Notes to IUSEP Lectures
in Mathematical Finance:

An Introduction to Derivative Pricing
and Portfolio Optimization

July 2021

Christoph Frei

www.math.ualberta.ca/∼cfrei
cfrei@ualberta.ca

1. Single-period binomial model

A single-period model for a financial market:

- Consider the following very simplistic model with one stock and a bank account over one period:

prices	initial value	terminal value	
stock	S_0	$\begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{with probability } 1-p \end{cases}$	
bank account	1	$1+r$	

  for numbers $S_0 > 0$, $0 \leq d < u$, $p \in (0,1)$ and r.

- Assumptions: agent can invest in or short sell (= negative investment) the stock, and they can invest in and borrow from the bank account at the same interest rate r.

- We also assume

$$d < 1 + r < u \quad \text{(no-arbitrage condition)}.$$

This assumption is reasonable. Example: if we had $d = 2$, $u = 3$ and $r = 0$, the agent could borrow a positive dollar amount x from the bank account and invest in stock to make a risk-free profit:

	initial value	terminal value
bank account	$-x$	$-x(1+r) = -x$
stock	x	$\begin{cases} ux = 3x \text{ with probability } p \\ dx = 2x \text{ with probability } 1-p \end{cases}$
total	0	$\begin{cases} 3x - x = 2x \text{ with probab. } p \\ 2x - x = x \text{ with probab. } 1-p \end{cases}$

Pricing financial derivatives:

- Consider now additionally a financial derivative with given payoff as follows:

initial value	terminal value
$x = ?$	$\begin{cases} f_u & \text{if stock price} = uS_0 \text{ (probab. } p) \\ f_d & \text{if stock price} = dS_0 \text{ (probab. } 1-p) \end{cases}$

for fixed numbers f_u and f_d.

- Example: (European) call option with strike K. *A call option gives the buyer the right, but not the obligation to buy the stock at maturity for the strike price K.* In our model the option payoff is

$$\begin{cases} \max\{uS_0 - K, 0\} & \text{with probability } p \\ \max\{dS_0 - K, 0\} & \text{with probability } 1 - p \end{cases}$$

because

- · two possibilities uS_0 or dS_0 for the stock price,
- · the buyer will use (= exercise) the option only if the stock price is higher than K.

- How can we find an initial value x for the financial derivative?
We can use a replication argument because we will see that we can obtain here the same payoff by investing in the stock and bank account.

Consider a portfolio consisting of Δ units of the stock and Ψ units of the bank account.

initial value	terminal value
$\Delta S_0 + \Psi$	$\begin{cases} \Delta uS_0 + \Psi(1 + r) & \text{if stock} = uS_0 \\ \Delta dS_0 + \Psi(1 + r) & \text{if stock} = dS_0 \end{cases}$

- We find Δ and Ψ of a replicating portfolio by setting its terminal value equal to that of the financial derivative, which implies

$$f_u = \Delta u S_0 + \Psi(1+r),$$
$$f_d = \Delta d S_0 + \Psi(1+r).$$

Solving this system of two linear equations for the unknowns gives

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}, \quad \Psi = \frac{u f_d - d f_u}{(1+r)(u-d)}$$

so that the initial value of the portfolio equals

$$\Delta S_0 + \Psi = \frac{f_u - f_d}{u-d} + \frac{u f_d - d f_u}{(1+r)(u-d)}$$
$$= \frac{1}{1+r}\left(\frac{1+r-d}{u-d} f_u + \frac{u-1-r}{u-d} f_d\right).$$

To avoid arbitrage ($=$ risk-free gains), this quantity must be equal to the initial value x of the financial derivative:

$$x = \frac{1}{1+r}\left(\frac{1+r-d}{u-d} f_u + \frac{u-1-r}{u-d} f_d\right).$$

Risk-neutral probabilities:

- Define

$$q_u = \frac{1 + r - d}{u - d}, \qquad q_d = \frac{u - 1 - r}{u - d}$$

  so that we can write

$$x = \frac{1}{1 + r}(q_u f_u + q_d f_d).$$

- Note that

  · $q_u + q_d = 1$,

  · $d < 1 + r < u \implies q_u > 0, q_d > 0.$

  Therefore, we can consider q_u, q_d as the probabilities of a probability measure Q and we have

$$x = \frac{1}{1 + r}E^Q[f] = \frac{1}{1 + r}(q_u f_u + q_d f_d),$$

  where f is the random variable of the option payoff and E^Q denotes the expectation under the measure Q. In other words,

$$\text{option value} = \begin{array}{l} \text{expectation of the discounted} \\ \text{payoff under a measure } Q \end{array}$$

  (discounted because payoff is divided by $1 + r$).

A crucial observation is that the probability measure Q used in the pricing formula does not equal the historical (from the model construction) probability measure because in general

$$q_u = \frac{1+r-d}{u-d} \neq p, \quad q_d = \frac{u-1-r}{u-d} \neq 1-p.$$

- If we calculate the expectation of the discounted stock price under Q, we obtain

$$
\begin{aligned}
E^Q &\left[\frac{\text{terminal value of stock}}{1+r}\right] \\
&= q_u \frac{uS_0}{1+r} + q_d \frac{dS_0}{1+r} \\
&= \frac{1+r-d}{u-d} \cdot \frac{uS_0}{1+r} + \frac{u-1-r}{u-d} \cdot \frac{dS_0}{1+r} \\
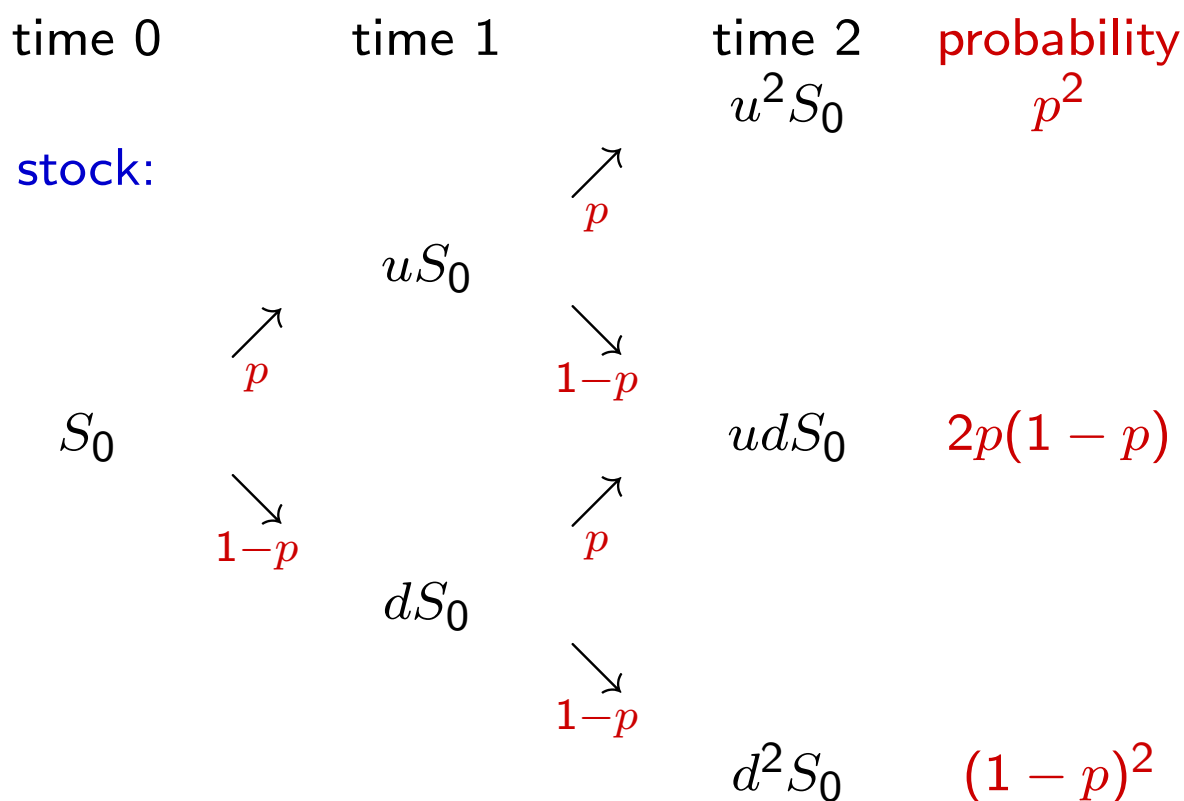&= S_0,
\end{aligned}
$$

which shows that the expectation of the discounted terminal stock price under Q equals its initial value. Therefore, q_u and q_d are called risk-neutral probabilities and Q a risk-neutral probability measure.

- Remark. A financial market (like that we considered here) where every payoff can be replicated is called complete. It can be proved that a risk-neutral probability measure exists if there is no arbitrage in the market model and it is unique if the market is complete.

- The pricing formula we derived was based on a replication argument: we replicated the payoff of the derivative by investing in the stock and bank account. As a byproduct, we also saw the right number of stocks we need for the replication, which is $\Delta = \frac{f_u - f_d}{S_0(u-d)}$.

  This means that as a buyer of the option, we can "neutralize" the option by investing $-\Delta$ in the stock. Conversely, as a writer ($=$ seller) of the option, we can buy Δ units of the stock to hedge against our risk. Consequently, this is called a replicating strategy or hedging strategy.

2. Two-period binomial model

- We can extend the model of Section 1 by adding a second period. We then have a tree of the form

time 0   time 1   time 2   probability

$$u^2 S_0 \qquad p^2$$

stock:

$$uS_0$$

$$p$$

$$1-p$$

$$S_0 \qquad\qquad udS_0 \qquad 2p(1-p)$$

$$1-p \qquad\qquad p$$

$$dS_0$$

$$1-p$$

$$d^2 S_0 \qquad (1-p)^2$$

bank account:

$$1 \quad \rightarrow \quad 1+r \quad \rightarrow \quad (1+r)^2$$

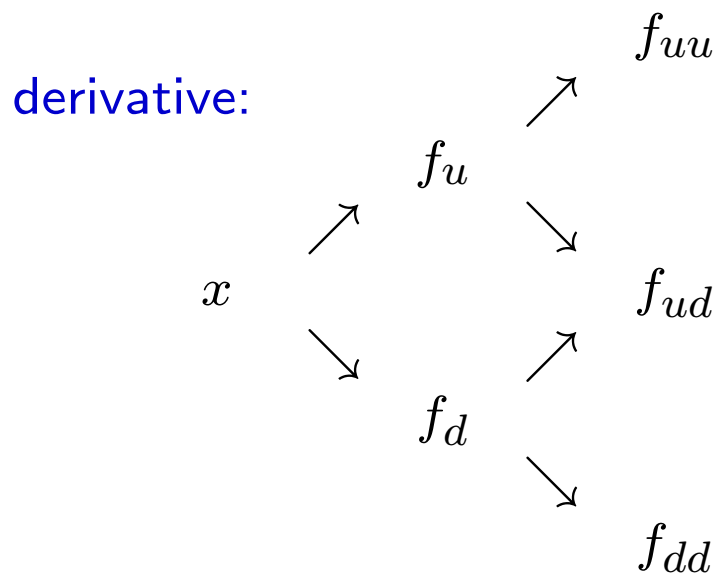- Trading is now also possible at the intermediate time 1. We still assume $d < 1 + r < u$.

- Let us consider a European call option with maturity 2. It has the following payoff at time 2:

$$\begin{cases} f_{uu} = \max\{u^2 S_0 - K, 0\} & \text{if stock } = u^2 S_0 \\ f_{ud} = \max\{ud S_0 - K, 0\} & \text{if stock } = ud S_0 \\ f_{dd} = \max\{d^2 S_0 - K, 0\} & \text{if stock } = d^2 S_0 \end{cases}$$

- By a replication argument similarly to that in Section 1 applied to both trading periods, the price of the derivative equals

$$\frac{1}{(1+r)^2}\left(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}\right),$$

where $q = \frac{1+r-d}{u-d}$. Indeed, we have

derivative:

$$\begin{array}{ccccc} & & & & f_{uu} \\ & & & \nearrow & \\ & & f_u & & \\ & \nearrow & & \searrow & \\ x & & & & f_{ud} \\ & \searrow & & \nearrow & \\ & & f_d & & \\ & & & \searrow & \\ & & & & f_{dd} \end{array}$$

Applying the reasoning of Section 1 to each branch of the tree gives

$$f_u = \frac{1}{1+r}\Big(qf_{uu} + (1-q)f_{ud}\Big),$$

$$f_d = \frac{1}{1+r}\Big(qf_{ud} + (1-q)f_{dd}\Big),$$

$$x = \frac{1}{1+r}\Big(qf_u + (1-q)f_d\Big)$$

$$= \frac{1}{(1+r)^2}\Big(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}\Big).$$

This means that the price equals $\frac{1}{(1+r)^2}E^Q[f]$, where $f = \max\{S_2 - K, 0\}$ is the option payoff and Q is the probability measure with probabilities q^2, $2q(1-q)$, $(1-q)^2$ corresponding to the different states $u^2 S_0$, udS_0, $d^2 S_0$, respectively, of the stock at time 2.

- One can also show that

$$\frac{1}{(1+r)^2}\Big(q^2 u^2 S_0 + 2q(1-q)udS_0 + (1-q)^2 d^2 S_0\Big)$$

equals S_0 so that Q is a risk-neutral measure.

3. Multiperiod binomial model

- We can further extend the model to n periods so that we have

	time n	probability
stock:		
	$u^n S_0$	p^n
	$u^{n-1} d S_0$	$np^{n-1}(1-p)$
	$\vdots$	$\vdots$
	$u^{n-j} d^j S_0$	$\binom{n}{j} p^{n-j}(1-p)^j$
	$\vdots$	$\vdots$
	$u d^{n-1} S_0$	$np(1-p)^{n-1}$
	$d^n S_0$	$(1-p)^n$

bank account:
$$(1+r)^n$$

- A European call with maturity n and strike K has the payoff $\max\{S_n - K, 0\}$, which means

$$
\begin{cases}
f_{u^n} = \max\{u^n S_0 - K, 0\} & \text{if } S_n = u^n S_0 \\
\quad\vdots & \quad\vdots \\
f_{u^{n-j} d^j} = \max\{u^{n-j} d^j S_0 - K, 0\} & \text{if } S_n = u^{n-j} d^j S_0 \\
\quad\vdots & \quad\vdots \\
f_{d^n} = \max\{d^n S_0 - K, 0\} & \text{if } S_n = d^n S_0
\end{cases}
$$

- Extending the pattern of the two-period, the fair price of the option is given by

$$\frac{1}{(1+r)^n}\Big(q^n f_{u^n} + \cdots + \binom{n}{j} q^{n-j}(1-q)^j f_{u^{n-j}d^j}$$
$$+ \cdots + (1-q)^n f_{d^n}\Big),$$

  where

$$q = \frac{1+r-d}{u-d}.$$

- Associating to q the corresponding measure Q, we can write the option price as

$$\frac{1}{(1+r)^n}E^Q[f] = \frac{1}{(1+r)^n}E^Q[\max\{S_n - K, 0\}],$$

  where we emphasize that it is the expectation under Q and not under the historical probability.

- Remark: Under the historical probability, S_n is related to a binomial distribution with parameters p and n. Under the probability measure Q, S_n is still related to a binomial distribution but with parameters $q = \frac{1+r-d}{u-d}$ and n. So for the

option pricing, we just change the parameters of the distribution of S_n and take then expectations of discounted values.

Pricing a call option by writing a MATLAB function `callnperiod.m`:

```matlab
function price = callnperiod(u,d,r,S0,K,n)
% calculate the price of a call option with ...
  strike K in an n period binomial model
if d<1+r && 1+r<u
    price=0;
    q = (1+r-d)/(u-d);
  for j=0:n
      price = price + ...
        nchoosek(n,j)*q^(n-j)*(1-q)^j*...
          max(u^(n-j)*d^j*S0-K,0)/(1+r)^n;
  end
else
error('wrong parameters')
end
```

```matlab
% plot the call price in dependence of the ...
  strike price
K = 0:0.05:10;
price = callnperiod(1.2,.95,.05,1,K,20);
plot(K,price,'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
```

```
ylabel('option price','fontsize',14);
axis([0 10 0 1]) % choosing suitable range ...
   for axes
```
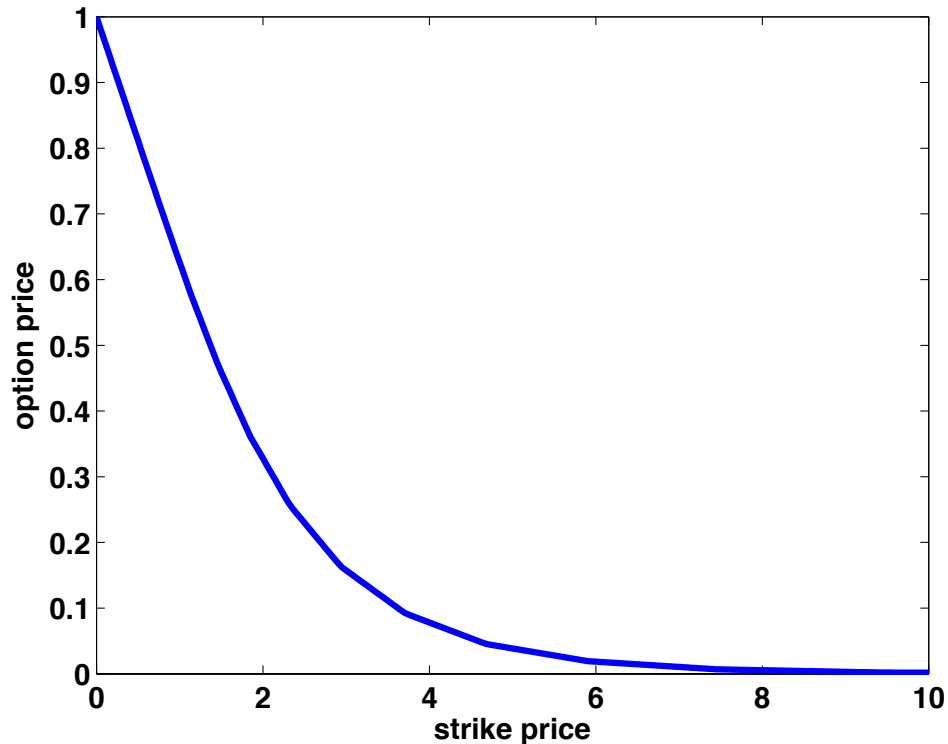

4. Transition to continuous time

- The binomial model can be used as approximation for a model with continuous trading possibilities on some time interval $[0, T]$.

To show convergence, one lets tend the number n of periods to infinity and, simultaneously, the length of each period tend to zero. This means that one makes specific choices for u, d and r depending on n; for details, please see the Appendix.

- The resulting continuous-time model has a bank account whose value at time T equals $\exp(\rho T)$ and a stock whose price at time T is given by

$$S_T = S_0 \exp\left((\mu - \sigma^2/2)T + \sigma\sqrt{T}N\right),$$

where ρ, μ and $\sigma > 0$ are constants and N is a standard normally distributed random variable.

```
function convergenceS(mu,sigma,T,n)
% compares the cumulative distribution ...
  function of S_n in a binomial model with ...
  that of the corresponding log-normal ...
  distribution

u = exp((mu-sigma^2/2)*T/n+sigma*(T/n)^.5);
% appropriate choice of u
d = exp((mu-sigma^2/2)*T/n-sigma*(T/n)^.5);
% appropriate choice of d
j=0:n;
```

```matlab
Sn = d.^(n—j).*u.^j; % S_n for p = 1/2
bin=binocdf(0:n,n,1/2); % cumulative ...
   distribution function of S_n
points = 0:.05:Sn(n+1); % choose equidistant ...
   points for plot
lognorm=logncdf(points,(mu—sigma^2/2)*T,...
    sigma*T^.5); % cumulative distribution ...
      function of S
plot(points,lognorm,Sn,bin,'r.','LineWidth',...
    3,'MarkerSize',18) % r = red, . = point
set(gca,'fontsize',14,'FontWeight','bold');
title('cumulative distribution functions');
legend('continuous time','discrete ...
   time','location','best');
```


cumulative distribution functions
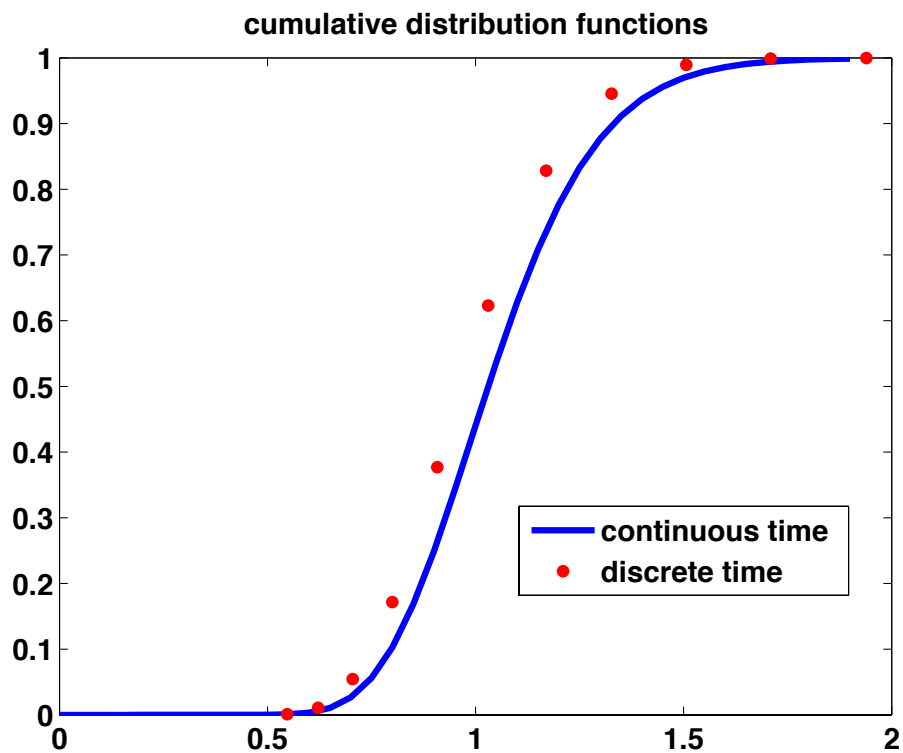
- Similarly to the binomial model, the price of a European call option with strike K and maturity T is given by

$$\frac{1}{\exp(\rho T)} E^Q[\max\{S_T - K, 0\}]$$

for some probability measure Q. This probability measure is such that

$$S_T = S_0 \exp\left((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\right)$$

for a random variable $\tilde{N}$ that is normally distributed under Q. Using this fact, we can rewrite the price of the European call as

$$c = S_0 \Phi(d_1) - K \exp(-\rho T)\Phi\left(d_1 - \sigma\sqrt{T}\right), \quad (\star)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\,du$$

is the standard-normal distribution function and $d_1 = \frac{\log\frac{S_0}{K} + \rho T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$.

Comments:

- $(\star)$ is the famous Black-Scholes formula. Note c depends on S_0, K, ρ, σ and T, but not on μ.

- In continuous time, the underlying process of the stock price dynamics is related to a Brownian motion.

- The partial derivatives of the Black-Scholes formula $(\star)$ with respect to its parameters are called Greeks.

  1. Delta $= \frac{\partial c}{\partial S_0} = \Phi(d_1) \in (0, 1)$ is the amount of the risky asset held in the replicating portfolio.

  2. Gamma $= \frac{\partial^2 c}{\partial S_0^2} = \Phi'(d_1) \frac{1}{S_0 \sigma \sqrt{T}} > 0$;
     if Gamma is big, frequent adjustments of the replicating portfolio are necessary.

  3. Theta $= -\frac{\partial c}{\partial T}$
     $= -\frac{S_0 \sigma \Phi'(d_1)}{2\sqrt{T}} - K\rho \exp(-\rho T)\Phi\left(d_1 - \sigma\sqrt{T}\right)$
     < 0.

  4. Rho $= \frac{\partial c}{\partial \rho} = KT \exp(-\rho T)\Phi\left(d_1 - \sigma\sqrt{T}\right) > 0$.

  5. Vega $= \frac{\partial c}{\partial \sigma} = S_0 \sqrt{T}\Phi'(d_1) > 0$.

- The principle of valuation under Q holds generally. The price of a derivative with payoff $f(S_T)$ is

$$\exp(-\rho T)\, E^Q[f(S_T)]$$

$$= \mathrm{e}^{-\rho T} E^Q\left[f\left(S_0 \exp\left((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\right)\right)\right]$$

for a normally distributed $\tilde{N}$ under Q.

Comparison of Black-Scholes with Binomial model:

```
function [priceBin,priceBS] = ...
  compareCall(rho,mu,sigma,T,K,n)
% compares the call option price in a ...
  binomial model with the continuous—time ...
  analogue from the Black—Scholes formula
u = exp((mu—sigma^2/2)*T/n+sigma*(T/n)^.5); ...
  % appropriate choice of u
d = exp((mu—sigma^2/2)*T/n—sigma*(T/n)^.5); ...
  % appropriate choice of d
r = rho*T/n; % appropriate choice of r
priceBin = callnperiod(u,d,r,1,K,n);
d1 = (log(1./K) + rho*T)/sigma/T^.5 + ...
  sigma*T^.5/2;
priceBS = normcdf(d1) — ...
  K.*exp(—rho*T).*normcdf(d1—sigma*T^.5);

% or, alternatively, by applying the ...
  Financial Toolbox, we could use
% priceBS = blsprice(1,K,rho,T,sigma);
```

Plot the comparison of the Call option prices using the script `compareCallPlot.m`:

```matlab
% plot comparison of Call option prices in …
  binomial model and Black—Scholes model
K=0:.05:1.5;
[a,b] = compareCall(.05,.5,.2,1,K,10);
plot(K,a,K,b,'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
ylabel('option price','fontsize',14);
legend('Binomial model','Black—Scholes')
```
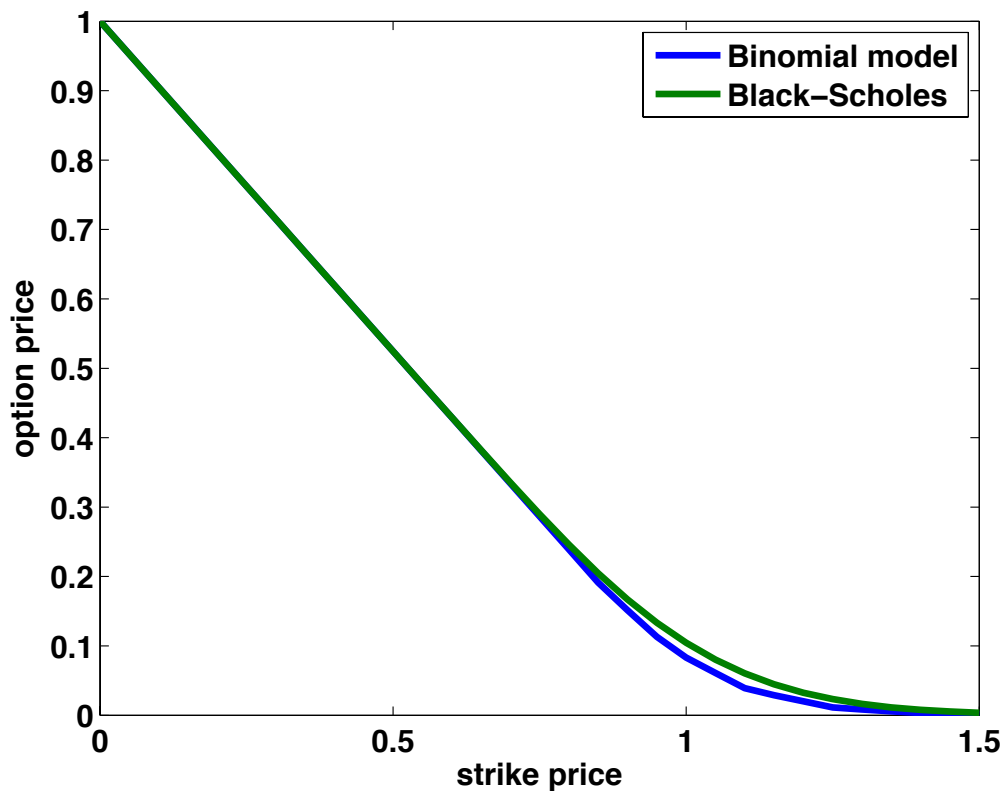
5. Implied volatility

- The value of σ is hard to determine
  $\rightarrow$ idea: find σ by inverting the Black-Scholes formula and using the market price of the option.

- The implied volatility σ_{impl} is defined as the unique σ such $c_{\text{BS}}(\sigma) = c_{\text{market}}$, where c_{market} is the market price of the option and c_{BS} is the value of the Black-Scholes formula $(\star)$ depending on σ.

- If the Black-Scholes model is correct, σ_{impl} does not depend on K, S_0, T and ρ. But in reality, one sees a strong dependence on K (volatility smile/skew).

```
% The financial toolbox has the function  ...
  blsimpv to calculate implied volatility: ...
  blsimpv(Current price of Stock S_0,  ...
  Strike K, Interest rate rho, Time to ...
  maturity T, Option price)

>> blsimpv(100, 95, 0, 0.25, 10)

ans =
    0.3722
```

We now calculate the implied volatility on EUR/USD Call options, writing a script `volaEURUSD.m`. The resulting plot shows a volatility smile.
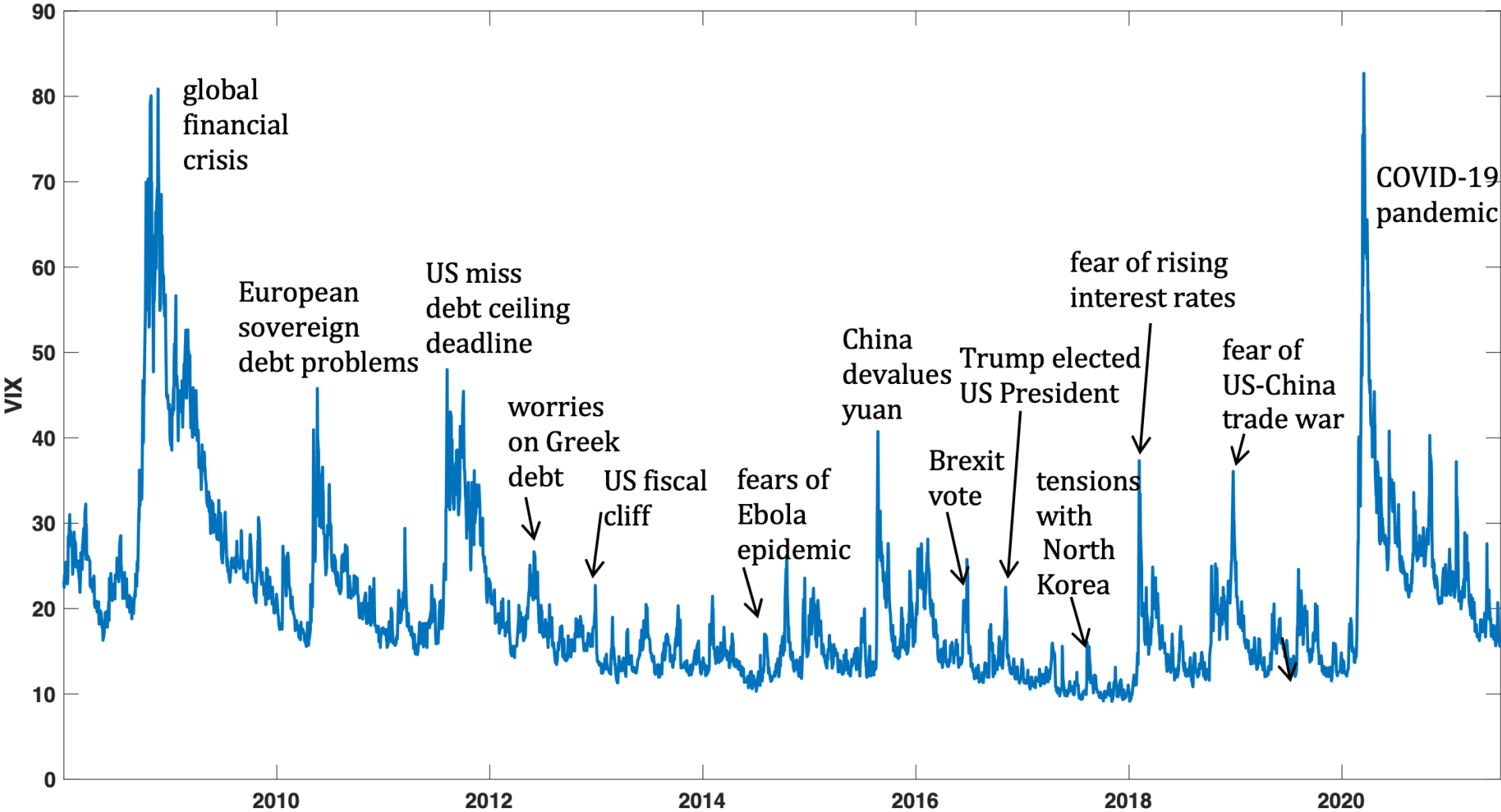
```matlab
% volaEURUSD.m needs financial toolbox
%
% Implied volatility of currency options.  ...
  We have the following data: 1 EUR = 1.1902 ...
  USD (June 30, 2021); the matrix A gives ...
  the prices (in USD) of call options with ...
  maturity end of September [such data can ...
  be found at http://www.cmegroup.com/]
A = [1.17 0.0258; 1.175 0.0218; 1.18 0.0184; ...
  1.185 0.0152; 1.19 0.0125; 1.195 0.0103; ...
  1.20 0.0086; 1.205 0.007; 1.21 0.0057; ...
  1.215 0.0046; 1.22 0.0037; 1.225 0.0029; ...
  1.23 0.0023];

% calculate the implied volatilities:
A(:,3) = blsimpv(1.1902,A(:,1),0,3/12,A(:,2));
% from June 30 until end of September = 3 ...
  months
% A(:,2) means all numbers of the 2nd column
plot(A(:,1),A(:,3),'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
ylabel('implied volatility','fontsize',14);
xlim([A(1,1),A(end,1)])
title('volatility smile')
```
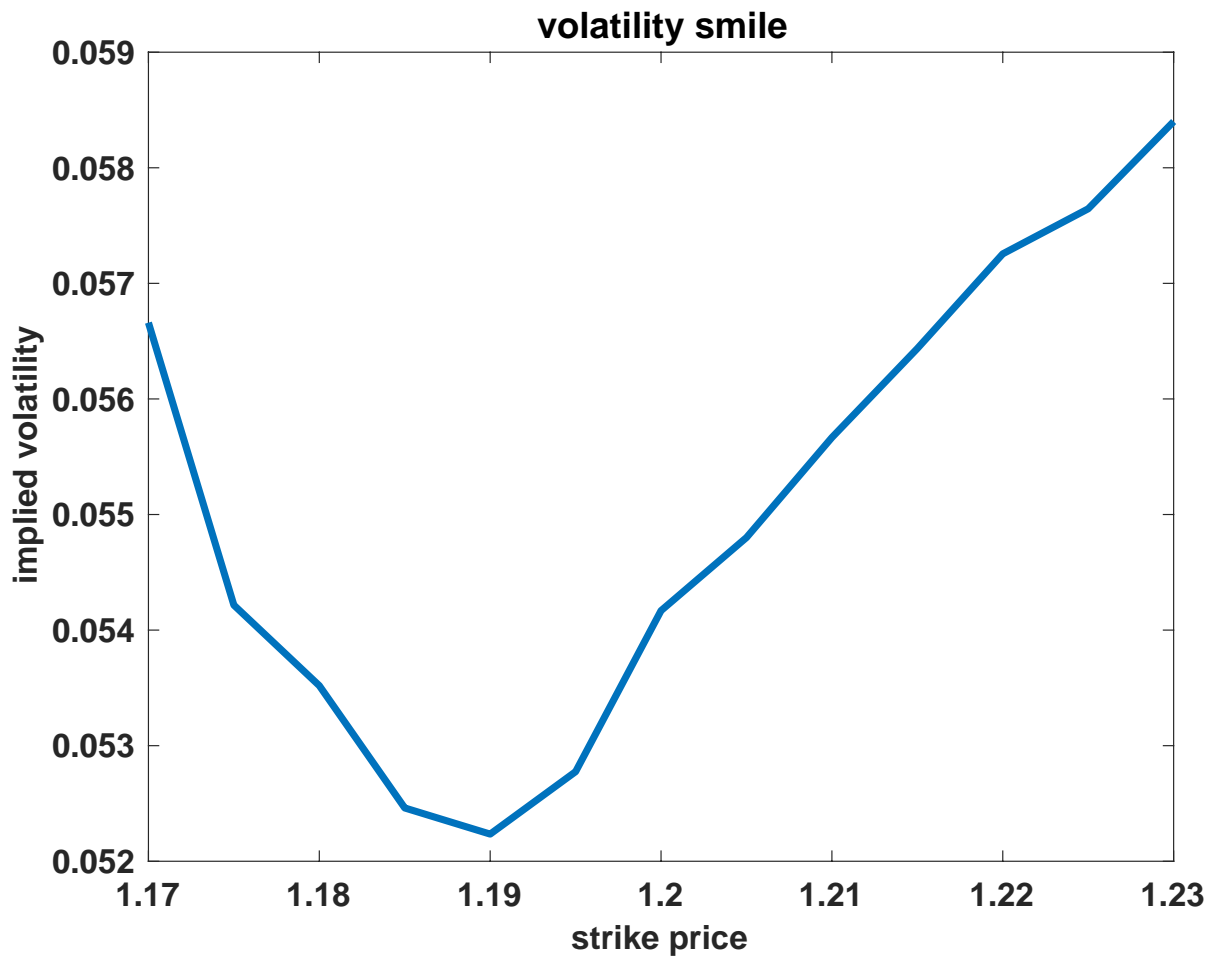
There exist indices which measure the implied volatility. A popular measure is VIX, which reflects the implied volatility of options on the stock index S&P 500. VIX is often referred to as "fear index", because a high level of VIX means a lot of uncertainty in the market; see the development of VIX on the next page.

VIX over the last twelve years

6. Portfolio optimization

This section uses a different model than the previous sections. The goal is to set up mean-variance portfolio optimization problems and solve them numerically using MATLAB.

A model for a financial market:

- **Single period:** the model has only one period

- **Risky assets:** we now consider n risky assets with returns $K_1, \ldots, K_n$ (random variables)

- **No risk-free asset**: all assets are risky and there is no risk-free asset

- The **portfolio weights** $\mathbf{w} = (w_1, \ldots, w_n)'$ give the percentage of initial wealth invested in each of the asset:

  $w_j = \%$ of initial wealth invested in asset j.

  Hence, we have $\sum_{j=1}^{n} w_j = 1$

- **No short selling**: We assume that short selling is not allowed, hence we require $w_j \geq 0$ for all j.

Return and risk:

- The portfolio return is

$$R = \frac{\text{terminal value} - \text{initial value}}{\text{initial value}},$$

which equals

$$R = \frac{\sum_{j=1}^{n} w_j V_0 S_1^j / S_0^j - V_0}{V_0} = \sum_{j=1}^{n} w_j S_1^j / S_0^j - 1$$

$$= \sum_{j=1}^{n} w_j \left(S_1^j / S_0^j - 1 \right) = \sum_{j=1}^{n} w_j K_j,$$

where

$$S_0^j = \text{price of asset } j \text{ at time } 0,$$
$$S_1^j = \text{price of asset } j \text{ at time } 1,$$
$$V_0 = \text{total initial wealth},$$
$$w_j V_0 = \text{initial wealth invested in asset } j,$$
$$w_j V_0 S_1^j / S_0^j = \text{terminal wealth invested in asset } j.$$

- As investor, we want **maximal expected return**

$$E[R] = E\left[\sum_{j=1}^{n} w_j K_j \right] = \sum_{j=1}^{n} w_j E[K_j] = \mathbf{w}' \mathbf{m}$$

where $\mathbf{m} = (E[K_1], \ldots, E[K_n])'$ is the vector of expected asset returns while the **risk should be minimal**. An often used measure for the risk is the standard deviation of the returns given by

$$\sigma = \sqrt{\mathsf{Var}(R)}.$$

Using the properties of (co-)variance, we have

$$\mathsf{Var}(R) = \mathsf{Var}\left(\sum_{j=1}^{n} w_j K_j \right)$$

$$= \sum_{i,j=1}^{n} w_i \mathsf{Cov}(K_i, K_j) w_j$$

$$= \mathbf{w}'\Sigma\mathbf{w},$$

where Σ is the covariance matrix which is defined by $\Sigma_{ij} = \mathsf{Cov}(K_i, K_j)$.

- Maximizing expected return and minimizing risk are conflicting objectives. It is reasonable to fix a target value r_T of expected return and minimize the risk. The optimization problem is

$$\min \mathbf{w}'\Sigma\mathbf{w}$$

$$\text{such that } \mathbf{w}'\mathbf{m} = r_T, \quad \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0.$$

- If for a given target return r_T the optimization problem has a solution, we get portfolio weights $\mathbf{w}$ with minimal risk $\sigma_{\min} = \sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}$, which is a function of r_T: $\sigma_{\min} = \sigma_{\min}(r_T)$. For different r_T, we get a curve $(\sigma_{\min}(r_T), r_T)$. The curve $(\sigma_{\min}(r_T), r_T)$ where there is no $\tilde{r}_T > r_T$ with $\sigma_{\min}(\tilde{r}_T) = \sigma_{\min}(r_T)$ is the **efficient frontier**.

We calculate the efficient frontier in a MATLAB script `efficientDJ.m`, which uses the Financial Toolbox:

```
% an example for the caclulation of the ...
  efficient frontier
load BlueChipStockMoments % consider monthly ...
  sample data of DJ stocks
AssetMeanAn = 12*AssetMean; % get the ...
  annualized mean returns
AssetCovarAn = 12*AssetCovar;
AssetRiskAn = sqrt(diag(AssetCovarAn)); % ...
  get the annualized risk
scatter(AssetRiskAn,AssetMeanAn,...
    'MarkerFaceColor','b') % plot single stocks
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('Standard deviation of returns ...
  (annualized)','fontsize',14);
ylabel('Mean of returns ...
  (annualized)','fontsize',14);
```

```matlab
title('Asset Risks and Returns');
dx = 0.005; % displacement so the text does …
  not overlay the data points
text(AssetRiskAn + dx, …
  AssetMeanAn,AssetList); % label data points
hold on
p = Portfolio; % set up an empty object for …
  the portfolio information
p = setAssetMoments(p,AssetMeanAn, …
  AssetCovarAn); % insert annualized …
  expected return and risk in object
p = setDefaultConstraints(p); % set the …
  constraints to default
plotFrontier(p); % plot the efficient frontier
hold off
```


Efficient Frontier
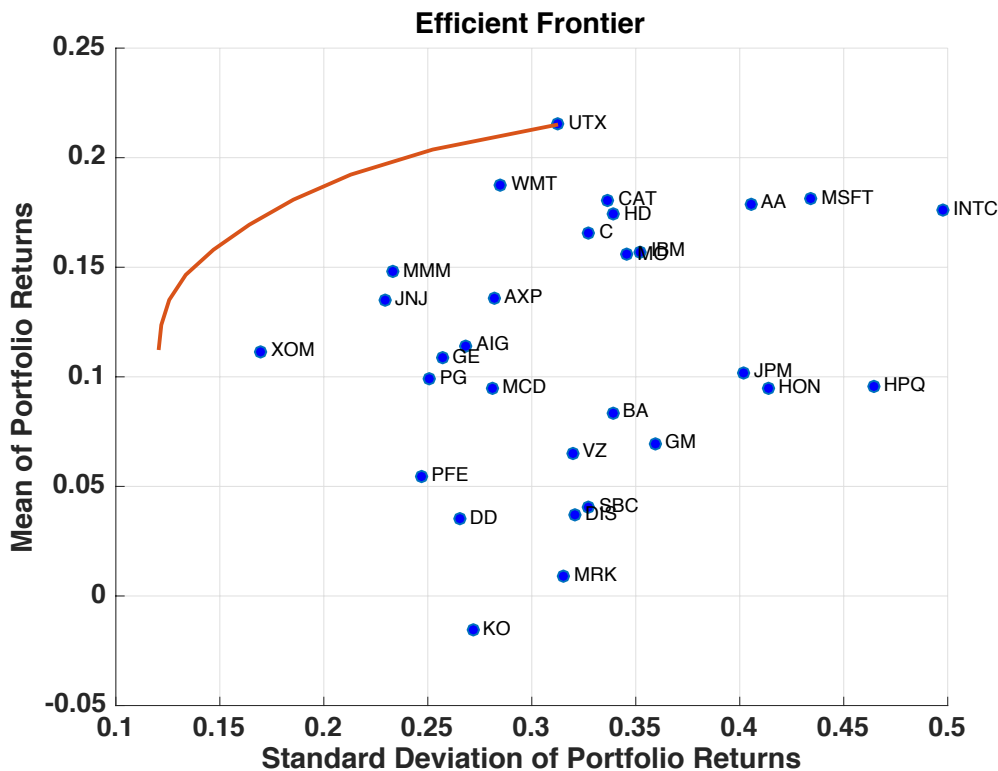
Appendix: additional explanations and proofs to Section 4

A.1 Choice in the continuous-time model

In the continuous-time situation, we model the terminal value of the bank account as $B_T = \exp(\rho T)$ and the terminal value of the stock as

$$S_T = S_0 \exp\left((\mu - \sigma^2/2)T + \sigma\sqrt{T}N\right), \qquad (1)$$

where ρ, μ and $\sigma > 0$ are constants and N is a standard normally distributed random variable.

A.2 Explanations behind choice

The reason behind these choices is as follows. In continuous time, the bank account models continuous interest, which means

$$\mathrm{d}B_t = \rho B_t \, \mathrm{d}t.$$

We can interpret this as that the infinitesimal change $\mathrm{d}B_t$ in the bank account is equal to the continuous interest rate ρ times the capital B_t. This equation is

equivalent to $\frac{\mathrm{d}B_t}{\mathrm{d}t} = \rho B_t$, which yields $B_T = \exp(\rho T)$ using that $B_0 = 1$.

To explain the form (1) of the stock price, we can say that on average (which means in expectation) the stock should have a similar growth form than the bank account. Hence, $E[S_T] = S_0 \exp(\mu T)$ for some constant μ (typically μ will be bigger than ρ to compensate for the risk in the stock), using that S starts at S_0 and not necessarily at 1, in contrast to the bank account. Now, S_T will not just be equal to the deterministic value $S_0 \exp(\mu T)$, but will also reflect some random factor because we do not know future prices. Hence, S_T is of the form

$$S_T = S_0 \exp(\mu T) \times (\text{positive random factor}). \quad (2)$$

The reason for this positive random factor is related to the so-called Brownian motion. At the moment, you should just accept that we can model it with a normally distributed random variable, but because it should be positive, we take the exponential of this normally distributed random variable so that

$$\text{positive random factor} = \exp(cN) \quad (3)$$

where c is some constant and N is a standard normally distributed random variable. The bigger T, the longer the time horizon is and more uncertain S_T is. Therefore, c should depend on T, and we will again see later that the right form is $c = \sigma\sqrt{T}$, hence it grows like square root in T times some constant σ, which gives us how big the fluctuation in S_T is. Combining this with (2) and (3), we get

$$S_T = S_0 \exp\left(\mu T + \sigma\sqrt{T}N\right) \qquad (4)$$

for some normally distributed N. Recall we wanted to have $E[S_T] = S_0 \exp(\mu T)$ so that μ has the interpretation of the mean growth rate, but we can calculate

$$E\left[S_0 \exp\left(\mu T + \sigma\sqrt{T}N\right)\right]$$
$$= S_0 \exp(\mu T)E\left[\exp\left(\sigma\sqrt{T}N\right)\right]$$
$$= S_0 \exp\left(\mu T + \sigma^2 T/2\right),$$

using the formula that $E[\exp(\alpha N)] = \exp(\alpha^2/2)$ for any constant α and standard normally distributed N. Therefore, to get $E[S_T] = S_0 \exp(\mu T)$, we need to divide (3) by $\exp(\sigma^2 T/2)$, which leads to (1).

A.3 Convergence proofs

We show now that under suitable choices of r_n, d_n and u_n, the terminal values of the bank account and stock in the binomial model converge to $B_T = \exp(\rho T)$ and S_T given in (1).

Proposition 1 *For $r_n = \rho T/n$, we have*

$$\lim_{n \to \infty} (1 + r_n)^n = \exp(\rho T).$$

Proof.

$$
\begin{aligned}
\lim_{n \to \infty} (1 + \rho T/n)^n &= \exp\left(\ln\left(\lim_{n \to \infty} (1 + \rho T/n)^n \right)\right) \\
&= \exp\left(\lim_{n \to \infty} \ln(1 + \rho T/n)^n \right) \\
&= \exp\left(\lim_{n \to \infty} n \ln(1 + \rho T/n) \right) \\
&= \exp\left(\lim_{n \to \infty} \frac{\ln(1 + \rho T/n)}{1/n} \right),
\end{aligned}
$$

which equals

$$
\begin{aligned}
\exp\left(\lim_{n \to \infty} \frac{\ln(1 + \rho T/n)}{1/n} \right) &\overset{(*)}{=} \exp\left(\lim_{s \searrow 0} \frac{\ln(1 + \rho T s)}{s} \right) \\
&\overset{(**)}{=} \exp\left(\lim_{s \searrow 0} \frac{\rho T}{1 + \rho T s} \right) \\
&= \exp(\rho T)
\end{aligned}
$$

$(*)$ set $s = 1/n$, then $n \to \infty \iff s \searrow 0$

$(**)$ L'Hôpital's rule using $\frac{d}{ds} \ln(1 + \rho Ts) = \frac{\rho T}{1 + \rho Ts}$ ∎

Proposition 2 *Set $p = 1/2$ and define*

$$u_n = \exp\left((\mu - \sigma^2/2)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}} \right),$$

$$d_n = \exp\left((\mu - \sigma^2/2)\frac{T}{n} - \sigma\sqrt{\frac{T}{n}} \right)$$

then S_n in the n-period binomial model converges to S_T in (1).

Proof. If S_n reflects j times u_n and $n - j$ times d_n, we have

$$S_n = S_0 u_n^j d_n^{n-j}$$

$$= S_0 \exp\left((\mu - \sigma^2/2)\frac{T}{n}j + \sigma\sqrt{\frac{T}{n}j} \right)$$

$$\times \exp\left((\mu - \sigma^2/2)\frac{T}{n}(n-j) - \sigma\sqrt{\frac{T}{n}}(n-j) \right)$$

$$= S_0 \exp\left((\mu - \sigma^2/2)T + \sigma\sqrt{T}\frac{2j - n}{\sqrt{n}} \right).$$

Comparing this with (1), it remains to show that $\frac{2j-n}{\sqrt{n}}$ converges to a standard normally distributed random variable. Define random variables X_i by

$$X_i = \begin{cases} 1, & \text{if we have } u_n \text{ in period } i \\ -1, & \text{if we have } d_n \text{ in period } i \end{cases} \tag{5}$$

and note that if we have j times u_n and $n-j$ times d_n, then

$$\sum_{i=1}^{n} X_i = j + (n-j)(-1) = 2j - n.$$

Therefore, we can write

$$\frac{2j-n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i. \tag{6}$$

We now apply the Central Limit Theorem, which says that for independent and identically distributed random variables $X_1, X_2, \ldots$ with mean $\mu = E[X_i]$ and finite variance $\sigma^2 = \text{Var}(X_i)$,

$$\frac{\sqrt{n}}{\sigma} \left(\frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right) \tag{7}$$

converges (in distribution) to a standard normally distributed random variable. In our case of X_i given by

(5) with equal probability $1/2$ for the two cases (because $p = 1/2$ by assumption), we have

$$\mu = E[X_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0,$$

$$\sigma^2 = \text{Var}(X_i) = E\left[X_i^2\right] = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot (-1)^2 = 1.$$

Therefore, (7) simplifies in our case to $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. Because of (6), this shows that $\frac{2j-n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ converges (in distribution) to a standard normally distributed random variable. ∎

A.4 Derivation of the Black-Scholes formula

Similarly to the binomial model, the price for a payoff f in the Black-Scholes model is given by $\frac{1}{e^{\rho T}} E^Q[f]$ where the terminal value of the stock price is

$$S_T = S_0 \exp\left((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\right)$$

with $\tilde{N}$ standard normally distributed under Q. In the case of a call option with strike price K, the price

equals

$$c = \frac{1}{e^{\rho T}} E^Q[\max\{S_T - K, 0\}]$$

$$= \frac{1}{e^{\rho T}} E^Q\left[\max\left\{S_0 e^{(\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}} - K, 0\right\}\right]$$

$$= \frac{1}{e^{\rho T}} \int_{-\infty}^{\infty} \max\left\{S_0 e^{(\rho - \sigma^2/2)T + \sigma\sqrt{T}x} - K, 0\right\} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \max\left\{S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - K e^{-\rho T}, 0\right\} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, \mathrm{d}x.$$

Now, we use the equivalences

$$S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - K e^{-\rho T} \geq 0$$

$$\iff S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \geq K e^{-\rho T}$$

$$\iff e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \geq \frac{K}{S_0} e^{-\rho T}$$

$$\iff -\sigma^2 T/2 + \sigma\sqrt{T}x \geq \log\left(\frac{K}{S_0}\right) - \rho T$$

$$\iff x \geq \frac{\log(K/S_0) - \rho T}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2.$$

Therefore, defining $d = \frac{\log(K/S_0) - \rho T}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2$ allows us to write

$$c = \int_{-\infty}^{\infty} \max\left\{S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - Ke^{-\rho T}, 0\right\} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x$$

$$= \int_{d}^{\infty} \left(S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - Ke^{-\rho T}\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x$$

$$= \int_{d}^{\infty} S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x$$

$$- \int_{d}^{\infty} Ke^{-\rho T} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x.$$

For the first term, we calculate

$$\int_{d}^{\infty} S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x$$

$$= S_0 e^{-\sigma^2 T/2} \int_{d}^{\infty} \frac{e^{\sigma\sqrt{T}x - x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x$$

$$= S_0 e^{-\sigma^2 T/2} \int_{d}^{\infty} \frac{e^{-(x-\sigma\sqrt{T})^2/2} e^{\sigma^2 T/2}}{\sqrt{2\pi}}\, \mathrm{d}x$$

$$= S_0 \int_{d-\sigma\sqrt{T}}^{\infty} \frac{e^{-y^2}}{\sqrt{2\pi}}\, \mathrm{d}y$$

$$= S_0 \left(1 - \Phi\left(d - \sigma\sqrt{T}\right)\right)$$

$$= S_0 \Phi\left(-d + \sigma\sqrt{T}\right).$$

For the second term, we have

$$\int_d^\infty K e^{-\rho T} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = K e^{-\rho T} \int_d^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$
$$= K e^{-\rho T} \Big(1 - \Phi(d) \Big)$$
$$= K e^{-\rho T} \Phi(-d).$$

Defining

$$d_1 = -d + \sigma\sqrt{T} = \frac{\log(K/S_0) + \rho T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

we obtain

$$c = \int_d^\infty S_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \, \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$
$$- \int_d^\infty K e^{-\rho T} \, \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$
$$= S_0 \Phi(d_1) - K e^{-\rho T} \Phi\Big(d_1 - \sigma\sqrt{T} \Big),$$

which is the Black-Scholes formula.