torsion.

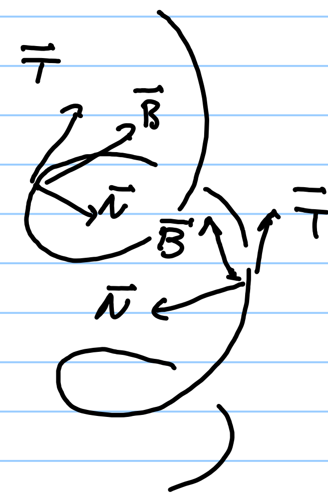
Exercise: For the circular helix

$$\vec{r}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}} \right)$$

$a \neq 0$, $b \neq 0$ are constants,

find $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$

and $\kappa(s)$, $\tau(s)$.



<u>Solution:</u>

$$\vec{r}\,' = \frac{1}{\sqrt{a^2+b^2}} \left(-a\sin\frac{s}{\sqrt{a^2+b^2}},\ a\cos\frac{s}{\sqrt{a^2+b^2}},\ b \right)$$

$|\vec{r}\,'| = 1$ so $\vec{T} = \vec{r}\,' = \frac{1}{\sqrt{a^2+b^2}}(\text{—"—})$

$$\frac{d\vec{T}}{ds} = -\frac{1}{a^2+b^2}\left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ 0 \right)$$

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{a}{a^2+b^2}$$

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = -\left(\cos\frac{s}{\sqrt{a^2+b^2}},\ \sin\frac{s}{\sqrt{a^2+b^2}},\ 0 \right)$$

so the vector pointing to centre of osculating circle is always horizontal

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\frac{s}{\sqrt{a^2+b^2}},\ -b\cos\frac{s}{\sqrt{a^2+b^2}},\ a \right).$$

Then $\dfrac{d\vec{B}}{ds} = \dfrac{b}{(a^2+b^2)}\left(\cos\dfrac{s}{\sqrt{a^2+b^2}},\ \sin\dfrac{s}{\sqrt{a^2+b^2}},\ 0 \right) = \dfrac{-b}{(a^2+b^2)}\vec{N}$

But $\dfrac{d\vec{B}}{ds} = -\tau\vec{N}$ from the matrix equation on the last page.

$$\Rightarrow \quad \tau = b/(a^2+b^2)$$

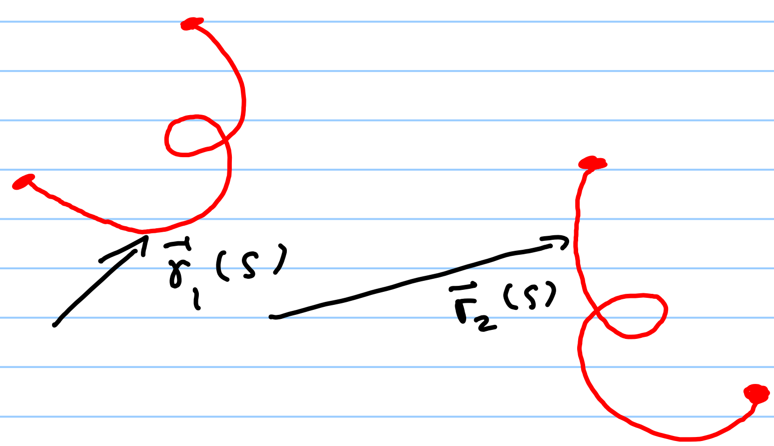
Fundamental Theorem of Space Curves:

Let $\vec{r}_1(s)$, $s \in I$, and $\vec{r}_2(s)$, $s \in I$ be two space curves parametrized by arclength s taking values in the same interval $I \subseteq \mathbb{R}$.

If the respective curvatures and torsions agree

$$\left.\begin{aligned} \kappa_1(s) &= \kappa_2(s) \\ \tau_1(s) &= \tau_2(s) \end{aligned}\right\} \text{ for all } s \in I$$

then $\vec{r}_1$ and r_2 are congruent (i.e., they are the same curve up to a rotation and translation.



Consider the *curve shortening flow*

$$\frac{\partial \vec{X}(\theta,t)}{\partial t} = \beta\kappa\vec{N} \tag{$*$}$$

Here $\vec{X}(\theta,t)$ is a flowing curve. That means that if t has a fixed constant value, say $t=0$ for example, then $\vec{X}(\theta,0) = \begin{pmatrix} x(\theta,0) \\ y(\theta,0) \end{pmatrix}$ is a *parametrized curve* in $\mathbb{R}^2$, with parameter θ. The curvature of the curve is κ and $\vec{N}$ is the unit normal vector pointing from the curve toward the centre of the asculating circle. Finally, $\beta > 0$ is a constant.
The first question should be easy, the next one medium, and the last one a bit more difficult:

1. Consider a circle, flowing according to equation $(*)$. Write the flowing circle as

$$\vec{X}(\theta,t) = a(t)(\cos\theta, \sin\theta)\ ,\ \theta \in [0, 2\pi]\ .$$

Say the flow begins at $t=0$, with initial condition $a(0) = a_0 > 0$. How long does it take for the circle to disappear? (i.e., at what t-value does the radius $a(t)$ become zero? Answer in terms of β and a_0.)

2. Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ ,\ 0 < a < b\ .$$

Now let this ellipse evolve under the mean curvature flow $(*)$. It disappears in some time T where $\frac{\beta}{b} < T < \frac{\beta}{a}$. Explain this (hint: use circles, and use the answer to the first question).

3. Let

$$\vec{X}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \end{pmatrix} = \begin{pmatrix} \arccos\frac{t}{\sqrt{t^2+s^2}} \\ \ln t - \ln\sqrt{t^2+s^2} - t \end{pmatrix}\ ,\ t \in [0,\infty)\ , s \in [0,\infty)\ . \tag{$\dagger$}$$

Here arccos means the inverse function to cos, sometimes denoted by $\cos^{-1}$. At each fixed t, this is a curve. For example, at $t=1$, this is the curve

$$\vec{X}(s,1) = \vec{X}(s) = \begin{pmatrix} \arccos\frac{1}{\sqrt{1+s^2}} \\ -1 - \ln\sqrt{1+s^2} \end{pmatrix}\ , s \in [0,\infty)\ .$$

(Try to sketch this curve.) Show that $\vec{X}(s,t)$ obeys equation $(*)$ (i.e., treating t as constant, compute κ and $\vec{N}$; then compare $\kappa\vec{N}$ to $\frac{\partial \vec{X}}{\partial t}$). As t increases, the curve $s \mapsto X(s,t)$ moves simply by translating downward. Can you see why?