Wavelet Application to Numerical PDEs

Bin Han

Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, Canada

Present at 2021 International Undergraduate

Summer Enrichment Program at UofA

July 12–August 13, 2021

Outline of Mini-course Talks

- Wavelet theory in the function setting.
- Multiresolution analysis.
- Riesz and biorthogonal wavelets in $L_2(\mathbb{R})$.
- Basics on Sobolev spaces.
- Basics on boundary value problems in PDEs.
- Wavelet applications to numerical PDEs.

Declaration: Some figures and graphs in this talk are from the book [Bin Han, Framelets and Wavelets: Algorithms, Analysis and Applications, Birkhäuser/Springer, 2017] and various other sources from Internet, or from published papers, or produced by `matlab`, `maple`, or `C` programming. [Details and sources of all graphs can be provided upon request of the audience.]

What Is a Wavelet in the Function Setting?

- Let $\phi = (\phi_1, \dots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \dots, \psi_s)^\mathsf{T}$ in $L_2(\mathbb{R})$.
- A system is derived from ϕ, ψ via dilates and integer shifts:

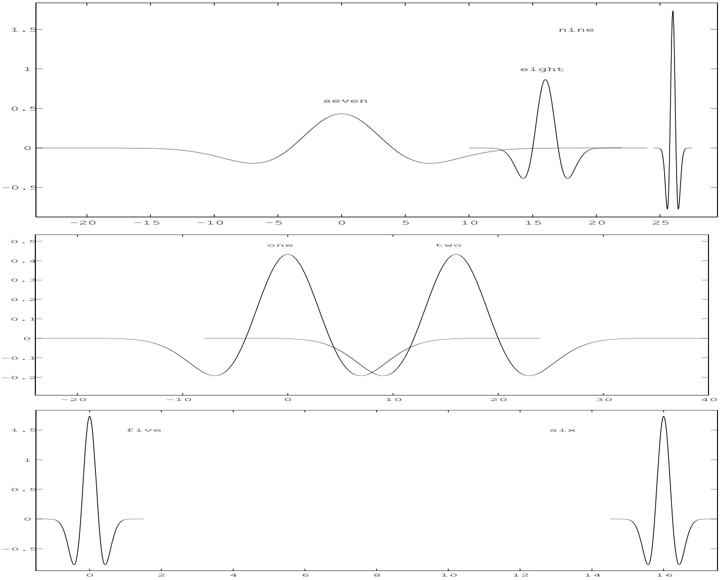
$$\begin{aligned} \mathsf{AS}(\phi;\psi) :=& \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \\ & \{\psi_{j;k} := 2^{j/2}\psi(2^j \cdot - k) \ : \ j \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}\}. \end{aligned}$$

- $\{\phi;\psi\}$ is called an orthogonal wavelet in $L_2(\mathbb{R})$ if $\mathsf{AS}(\phi;\psi)$ is an orthonormal basis of $L_2(\mathbb{R})$.
- Wavelet representation:

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j;k}\rangle \psi_{j;k}, \quad f \in L_2(\mathbb{R}),$$

where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}^\mathsf{T} dx$ is the inner product.
- For $r = 1$, $\{\phi;\psi\}$ is called a scalar wavelet. For $r > 1$ (i.e., ϕ is a vector function), $\{\phi;\psi\}$ is called a multiwavelet.

Dilates and Shifts of Affine Systems



Multiresolution Analysis

Definition

A sequence $\{\mathscr{V}_j\}_{j\in\mathbb{Z}}$ of closed subspaces in $L_2(\mathbb{R})$ forms a (wavelet) multiresolution analysis (MRA) of $L_2(\mathbb{R})$ if

1. $\mathscr{V}_j = \{f(2^j\cdot) \ : \ f \in \mathscr{V}_0\}$ and $\mathscr{V}_j \subseteq \mathscr{V}_{j+1}$ for all integers $j \in \mathbb{Z}$;
2. $\overline{\cup_{j\in\mathbb{Z}}\mathscr{V}_j} = L_2(\mathbb{R})$ (that is, $\cup_{j\in\mathbb{Z}}\mathscr{V}_j$ is dense in $L_2(\mathbb{R})$) and $\cap_{j\in\mathbb{Z}}\mathscr{V}_j = \{0\}$;
3. there exists a set Φ of functions in $L_2(\mathbb{R})$ such that $\{\phi(\cdot - k) \ : \ k \in \mathbb{Z}, \phi \in \Phi\}$ is a Riesz basis for $\mathscr{V}_0$.

Note that the set Φ of functions in item (3) completely determines a multiresolution analysis by

$$\mathscr{V}_j = \mathsf{S}_{2^j}(\Phi|L_2(\mathbb{R})) := \overline{\operatorname{span}\{\phi(2^j \cdot -k) \ : \ k \in \mathbb{Z}, \phi \in \Phi\}}^{L_2(\mathbb{R})}.$$

For scalar wavelets, Φ is a singleton and $\Phi = \phi$.

MRA of Orthogonal (Multi)Wavelets

- Let $\phi = (\phi^1, \dots, \phi^r)^\mathsf{T}$ and $\psi = (\psi^1, \dots, \psi^s)^\mathsf{T}$ in $L_2(\mathbb{R})$.
- $\{\phi; \psi\}$ is called an orthogonal wavelet if $\mathsf{AS}(\phi; \psi)$ is an orthonormal basis of $L_2(\mathbb{R})$, where as

$$\begin{aligned} \mathsf{AS}(\phi;\psi) &:= \{\phi^\ell(\cdot - k) \;:\; k \in \mathbb{Z}, \ell = 1, \dots, r\} \\ &\cup \{\psi^\ell_{2^j;k} := 2^{j/2}\psi^\ell(2^j \cdot - k) \;:\; j \geqslant 0, k \in \mathbb{Z}, \ell = 1, \dots, s\}. \end{aligned}$$

- Define $\mathscr{V}_j := \mathsf{S}_{2^j}(\{\phi^1, \dots, \phi^r\}|L_2(\mathbb{R}))$ and $\mathscr{W}_j := \mathsf{S}_{2^j}(\{\psi^1, \dots, \psi^s\}|L_2(\mathbb{R}))$ for $j \in \mathbb{Z}$.
- Then we have the space decomposition of $L_2(\mathbb{R})$:

$$\mathscr{V}_{J+1} = \mathscr{V}_J \oplus \mathscr{W}_J \quad \text{and} \quad L_2(\mathbb{R}) = \mathscr{V}_J \oplus_{j=J}^{\infty} \mathscr{W}_j = \oplus_{j \in \mathbb{Z}} \mathscr{W}_j, \; \forall\, J \in \mathbb{Z},$$

where $\oplus$ means the orthogonal sum of closed subspaces in $L_2(\mathbb{R})$.

Vanishing Moments and Sparsity

- A function ψ has m vanishing moments if $\widehat{\psi}(\xi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$, i.e., $\widehat{\psi}(0) = \widehat{\psi}'(0) = \cdots = \widehat{\psi}^{(m-1)}(0) = 0$.
- If ψ has decay, then the above is equivalent to $\int_{\mathbb{R}} \psi(x) x^j dx = 0$ for all $j = 0, \dots, m-1$.
- If $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ and $\widehat{\phi}(0) \neq 0$, then ψ has m vanishing moments if and only if the filter b has m vanishing moments: $\widehat{b}(\xi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$.
- Let $\{\phi; \psi\}$ be an orthogonal wavelet in $L_2(\mathbb{R})$. Then every function $f \in L_2(\mathbb{R})$ can be represented as

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi_{2^j;k}\rangle \psi_{2^j;k}.$$

- If $f \approx$ p on the support of $\psi_{2^j;k}$ and deg(p) $< m$, then ψ has m vanishing moments implies $\langle f, \psi_{2^j;k}\rangle \approx 0$.

Fast Wavelet Transform in Function Setting

Let $\{\phi; \psi\}$ be an orthogonal wavelet in $L_2(\mathbb{R})$ with an orthogonal wavelet filter bank $\{a; b\}$ such that

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi).$$

Then every function $f \in L_2(\mathbb{R})$ can be represented as

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{2^J;k}\rangle \phi_{2^J;k} + \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi_{2^j;k}\rangle \psi_{2^j;k}.$$

Choose j large so that

$$f \approx f_j := \sum_{k\in\mathbb{Z}} v^j(k)\phi_{2^j;k} \quad \text{with} \quad v^j(k) := \langle f, \phi_{2^j;k}\rangle.$$

On the other hand, we also have

$$f_j = f_{j-1} + \sum_{k\in\mathbb{Z}} w^{j-1}(k)\rangle\psi_{2^{j-1};k} \quad \text{with} \quad w^{j-1}(k) := \langle f, \psi_{2^{j-1};k}\rangle.$$

Fast Wavelet Transform in Function Setting...

Then the coefficients v^{j-1}, w^{j-1} can be computed from v^j as follows:

$$v^{j-1} = \frac{\sqrt{2}}{2}\mathcal{T}_a v^j, \quad w^{j-1} = \frac{\sqrt{2}}{2}\mathcal{T}_b v^j,$$

which is exactly the same discrete wavelet decomposition in the discrete setting.

Conversely, we can obtained v^j from v^{j-1} and w^{j-1} by

$$v^j = \frac{\sqrt{2}}{2}\mathcal{S}_a v^{j-1} + \frac{\sqrt{2}}{2}\mathcal{S}_b w^{j-1},$$

which is exactly the same discrete wavelet reconstruction.

Explanation

Note $\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$ is equivalent to

$$\psi = 2\sum_{n\in\mathbb{Z}} b(n)\phi(2\cdot -n).$$

Then

$$\psi_{2^{j-1};k} = 2^{(j-1)/2}\psi(2^{j-1}\cdot -k) = 2^{(j+1)/2}\sum_{n\in\mathbb{Z}} b(n)\phi(2^j\cdot -2k-n)$$

$$= \sqrt{2}\sum_{n\in\mathbb{Z}} b(n)\phi_{2^j;2k+n}.$$

Therefore,

$$w^{j-1}(k) = \langle f, \psi_{2^{j-1};k}\rangle = \sqrt{2}\sum_{n\in\mathbb{Z}} b(n)\langle f, \phi_{2^j;2k+n}\rangle$$

$$= \sqrt{2}\sum_{n\in\mathbb{Z}} \overline{b(n)} v^j(2k+n) = \frac{\sqrt{2}}{2}[\mathcal{T}_b v^j](k).$$

Explanation...

Recall

$$\psi_{2^{j-1};k} = \sqrt{2}\sum_{n\in\mathbb{Z}} b(n)\phi_{2^j;2k+n}.$$

Conversely,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} w^{j-1}(k)\psi_{2^{j-1};k} &= \sum_{k\in\mathbb{Z}}\sqrt{2}\sum_{n\in\mathbb{Z}} w^{j-1}(k)b(n)\phi_{2^j;2k+n}\\ &= \sqrt{2}\sum_{k\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} w^{j-1}(k)b(2k-n')\phi_{2^j;n'}\\ &= \sum_{n'\in\mathbb{Z}}\sqrt{2}\sum_{k\in\mathbb{Z}} b(2k'-n)w^{j-1}(k)\\ &= \sum_{n'\in\mathbb{Z}}\frac{\sqrt{2}}{2}[\mathcal{S}_b w^{j-1}](n')\end{aligned}$$

Explanation...

Hence,

$$\sum_{k\in\mathbb{Z}} v^j(k)\phi_{2^j;k} = \sum_{k\in\mathbb{Z}} v^{j-1}(k)\phi_{2^{j-1};k} + \sum_{k\in\mathbb{Z}} w^{j-1}(k)\psi_{2^{j-1};k}$$
$$= \sum_{n'\in\mathbb{Z}} \left(\frac{\sqrt{2}}{2}[\mathcal{S}_a v^{j-1}](n') + \frac{\sqrt{2}}{2}[\mathcal{S}_b w^{j-1}](n')\right)\phi_{2^j;n'}$$
$$= \sum_{k\in\mathbb{Z}} \left(\frac{\sqrt{2}}{2}[\mathcal{S}_a v^{j-1}](k) + \frac{\sqrt{2}}{2}[\mathcal{S}_b w^{j-1}](k)\right)\phi_{2^j;k}.$$

Since $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal system, we must have

$$v^j(k) = \frac{\sqrt{2}}{2}[\mathcal{S}_a v^{j-1}](k) + \frac{\sqrt{2}}{2}[\mathcal{S}_b w^{j-1}](k).$$

Construction of Orthogonal Scalar Wavelets

Theorem

Let $a, b \in l_0(\mathbb{Z})$ *with* $\widehat{a}(0) = 1$. *Define*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \qquad \widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2).$$

Then $\phi, \psi \in L_2(\mathbb{R})$ *and* $\{\phi; \psi\}$ *is an orthogonal wavelet in* $L_2(\mathbb{R})$ *if and only if*

- $[\widehat{\phi}, \widehat{\phi}] = 1$, *i.e.*, $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ *is an orthonormal system.*
- $\{a; b\}$ *is an orthogonal wavelet filter bank:*

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} = I_2.$$

Orthogonal Wavelet Filter Bank

Proposition

Let $a, b \in l_0(\mathbb{Z})$. Then $\{a; b\}$ is an orthogonal wavelet filter bank:

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}^\star = I_2$$

if and only if a is an orthogonal low-pass filter:

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 = 1$$

and there exist $c \in \mathbb{T}$ and $n \in \mathbb{Z}$ such that

$$\widehat{b}(\xi) = ce^{i(2n-1)\xi}\overline{\widehat{a}(\xi+\pi)}.$$

For $c = 1$ and $n = 0$, $\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}$.

Daubechies Orthogonal Wavelets

Define interpolatory filter $\widehat{a^I_{2m}}(\xi) := \cos^{2m}(\xi/2)\mathsf{P}_{m,m}(\sin^2(\xi/2))$ with $\mathsf{P}_{m,m}(x) := \sum_{j=0}^{m-1} \binom{m+j-1}{j} x^j$. Since $\widehat{a^I_{2m}}(\xi) \geqslant 0$, by Fejér-Riesz lemma, there exists $a^D_m \in l_0(\mathbb{Z})$ such that $\widehat{a^D_m}(0) = 1$.

$$|\widehat{a^D_m}(\xi)|^2 = \widehat{a^I_{2m}}(\xi) := \widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2)\mathsf{P}_{m,m}(\sin^2(\xi/2)).$$

Define ϕ through $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a^D_m}(2^{-j}\xi)$. Then $[\widehat{\phi}, \widehat{\phi}] = 1$ and $\{a^D_m; b^D_m\}$ is an orthogonal wavelet filter bank with

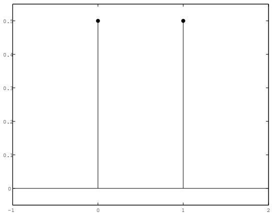
$$\widehat{b^D_m}(\xi) := e^{-i\xi}\overline{\widehat{a^D_m}(\xi + \pi)}, \quad \widehat{\psi}(\xi) := \widehat{b^D_m}(\xi/2)\widehat{\phi}(\xi/2).$$

Then $\{\phi; \psi\}$ is a compactly supported orthogonal wavelet such that the low-pass filter a^D_m has order m sum rules and the high-pass filter b^D_m has m vanishing moments, called the Daubechies orthogonal wavelet of order m.

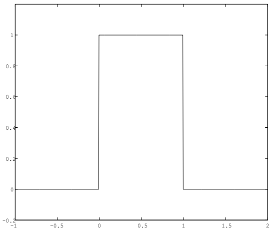
Daubechies Orthogonal Filters

$$
\begin{aligned}
a_1^D &= \{\underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{2}\}_{[0,1]},\\
a_2^D &= \{\tfrac{1+\sqrt{3}}{8}, \underline{\tfrac{\mathbf{3}+\sqrt{\mathbf{3}}}{\mathbf{8}}}, \tfrac{3-\sqrt{3}}{8}, \tfrac{1-\sqrt{3}}{8}\}_{[-1,2]}\\
a_3^D &= \{\tfrac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{32}, \tfrac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{32}, \underline{\tfrac{\mathbf{5}-\sqrt{\mathbf{10}}+\sqrt{\mathbf{5}+\mathbf{2}\sqrt{\mathbf{10}}}}{\mathbf{16}}},\\
&\quad \tfrac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{32}, \tfrac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{32}\}_{[-2,3]},\\
a_4^D &= \{-0.0535744507091, -0.0209554825625, 0.351869534328,\\
&\quad \underline{\mathbf{0.568329121704}}, 0.210617267102, -0.0701588120893,\\
&\quad -0.00891235072084, 0.0227851729480\}_{[-3,4]}.
\end{aligned}
$$

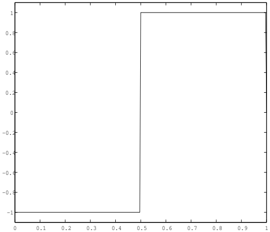
Daubechies Orthogonal Wavelets



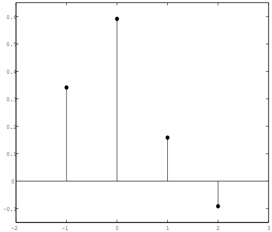
(a) Filter a_1^D



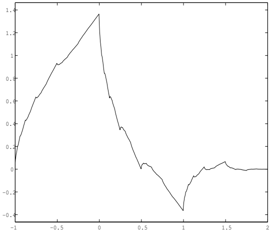
(b) $\phi^{a_1^D}$



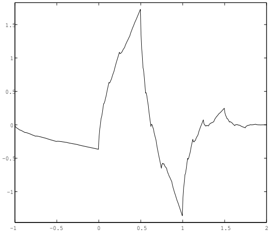
(c) $\psi^{a_1^D}$



(d) Filter a_2^D



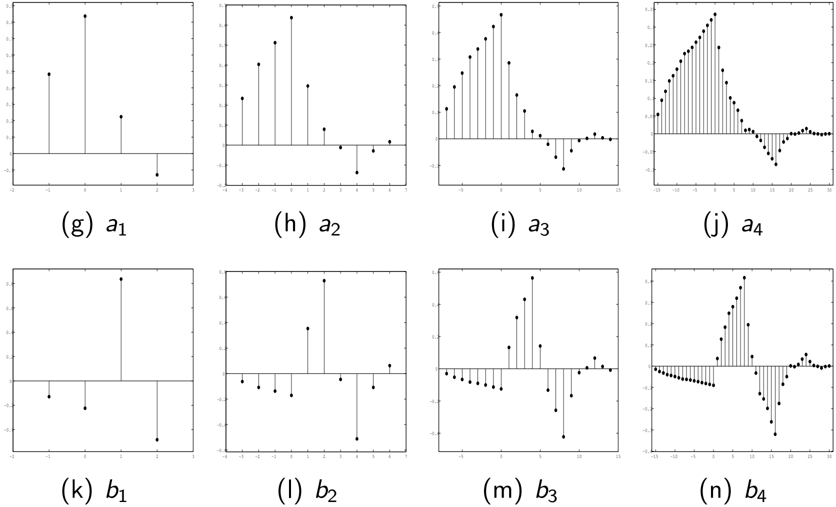
(e) $\phi^{a_2^D}$



(f) $\psi^{a_2^D}$

Plot Refinable Function ϕ and Wavelet Function ψ

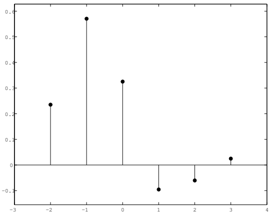
$$a = \left\{\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8}\right\}, \qquad b = \left\{-\frac{1-\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, -\frac{3+\sqrt{3}}{8}, \frac{1+\sqrt{3}}{8}\right\}.$$



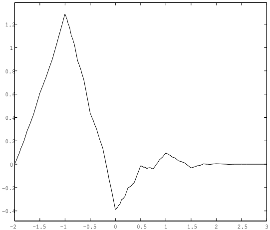
(g) a_1 (h) a_2 (i) a_3 (j) a_4

(k) b_1 (l) b_2 (m) b_3 (n) b_4

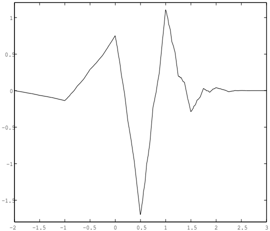
Daubechies Orthogonal Wavelets



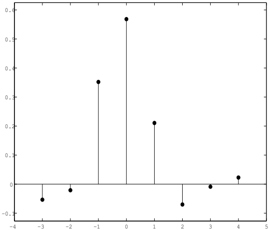
(a) Filter a_3^D



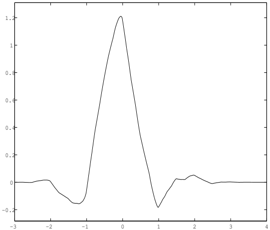
(b) $\phi^{a_3^D}$



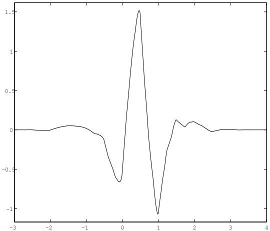
(c) $\psi^{a_3^D}$



(d) Filter a_4^D



(e) $\phi^{a_4^D}$



(f) $\psi^{a_4^D}$

Riesz (Multi)Wavelets in $L_2(\mathbb{R})$

- Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_s)^\mathsf{T}$ in $L_2(\mathbb{R})$.
- $\{\phi; \psi\}$ is a Riesz wavelet if $\mathrm{AS}(\phi; \psi)$ is a Riesz basis in $L_2(\mathbb{R})$, that is, (1) there exists positive constants C_1 and C_2 such that

$$C_1 \sum_{h \in \mathrm{AS}(\phi;\psi)} |c_h|^2 \leqslant \Big\| \sum_{h \in \mathrm{AS}} c_h h \Big\|^2_{L_2(\mathbb{R})} \leqslant C_2 \sum_{h \in \mathrm{AS}(\phi;\psi)} |c_h|^2,$$

and the linear span of $\mathrm{AS}(\phi; \psi)$ is dense in $L_2(\mathbb{R})$, where

$$\begin{aligned}\mathrm{AS}(\phi; \psi) := &\{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \\ &\cup \{\psi_{2^j;k} := 2^{j/2}\psi(2^j \cdot - k) \ : \ j \geqslant 0, k \in \mathbb{Z}\}.\end{aligned}$$

- If $C_1 = C_2 = 1$, then a Riesz wavelet becomes an orthogonal wavelet. That is, an orthogonal wavelet is a special case of Riesz wavelets.

Biorthogonal Wavelets in $L_2(\mathbb{R})$

- Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_s)^\mathsf{T}$ in $L_2(\mathbb{R})$.
- Let $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^\mathsf{T}$ and $\tilde{\psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_s)^\mathsf{T}$ in $L_2(\mathbb{R})$.
- $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a biorthogonal (multi)wavelet in $L_2(\mathbb{R})$ if
  1. Both $\{\tilde{\phi}; \tilde{\psi}\}$ and $\{\phi; \psi\}$ are Riesz wavelets in $L_2(\mathbb{R})$;
  2. $\mathrm{AS}(\tilde{\phi}; \tilde{\psi})$ and $\mathrm{AS}(\phi; \psi)$ are biorthogonal to each other:

$$\langle h, \tilde{h} \rangle = 1 \quad \text{and} \quad \langle h, g \rangle = 0, \quad \forall\, g \in \mathrm{AS}(\phi; \psi) \backslash \{h\}.$$

- Every function $f \in L_2(\mathbb{R})$ has the wavelet representation:

$$f = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k) \rangle \phi(\cdot - k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{2^j;k} \rangle \psi_{2^j;k}.$$

- An orthogonal wavelet $\{\phi; \psi\}$ is just a biorthogonal wavelet $(\{\phi; \psi\}, \{\phi; \psi\})$ (i.e., its dual is itself.)
- Biorthogonal wavelets and multiwavelets are widely used in image processing and numerical solutions of PDEs.

MRA of Biorthogonal Wavelets

- Let $(\{\tilde{\phi};\tilde{\psi}\},\{\phi;\psi\})$ be a biorthogonal wavelet in $L_2(\mathbb{R})$.
- Define $\mathscr{V}_j := \mathsf{S}_{2^j}(\phi|L_2(\mathbb{R}))$ and $\mathscr{W}_j := \mathsf{S}_{2^j}(\psi|L_2(\mathbb{R}))$ for $j \in \mathbb{Z}$.
- Define $\tilde{\mathscr{V}}_j := \mathsf{S}_{2^j}(\tilde{\phi}|L_2(\mathbb{R}))$ and $\tilde{\mathscr{W}}_j := \mathsf{S}_{2^j}(\tilde{\psi}|L_2(\mathbb{R}))$ for $j \in \mathbb{Z}$.
- Then we have two intertwined MRAs: For $J \in \mathbb{Z}$,

$$\mathscr{V}_{J+1} = \mathscr{V}_J \oplus \mathscr{W}_J \quad \text{and} \quad L_2(\mathbb{R}) = \mathscr{V}_J \oplus_{j=J}^{\infty} \mathscr{W}_j = \oplus_{j\in\mathbb{Z}} \mathscr{W}_j,$$
$$\tilde{\mathscr{V}}_{J+1} = \tilde{\mathscr{V}}_J \oplus \tilde{\mathscr{W}}_J \quad \text{and} \quad L_2(\mathbb{R}) = \tilde{\mathscr{V}}_J \oplus_{j=J}^{\infty} \mathscr{W}_j = \oplus_{j\in\mathbb{Z}} \mathscr{W}_j$$

where $\oplus$ means the direct sum of closed subspaces in $L_2(\mathbb{R})$ and

$$\mathscr{W}_j = \mathscr{V}_{j+1} \cap \tilde{\mathscr{V}}_j^{\perp}, \quad \tilde{\mathscr{W}}_j = \tilde{\mathscr{V}}_{j+1} \cap \mathscr{V}_j^{\perp}, \qquad j \in \mathbb{Z}.$$

- Wavelet coefficients $\langle f, \tilde{\psi}_{2^j;k}\rangle$ can be computed through MRAs using filter banks by fast wavelet transform as in the case of orthogonal wavelets.

Construction of Scalar Biorthogonal Wavelets

Theorem: Let $\phi, \psi \in L_2(\mathbb{R})$ and $\tilde{\phi}, \tilde{\psi} \in L_2(\mathbb{R})$. Then $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$ if and only if

1. There exist $a, b, \tilde{a}, \tilde{b} \in l_2(\mathbb{Z})$ such that

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi),$$
$$\widehat{\tilde{\phi}}(2\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi), \qquad \widehat{\tilde{\psi}}(2\xi) = \widehat{\tilde{b}}(\xi)\widehat{\tilde{\phi}}(\xi).$$

2. $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ is a biorthogonal wavelet filter bank:

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}}^T = I_2.$$

3. $[\widehat{\phi}, \widehat{\phi}] \in L_\infty(\mathbb{R})$, $[\widehat{\tilde{\phi}}, \widehat{\tilde{\phi}}] \in L_\infty(\mathbb{R})$, and $[\widehat{\tilde{\phi}}, \widehat{\phi}] = 1$, where

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad [\widehat{f}, \widehat{g}](\xi) := \sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi k)}.$$

Scalar Biorthogonal Wavelet Filter Bank

Proposition

Let $a, b, \tilde{a}, \tilde{b} \in l_0(\mathbb{Z})$. Then $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ is a biorthogonal wavelet filter bank:

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} = I_2$$

if $(\tilde{a}, a)$ is a biorthogonal low-pass filter:

$$\widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi)} + \widehat{\tilde{a}}(\xi+\pi)\overline{\widehat{a}(\xi+\pi)} = 1$$

with the choice $\widehat{\tilde{b}}(\xi) = e^{i\xi}\overline{\widehat{a}(\xi+\pi)}$ and $\widehat{b}(\xi) = e^{i\xi}\overline{\widehat{\tilde{a}}(\xi+\pi)}$.

If $\widehat{a}(0) = \sum_{k\in\mathbb{Z}} a(k) = 1$, then $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ is well defined and satisfies $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$.

Example of Scalar Biorthogonal Wavelets

We can obtain a pair of biorthogonal wavelet filters by splitting the interplatory filter

$$\overline{\widehat{\tilde{a}_m}(\xi)}\widehat{a_m}(\xi) := \widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2)\mathsf{P}_{m,m}(\sin^2(\xi/2))$$

as follows: $\mathsf{P}(x)\tilde{\mathsf{P}}(x) = \mathsf{P}_{m,m}(x)$ and

$$\widehat{a_m}(\xi) = 2^{-m}(1+e^{-i\xi})^m\mathsf{P}(\sin^2(\xi/2)), \qquad \widehat{b_m}(\xi) := e^{-i\xi}\overline{\widehat{\tilde{a}_m}(\xi+\pi)},$$

$$\widehat{\tilde{a}_m}(\xi) = 2^{-m}(1+e^{-i\xi})^m\tilde{\mathsf{P}}(\sin^2(\xi/2)), \qquad \widehat{\tilde{b}_m}(\xi) := e^{-i\xi}\overline{\widehat{a_m}(\xi+\pi)}.$$

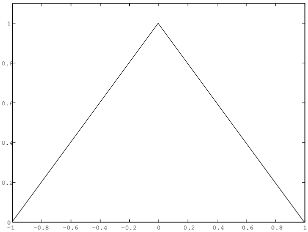
For $m = 2$, we have the LeGall biorthogonal wavelet filter bank:

$$a_2 = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}$$

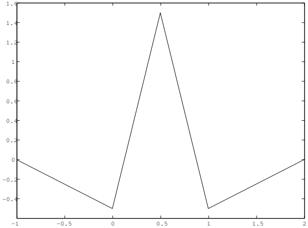
and

$$\tilde{a}_2 = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}.$$

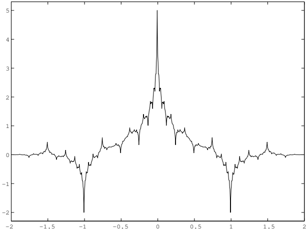
Examples of Biorthogonal Wavelets



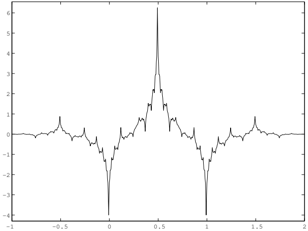
(g) ϕ^{a_2}



(h) ψ^{a_2, b_2}



(i) $\phi^{\tilde{a}_2}$



(j) $\psi^{\tilde{a}_2, \tilde{b}_2}$

The Most Famous Biorthogonal Wavelet

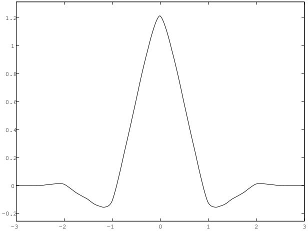
For $m = 4$,

$$a_4 = \left\{-\frac{t}{64}, \frac{2-t}{32}, \frac{16+t}{64}, \underline{\mathbf{\frac{6+t}{16}}}, \frac{16+t}{64}, \frac{2-t}{32}, -\frac{t}{64}\right\}_{[-3,3]},$$

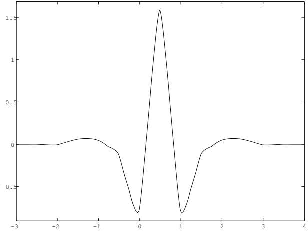
$$\tilde{a}_4 = \left\{\frac{t^2-4t+10}{256}, \frac{t-4}{64}, \frac{-t^2+6t-14}{64}, \frac{20-t}{64}, \underline{\mathbf{\frac{3t^2-20t+110}{128}}}, \frac{20-t}{64}, \frac{-t^2+6t-14}{64}, \frac{t-4}{64}, \frac{t^2-4t+10}{256}\right\}_{[-4,4]},$$

where $t \approx 2.92069$. The derived biorthogonal wavelet is called Daubechies 7/9 filter and has very impressive performance in many applications.

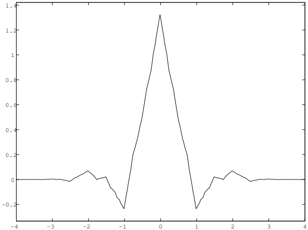
Example of Biorthogonal Wavelets



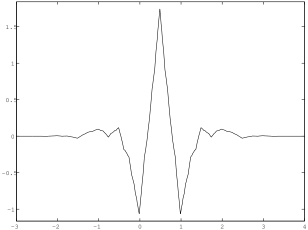
(k) ϕ^{a_4}



(l) ψ^{a_4,b_4}



(m) $\phi^{\tilde{a}_4}$



(n) $\psi^{\tilde{a}_4,\tilde{b}_4}$

B-spline Functions

- For $m \in \mathbb{N}$, the B-spline function B_m of order m is defined to be

$$B_1 := \chi_{(0,1]} \quad \text{and} \quad B_m := B_{m-1} * B_1 = \int_0^1 B_{m-1}(\cdot - t)dt.$$

- $\operatorname{supp}(B_m) = [0, m]$ and $B_m(x) > 0$ for all $x \in (0, m)$.
- $B_m = B_m(m - \cdot)$ and $B_m \in \mathscr{C}^{m-2}(\mathbb{R})$.
- $B_m|_{(k,k+1)} \in \mathbb{P}_{m-1}$ for all $k \in \mathbb{Z}$.
- $\widehat{B_m}(\xi) = (\frac{1-e^{-i\xi}}{i\xi})^m$ and B_m is refinable:

$$B_m = 2\sum_{k\in\mathbb{Z}} a_m^B(k)B_m(2 \cdot -k),$$

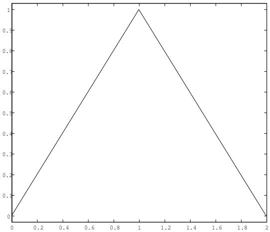
where a_m^B is the B-spline filter of order m:

$$\widehat{a_m^B}(\xi) := 2^{-m}(1 + e^{-i\xi})^m.$$

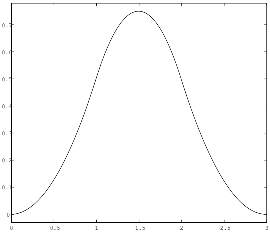
- Note that $\widehat{B_m}(2\xi) = \widehat{a_m^B}(\xi)\widehat{B_m}(\xi)$ and $\widehat{B_m}(\xi) = \prod_{j=1}^{\infty} \widehat{a_m^B}(2^{-j}\xi)$.

Graphs of B-spline Functions

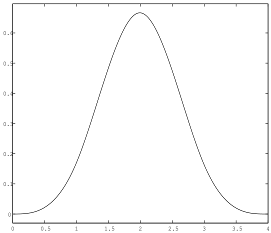
(a) B_1



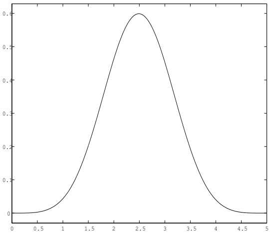
(b) B_2



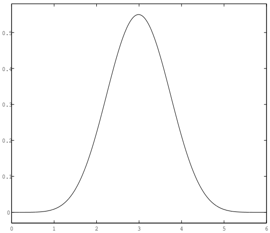
(c) B_3



(d) B_4



(e) B_5



(f) B_6

B-spline Filters a_m^B

$$
\begin{aligned}
a_1^B &= \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]},\\
a_2^B &= \{\underline{\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]},\\
a_3^B &= \{\underline{\mathbf{\tfrac{1}{8}}}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]},\\
a_4^B &= \{\underline{\mathbf{\tfrac{1}{16}}}, \tfrac{1}{4}, \tfrac{3}{8}, \tfrac{1}{4}, \tfrac{1}{16}\}_{[0,4]},\\
a_5^B &= \{\underline{\mathbf{\tfrac{1}{32}}}, \tfrac{5}{32}, \tfrac{5}{16}, \tfrac{5}{16}, \tfrac{15}{32}, \tfrac{1}{32}\}_{[0,5]},\\
a_6^B &= \{\underline{\mathbf{\tfrac{1}{64}}}, \tfrac{3}{32}, \tfrac{15}{64}, \tfrac{5}{16}, \tfrac{15}{64}, \tfrac{3}{32}, \tfrac{1}{64}\}_{[0,6]}.
\end{aligned}
$$

Basics on Sobolev Spaces

- A function f on $I := [a, b]$ is absolutely continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all nonoverlapping $(a_j, b_j), j = 1, \dots, n$ in I such that

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon \quad \text{as long as} \quad \sum_{j=1}^{n} (b_j - a_j) < \delta.$$

- If f is absolutely continuous, then f' exists almost everywhere and f is uniformly continuous.
- The Sobolev space $H^m(I)$ with $m \in \mathbb{N} \cup \{0\}$ consists of all functions f on I such that $f, f', \dots, f^{(m-1)}$ are absolutely continuous on I, $f, f', \dots, f^{(m-1)}, f^{(m)} \in L_2(I)$.
- Sobolev norm: $\|f\|_{H^m}^2 := \sum_{j=0}^{m} \|f^{(j)}\|_{L_2}^2$.
- Sobolev spaces $H^m(I)$ are widely used for studying and solving PDEs.

Basics on Boundary Value Problems (BVP)

- Poisson equation: $u''(x) = f(x)$ for $x \in (0,1)$ with boundary conditions $u(0) = \alpha$ (Dirichlet boundary condition) and $u'(0) = \beta$ (Neumann boundary condition).
- Helmholtz equation: $u'' + \kappa^2 u = f$ on $(0,1)$ with boundary conditions $u(0) = \alpha$ and $u'(1) - iku(1) = 0$ (Robin boundary conditions).
- A weak solution $u \in H^1(0,1)$ must satisfy

$$\langle u'', v\rangle = \langle f, v\rangle, \qquad \forall\, v \in H^1(0,1).$$

Using integration by parts and boundary conditions,

$$\langle u'', v\rangle = \int_0^1 u''(x)v(x)dx = u'(x)v(x)|_{x=0}^{x=1} - \int_0^1 u'(x)v'(x)dx.$$

- Hence, a weak solution u is given by

$$\langle u', v'\rangle = -\langle f, v\rangle - u'(1)v(1) + u'(0)v(0), \quad \forall\, v \in H^1(0,1).$$

Galerkin Scheme

- Poisson equation: $u'' = f$ on $(0, 1)$ with $u(0) = u(1) = 0$.
- Let V_h be a finite dimensional subspace of $H^1(0, 1)$ with suitable boundary conditions. For the above Poisson equation, we consider $H^1_0(0, 1)$ by requiring $\phi(0) = \phi(1) = 0$ for all $\phi \in V_h$.
- Galerkin scheme: Seek $u_h \in V_h \subseteq H^1_0(0, 1)$ such that

$$\langle u_h', v' \rangle = \int_0^1 u_h' v' = -\langle f, v \rangle, \qquad \forall v \in V_h.$$

- Let $\{\phi_1, \ldots, \phi_N\}$ be a basis of V_h. Then we can write $u_h = \sum_{j=1}^N c_j \phi_j \in V_h$ such that $\{c_1, \ldots, c_N\}$ must satisfy

$$\sum_{j=1}^N c_j \langle \phi_j', \phi_k' \rangle = \langle f, \phi_k \rangle, \qquad k = 1, \ldots, N.$$

- Solve the linear system $Ac = b$ for $c = (c_1, \ldots, c_N)^T$, where $A = (\langle \phi_j', \phi_k' \rangle)_{1 \leqslant j,k \leqslant N}$, and $b = (-\langle f, \phi_k \rangle)_{1 \leqslant k \leqslant N}$.

Finite Element Method

- Let $N \in \mathbb{N}$ and $h := \frac{1}{N}$.
- Consider partition

$$0 = x_0 < x_1 < \cdots < x_N = 1$$

with $x_j := \frac{j}{N}$ for $j = 0, \ldots, N$.

- Define a piecewise linear function ϕ_j with support $[x_{j-1}, x_{j+1}]$ such that $\phi_j(x_{j-1}) = \phi(x_{j+1}) = 0$ and $\phi_j(x_j) = 1$.
- The Finite Element Method uses the basis $\{\phi_1, \ldots, \phi_{N-1}\}$ which spans V_h (a spline space generated by the linear spline)
- Note that all basis elements $\phi_j \in H_0^1(0,1)$ and hence $V_h \subseteq H_0^1(0,1)$.

Collocation Scheme

- Poisson equation: $u'' = f$ on $(0,1)$ with $u(0) = u(1) = 0$.
- Let V_h be a finite dimensional subspace of $H^1(0,1) \cap C^2$ with suitable boundary conditions. For the above Poisson equation, we consider $H_0^1(0,1)$ by requiring $\phi(0) = \phi(1) = 0$ for all $\phi \in V_h$.
- Let $\{\phi_1, \ldots, \phi_N\}$ be a basis of V_h. Then we seek $u_h = \sum_{j=1}^N c_j \phi_j \in V_h$ with coefficients $c_1, \ldots, c_N$ to be determined.
- Collocation scheme: (1) suitably pick up N sampling points $z_1, \ldots, z_N$ inside $[0,1]$. (2) obtain the linear system through

$$\sum_{j=1}^{N} c_j \phi_j''(z_k) = f(z_k), \qquad k = 1, \ldots, N.$$

- Solve the above linear equations to determine the coefficients $c_1, \ldots, c_N$.

Wavelet Method

- Let $\{\phi; \psi\}$ be a Riesz wavelet for $H^1(\mathbb{R})$ such that ϕ, ψ belong to the Sobolev space $H^1(0,1)$.
- Adapt the Riesz wavelet $\{\phi; \psi\}$ on the real line into the interval $[0,1]$ with prescribed boundary conditions to obtain a Riesz basis $\Phi_0 \cup \cup_{j=0}^{\infty} \Psi_j$ for $H_0^1(0,1)$, where

$$\Phi_0 = \{\phi^L\} \cup \{\phi(\cdot - k) \ : \ l_\phi \leqslant k \leqslant h_\phi\} \cup \{\phi^R\}$$

and

$$\Psi_j = \{\psi^L_{2^j;0}\} \cup \{\psi_{2^j;k} \ : \ l_\psi \leqslant k \leqslant 2^j - h_\psi\} \cup \{\psi^R_{2^j;2^j-1}\}.$$

- Take a large integer $J \in \mathbb{N}$ and consider $\{\eta_j\}_{j \in N_J} = \Phi_0 \cup \cup_{j=0}^{J} \Psi_j$ which spans V_h.
- Now apply Galerkin scheme or collocation scheme.

Riesz Wavelets in Sobolev Spaces

- For $\tau \in \mathbb{R}$, the Sobolev space $H^\tau(\mathbb{R})$ consists of all tempered distributions f satisfying

$$\|f\|^2_{H^\tau(\mathbb{R})} := \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1+|\xi|^2)^\tau d\xi < \infty,$$

For $m \in \mathbb{N} \cup \{0\}$, $f \in H^m(\mathbb{R})$ if $f, f', \ldots, f^{(m)} \in L_2(\mathbb{R})$.

- For $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_r)^\mathsf{T}$, we define

$$\mathsf{AS}_0^\tau(\phi;\psi) = \{\phi(\cdot - k) : k \in \mathbb{Z}\} \cup \{2^{j(\frac{1}{2}-\tau)}\psi(2^j \cdot -k) : k \in \mathbb{Z}\}_{j=0}^\infty.$$

- $\{\phi;\psi\}$ is a Riesz wavelet in the Sobolev space $H^\tau(\mathbb{R})$ if $\mathsf{AS}_0^\tau(\phi;\psi)$ is Riesz basis for $H^\tau(\mathbb{R})$: the linear span of $\mathsf{AS}_0^\tau(\phi;\psi)$ is dense in $H^\tau(\mathbb{R})$ and there exist $C_1, C_2 > 0$ such that

$$C_1 \sum_{h \in \mathsf{AS}_0^\tau(\phi;\psi)} |c_h|^2 \leqslant \Big\| \sum_{h \in \mathsf{AS}_0^\tau(\phi;\psi)} c_h h \Big\|_{H^\tau(\mathbb{R})} \leqslant C_2 \sum_{h \in \mathsf{AS}_0^\tau(\phi;\psi)} |c_h|^2$$

for all finitely supported sequences $\{c_h\}_{h \in \mathsf{AS}_0^\tau(\phi;\psi)}$.

Derivative-Orthogonal Riesz Wavelets

- Let $m \in \mathbb{N} \cup \{0\}$ be a nonnegative integer.
- Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_r)^\mathsf{T}$ in $H^m(\mathbb{R})$.
- We say that $\{\phi; \psi\}$ is an mth-order derivative-orthogonal Riesz wavelet in the Sobolev space $H^m(\mathbb{R})$ if
  1. $\mathsf{AS}_0^m(\phi; \psi)$ is a Riesz wavelet in $H^m(\mathbb{R})$;
  2. The mth-order derivatives are orthogonal between levels:

$$\langle \psi^{(m)}, \phi^{(m)}(\cdot - k) \rangle = 0, \qquad \forall\, k \in \mathbb{Z},$$

and

$$\langle \psi^{(m)}(2^j \cdot -k), \psi^{(m)}(2^{j'} \cdot -k') \rangle = 0,$$

for all $k, k' \in \mathbb{Z}, j, j' \in \mathbb{N}_0$ with $j \neq j'$.

For $m = 0$, they are called semi-orthogonal (or pre-) wavelets and well studied, e.g., [Chui-Wang,Jia-Micchelli,Shen-Riemanschneider] through a simple orthogonalization procedure.

Why Derivative-Orthogonal Riesz Wavelets?

- Differential equation with homogeneous boundary condition:

$$u^{(2m)}(x) + \alpha u(x) = f(x), \quad x \in I = [0, 1].$$

- The Galerkin formulation of a weak solution $u \in H^m(I)$ is

$$(-1)^m \langle u^{(m)}, v^{(m)} \rangle + \alpha \langle u, v \rangle = \langle f, v \rangle, \quad v \in H^m(I).$$

- Let S be a Riesz wavelet basis of $H^m(I)$ derived from mth-order derivative-orthogonal wavelet $\{\phi; \psi\}$. Then $u = \sum_{h \in S} c_h h$ and

$$\sum_{h \in S} \Big((-1)^m A_{h,g} + \alpha B_{h,g} \Big) c_h = \langle f, g \rangle, \qquad g \in S$$

with $A = (\langle h^{(m)}, g^{(m)} \rangle)_{h,g \in S}$ and $B = (\langle h, g \rangle)_{h,g \in S}$.

(1) The matrix A is sparse and is almost diagonal.

(2) The condition number of A dominates that of $(-1)^m A + \alpha B$ and is often very small (can be the optimal condition number 1).

Stable Integer Shifts of Vector Functions

- Let $\phi = (\phi_1, \dots, \phi_r)^\mathsf{T}$ in $H^m(\mathbb{R})$ be compactly supported.
- We say that the integer shifts of ϕ are stable if

$$\text{span}\{\widehat{\phi}(\xi + 2\pi k) \ : \ k \in \mathbb{Z}\} = \mathbb{C}^r, \qquad \forall \xi \in \mathbb{R}.$$

- For $f = (f_1, \dots, f_r)^\mathsf{T}$ and $g = (g_1, \dots, g_r)^\mathsf{T}$, bracket product is

$$[\widehat{f}, \widehat{g}](\xi) := \sum_{k \in \mathbb{Z}} \widehat{f}(\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi k)}^\mathsf{T}, \quad \xi \in \mathbb{R}.$$

- The integer shifts of ϕ in $H^m(\mathbb{R})$ are stable $\Longleftrightarrow$ $[\widehat{\phi}, \widehat{\phi}](\xi) > 0$ for all $\xi \in \mathbb{R}$ $\Longleftrightarrow$ $\{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\}$ is a Riesz sequence in $H^m(\mathbb{R})$.
- Smoothness of a function is measured by

$$\text{sm}(\phi) := \sup\{\tau \in \mathbb{R} \ : \ \phi \in H^\tau(\mathbb{R})\}.$$

Semi-orthogonal (or Pre-) Wavelets

For $m = 0$, mth order derivative-orthogonal wavelets are called semi-orthogonal (or pre-) wavelets and well studied, e.g., [Chui-Wang,Jia-Micchelli,Shen-Riemanschneider] through

- $\phi \in L_2(\mathbb{R})$ has compact support and satisfies
$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$$
where $\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}$ is a 2π-periodic trigonometric polynomial and $\widehat{\phi}(\xi) := \int_{\mathbb{R}} \phi(x)e^{-ix\xi}dx$.
- Assume the integer shifts of ϕ are stable: $[\widehat{\phi}, \widehat{\phi}](\xi) > 0$.
- Define $\widehat{\psi}(2\xi) := e^{-i\xi}\overline{\widehat{a}(\xi+\pi)[\widehat{\phi}, \widehat{\phi}]}(\xi+\pi)\widehat{\phi}(\xi)$.
- Then $\{\phi; \psi\}$ is a semi-orthogonal wavelet in $L_2(\mathbb{R})$, that is,
$$\mathrm{AS}_0^0(\phi;\psi) = \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \cdot -k) \ : \ k \in \mathbb{Z}\}_{j=0}^{\infty}$$
is almost an orthonormal basis for $L_2(\mathbb{R})$, except orthogonality among the same scale level j.

Sum Rules of a Matrix-valued Filter

- A filter $a \in (l_0(\mathbb{Z}))^{r\times r}$ has order m sum rules if there exists a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{\upsilon}(0) \ne 0$ and

$$\widehat{\upsilon}(2\xi)\widehat{a}(\xi) = \widehat{\upsilon}(\xi) + \mathscr{O}(|\xi|^m), \quad \widehat{\upsilon}(2\xi)\widehat{a}(\xi+\pi) = \mathscr{O}(|\xi|^m), \ \ \xi \to 0.$$

- $f(\xi) = g(\xi) + \mathscr{O}(|\xi|^m), \xi \to 0 \Leftrightarrow f^{(j)}(0) = g^{(j)}(0), 0 \leqslant j < m.$
- $\mathrm{sr}(a)$ denotes the highest order of sum rules satisfied by a.
- For $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with stable integer shifts, TFAE
  - The filter a has order m sum rules (i.e., $\mathrm{sr}(a) \geqslant m$).
  - All the polynomials of degree $< m$ are contained inside the shift-invariant space

$$S_j(\phi) := \left\{ \sum_{k\in\mathbb{Z}} v(k)\phi(2^j\cdot - k) \ : \ \text{all sequences } v \text{ on } \mathbb{Z} \right\}$$

  - The vector function ϕ has approximation order m:

$$\inf_{g\in S_j(\phi)\cap L_2(\mathbb{R})} \|f-g\|_{L_2(\mathbb{R})} \leqslant C2^{-jm}\|f\|_{H^m(\mathbb{R})}, \quad f \in H^m(\mathbb{R}).$$

Main Result: Existence

Theorem

Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ be a compactly supported refinable vector function in $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ for some $a \in (l_0(\mathbb{Z}))^{r\times r}$. Then there exists a finitely supported high-pass filter $b \in (l_0(\mathbb{Z}))^{r\times r}$ such that $\{\phi; \psi\}$ with

$$\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$$

is an mth-order derivative-orthogonal Riesz wavelet in the Sobolev space $H^m(\mathbb{R})$ $\iff$

1. *the integer shifts of ϕ are stable;*
2. *the filter a has at least order $2m$ sum rules (i.e., $\operatorname{sr}(a) \geqslant 2m$).*

Remark: The proof is quite complicated for $r > 1$ and $m > 0$.

Main Result: Construction

Theorem

Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ be a compactly supported refinable vector function in $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ for some $a \in (l_0(\mathbb{Z}))^{r\times r}$. Suppose that the integer shifts of ϕ are stable. For any $b \in (l_0(\mathbb{Z}))^{r\times r}$, $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ is an mth-order derivative-orthogonal Riesz wavelet in $H^m(\mathbb{R})$ $\Longleftrightarrow$

$$\widehat{b}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^\mathsf{T} + \widehat{b}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^\mathsf{T} = 0,$$

$$\det(\{\widehat{a}; \widehat{b}\})(\xi) := \det\left(\begin{bmatrix} \widehat{a}(\xi) & \widehat{a}(\xi+\pi) \\ \widehat{b}(\xi) & \widehat{b}(\xi+\pi) \end{bmatrix}\right) \neq 0, \qquad \forall\, \xi \in \mathbb{R}.$$

Moreover, $\mathrm{AS}_0^\tau(\phi; \psi)$ is a Riesz basis in the Sobolev space $H^\tau(\mathbb{R})$ for all τ in the nonempty open interval $(2m - \mathrm{sm}(\phi), \mathrm{sm}(\phi))$.

Construction for the Scalar Case $r = 1$

Theorem

Let $m \in \mathbb{N}_0$ *and* $a \in l_0(\mathbb{Z})$ *such that* $\widehat{a}(\xi) = 2^{-2m}(1+e^{-i\xi})^{2m}\widehat{\breve{a}}(\xi)$ *with* $\breve{a} \in l_0(\mathbb{Z})$ *and* $\widehat{\breve{a}}(0) = 1$. *Define* $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ *and* $\widehat{\mathring{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\mathring{a}}(2^{-j}\xi)$ *with* $\widehat{\mathring{a}}(\xi) := 2^{-m}(1+e^{-i\xi})^{m}\widehat{\breve{a}}(\xi)$. *Suppose that* $\phi \in H^m(\mathbb{R})$ *and the integer shifts of* ϕ *are stable. Then for* $b \in l_0(\mathbb{Z})$, $\{\phi; \psi\}$ *with* $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ *is an mth-order derivative-orthogonal Riesz wavelet in* $H^m(\mathbb{R})$ $\iff$

$$\widehat{b}(\xi) = e^{i(m-1)\xi}\overline{\widehat{\breve{a}}(\xi+\pi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi+\pi)\widehat{\theta}(2\xi)/\widehat{c}(2\xi),$$

where $\theta \in l_0(\mathbb{Z})$ *satisfying* $\widehat{\theta}(\xi) \neq 0\ \forall \xi \in \mathbb{R}$ *and* $c \in l_0(\mathbb{Z})$ *is*

$$\widehat{c}(2\xi) := \gcd\left(\overline{\widehat{\breve{a}}(\xi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi), \overline{\widehat{\breve{a}}(\xi+\pi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi+\pi)\right).$$

Example of Refinable Functions: B-Splines

- For $n \in \mathbb{N}$, the B-spline function B_n of order n is defined to be

$$B_1 := \chi_{(0,1]}, \quad B_n := B_{n-1} * B_1 = \int_0^1 B_{n-1}(\cdot - t)dt.$$

- $B_n = B_n(n - \cdot)$ (symmetry), $\text{sm}(B_n) = n - 1/2$ and B_n is a piecewise polynomial: $B_n|_{(k,k+1)} \in \mathbb{P}_{n-1}$ for all $k \in \mathbb{Z}$.
- $B_n = 2\sum_{k\in\mathbb{Z}} a(k)B_n(2\cdot -k)$, i.e., $\widehat{B_n}(2\xi) = \widehat{a}(\xi)\widehat{B_n}(\xi)$ with

$$\widehat{a}(\xi) := 2^{-n}(1 + e^{-i\xi})^n \quad \text{with} \quad \text{sr}(a) = n.$$

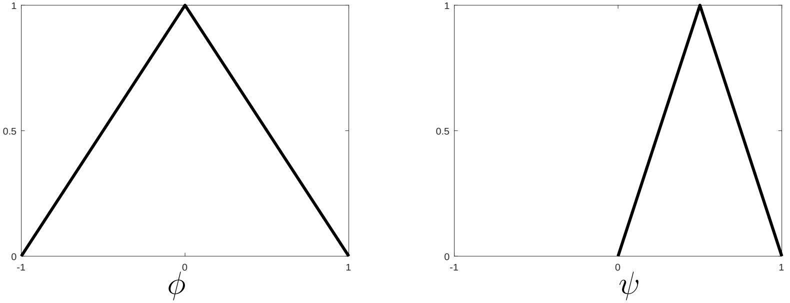
- A compactly supported mth-order derivative-orthogonal Riesz wavelet $\{B_n; \psi\}$ in $H^m(\mathbb{R})$ can be derived from B_n $\iff$

$$n \geqslant \max(m + 1, 2m).$$

Example from B-Spline B_2: $m = 1$

$$a = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}_{[-1,1]}, \qquad b = \left\{\frac{1}{2}\right\}_{[1,1]}.$$

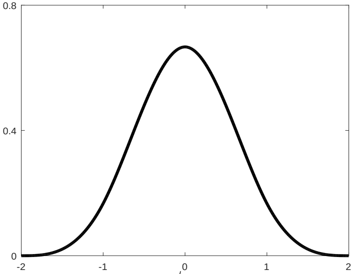
$\mathrm{sr}(a) = 2$ and $\phi = B_2(\cdot - 1)$ is the centered piecewise linear spline B_2 of order 2 satisfying $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. The wavelet function $\psi = \phi(2x)$. Then $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$, which is closely linked to finite element methods.



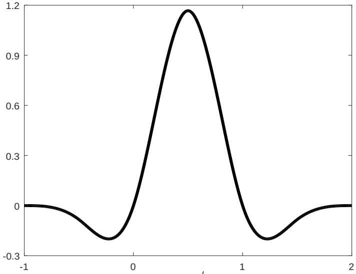
Example from B-Spline B_4: $m = 2$

$$a = \left\{ \frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16} \right\}_{[-2,2]}, \quad b = \left\{ -\frac{1}{4}, 1, -\frac{1}{4} \right\}_{[0,2]}$$

$\mathrm{sr}(a) = 4$ and $\phi = B_4(\cdot - 2)$ is the centered B-spline B_4 of order 4 satisfying $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. The wavelet function $\psi = \phi(2x) - \frac{1}{2}\phi(2x+1) - \frac{1}{2}\phi(2x-1)$. Then $\{\phi; \psi\}$ is a second-order derivative-orthogonal Riesz wavelet in $H^2(\mathbb{R})$.



ϕ



ψ

Hermite Cubic Splines

The Hermite cubic splines are given by

$$\phi_1 = \begin{cases} (1-x)^2(1+2x), & x \in [0,1], \\ (1+x)^2(1-2x), & x \in [-1,0), \\ 0, & \text{otherwise}, \end{cases} \quad \phi_2 = \begin{cases} (1-x)^2 x, & x \in [0,1], \\ (1+x)^2 x, & x \in [-1,0), \\ 0, & \text{otherwise}. \end{cases}$$

Then $\phi = (\phi_1, \phi_2)^\mathsf{T}$ in $H^2(\mathbb{R})$ satisfies the refinement equation $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with the filter $a \in (l_0(\mathbb{Z}))^{2\times 2}$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{16} & -\frac{1}{16} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ \frac{1}{16} & -\frac{1}{16} \end{bmatrix} \right\}_{[-1,1]}.$$

The filter a has order 4 sum rules with $\operatorname{sr}(a) = 4$ and ϕ has the Hermite interpolation property with $\operatorname{sm}(\phi) = 5/2$:

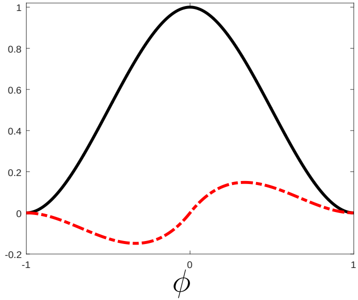
$$\phi_1(k) = \boldsymbol{\delta}(k), \quad \phi_1'(k) = 0, \quad \phi_2(k) = 0, \quad \phi_2'(k) = \boldsymbol{\delta}(k), \quad \forall\, k \in \mathbb{Z},$$

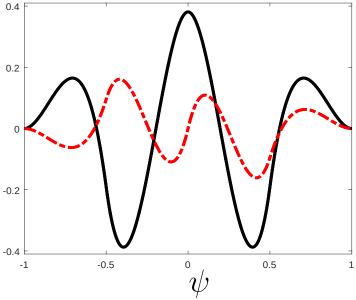
where $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for $k \neq 0$.

Example from Hermite Cubic Splines: $m = 1$

$$b = \left\{ \begin{bmatrix} \frac{2}{21} & 1 \\ \frac{1}{9} & 1 \end{bmatrix}, \begin{bmatrix} -\frac{4}{21} & 0 \\ 0 & \frac{4}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{21} & -1 \\ -\frac{1}{9} & 1 \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$, which is used in [Jia-Liu, Adv. Comput. Math., (2006)] for Sturm-Liouville equations.

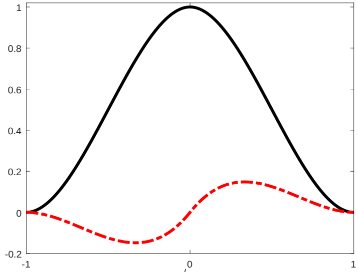




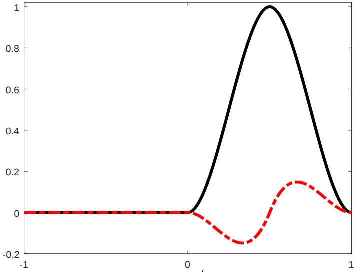
Example from Hermite Cubic Splines: $m = 2$

$$b = \left\{ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right\}_{[1,1]}.$$

Then $\{\phi; \psi\}$ with $\psi = \phi(2\cdot)$ is a second-order derivative-orthogonal Riesz wavelet in $H^2(\mathbb{R})$.



ϕ



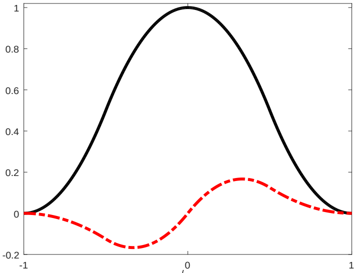
ψ

Example from Hermite Quadratic Splines: $m = 1$

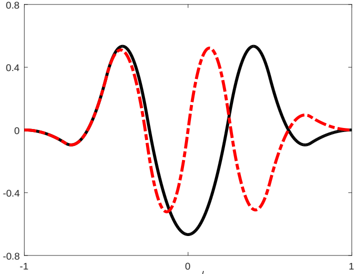
$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{16} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{16} & -\frac{1}{8} \end{bmatrix} \right\}_{[-1,1]},$$

$$b = \left\{ \begin{bmatrix} \frac{1}{6} & 1 \\ \frac{1}{6} & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & -1 \\ -\frac{1}{6} & 1 \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\mathrm{sr}(a) = 3$, $\mathrm{sm}(\phi) = 2.5$ and $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$.



ϕ



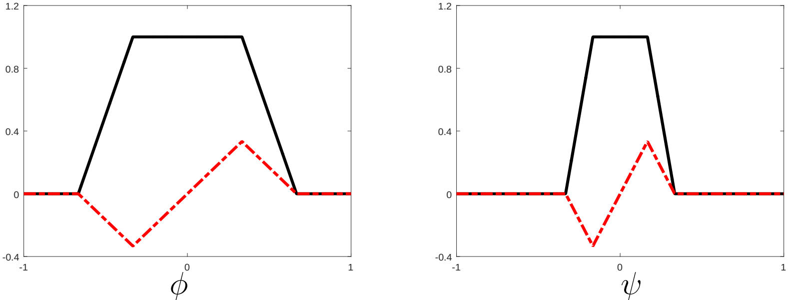
ψ

Example from Hermite Linear Splines: $m = 1$

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{12} & -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{12} & -\frac{1}{4} \end{bmatrix} \right\}_{[-1,1]},$$

$$b = \left\{ \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{bmatrix} \right\}_{[0,0]}.$$

Then $\mathrm{sr}(a) = 2$, $\mathrm{sm}(\phi) = 1.5$ and $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$.



Wavelets on [0, 1] with Very Simple Structure

- Let $n > 1$ be the scale/resolution level.
- Let $g_1 = \phi^L(2\cdot), g_2 = \phi_1(2\cdot -1), g_3 = \phi_2(2\cdot -1), g_4 = \phi^R(2\cdot)$.
- For $1 < j \leqslant n$, the boundary wavelets at level j are given by

$$g_{2^j+1} := 2^{j(1/2-m)}\psi^L(2^j\cdot), \quad g_{2^j+2^j} := 2^{j(1/2-m)}\psi^R(2^j\cdot -2^j)$$

with only one boundary wavelet at each endpoint:

$$\phi^L \in \{\phi_1|_{[0,1]}, \phi_2|_{[0,1]}\}, \qquad \phi^R \in \{\phi_1(\cdot-1)|_{[0,1]}, \phi_2(\cdot-1)|_{[0,1]}\},$$
$$\psi^L \in \{\psi_1|_{[0,1]}, \psi_2|_{[0,1]}\}, \qquad \psi^R \in \{\psi_1(\cdot-1)|_{[0,1]}, \psi_2(\cdot-1)|_{[0,1]}\}$$

- For $k = 1, \dots, 2^j - 1$, the inner wavelets at level j are

$$g_{2^j+(2k-1)} := 2^{j(1/2-m)}\psi_1(2^j\cdot -k), \quad g_{2^j+2k} := 2^{j(1/2-m)}\psi_2(2^j\cdot -k).$$

- Normalize $\{g_k\}_{k=1}^{2^{n+1}}$ so that $\|g_k^{(m)}\|_{L_2(\mathbb{R})} = 1$ for $k = 1, \dots, 2^{n+1}$.

Example: 1D Sturm-Liouville Equations

- Consider the following differential equation:

$$\begin{cases} -u'' + \alpha u = f \quad \text{on} \ (0,1), \\ u'(0) = 100(1-e^{-1}), \quad u(1) = 200e^{-1} - 100, \end{cases}$$

where $\alpha = 5$ and $f(x) = -100e^{-x} - 500(1-e^{-x}) - 500e^{-1}x$. The exact solution is $u(x) = 100(1-e^{-x}) - 100e^{-1}x$.

- Its corresponding Galerkin formulation is

$$\sum_{k=1}^{2^{n+1}} A_{j,k} c_k = \langle g_j, f \rangle, \qquad j = 1, \ldots, 2^{n+1}$$

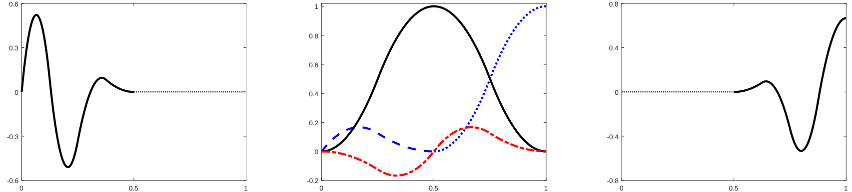
with the coefficient matrix $A_{j,k} := \langle g_j', g_k' \rangle + \alpha \langle g_j, g_k \rangle$.

- According to boundary conditions, the boundary elements are

$$\phi^L := \phi_2|_{[0,1]}, \qquad \phi^R := \phi_1(\cdot - 1)|_{[0,1]},$$
$$\psi^L := \psi_2|_{[0,1]}, \qquad \psi^R := \psi_1(\cdot - 1)|_{[0,1]}.$$

Boundary Elements and Numerical Performance

Use the first-order derivative-orthogonal Riesz wavelet derived from Hermite quadratic splines with $m = 1$.



Level	Size	Iteration	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$
5	128	15	3.2106	4.2213e-07	1.8141e-07	—
6	256	15	3.2106	5.2757e-08	2.2607e-08	3.0044
7	512	15	3.2106	6.5944e-09	2.8215e-09	3.0022
8	1024	15	3.2106	8.2365e-10	3.5242e-10	3.0011
9	2048	16	3.2106	1.1561e-10	4.6097e-11	2.9346

Table: Size of linear system, conjugate gradient iterations, condition number κ of the coefficient matrices A, errors, and convergence rates

Example: 1D Sturm-Liouville Equations

- Consider the following differential equation:

$$\begin{cases} -u'' = f \quad \text{on} \quad (0,1), \\ u'(0) = 0, \quad u(1) = 0, \end{cases}$$

where $f(x) = x \ln(x+1)$. The exact solution is $u(x) = (5x^3 - 12x^2 - 12x + 19 - 24\ln(2) - 6(x-2)(x+1)^2 \ln(x+1))/36$.

- Its corresponding Galerkin formulation is

$$\sum_{k=1}^{2^{n+1}} A_{j,k} c_k = \langle g_j, f \rangle, \qquad j = 1, \dots, 2^{n+1}$$

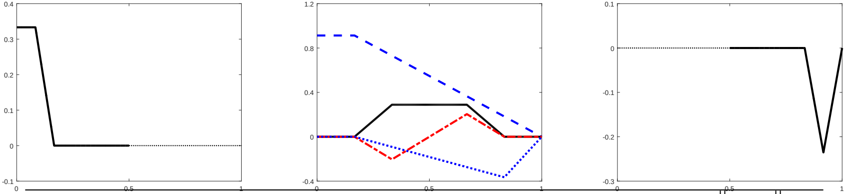
with the coefficient matrix $A_{j,k} := \langle g_j', g_k' \rangle$.

- According to boundary conditions, the boundary elements are

$$\phi^L := \phi_1|_{[0,1]}, \qquad \phi^R := \phi_2(\cdot - 1)|_{[0,1]},$$
$$\psi^L := \psi_1|_{[0,1]}, \qquad \psi^R := \psi_2(\cdot - 1)|_{[0,1]}.$$

Boundary Elements and Numerical Performance

Use the first-order derivative-orthogonal Riesz wavelet derived from Hermite linear splines with $m = 1$.



Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$
8	1024	1	1.4640e-07	4.1807e-08	1.9971
9	2048	1	3.6662e-08	1.0462e-08	1.9986
10	4096	1	9.1733e-09	2.6168e-09	1.9995
11	8192	1	2.2943e-09	6.5284e-10	2.0029
12	16384	1	5.7369e-10	1.5988e-10	2.0298

Table: Size of linear system, no iteration is needed, condition number κ of the matrices A (A is the identity matrix), errors, and convergence rates.

Example: 1D Biharmonic Equations

- Consider the following differential equation:

$$\begin{cases} u^{(4)} - \alpha u = f \quad \text{on} \ \ (0,1), \\ u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \end{cases}$$

  where $\alpha = 11$ and $f(x) = -4\pi^4 \cos(2\pi x) - \frac{11}{4}(1 - \cos(2x))$. The exact solution is $u(x) = \frac{1}{4}(1 - \cos(2\pi x))$.

- Its corresponding Galerkin formulation is

$$\sum_{k=1}^{2^{n+2}} B_{j,k} c_k = \langle g_j, f \rangle, \qquad j = 1, \dots, 2^{n+2} - 2$$

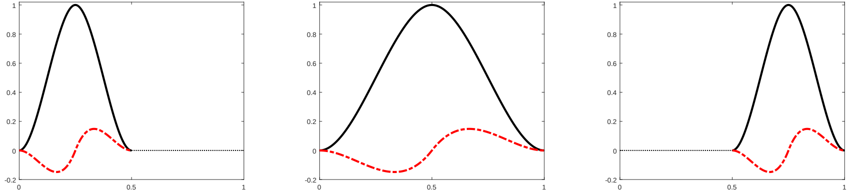
  with the coefficient matrix $B_{j,k} := \langle g_j'', g_k'' \rangle + \alpha \langle g_j, g_k \rangle$.

- According to boundary conditions, the boundary elements are

$$\phi^L := \emptyset, \qquad \phi^R := \emptyset,$$
$$\psi^L := \emptyset, \qquad \psi^R := \emptyset.$$

Boundary Elements and Numerical Performance

Use the second-order derivative-orthogonal Riesz wavelet derived from Hermite cubic splines with $m = 2$.



Level	Size	Iteration	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H^1}$	$\log_2 \frac{\|e_{n-1}\|_{H_1}}{\|e_n\|_{H_1}}$
2	14	4	1.0225	2.2655e-04	1.1050e-04	—	3.0679e-03	—
3	30	4	1.0225	1.5147e-05	6.9610e-06	3.9886	3.8598e-04	2.9907
4	62	4	1.0225	9.6244e-07	4.3628e-07	3.9971	4.8325e-05	2.9977
5	126	4	1.0225	6.0411e-08	2.7653e-08	3.9993	6.0431e-06	2.9994
6	254	4	1.0225	3.9699e-09	2.1493e-09	4.0000	7.5554e-07	2.9999
7	510	4	1.0225	1.1049e-09	6.8832e-10	4.0650	9.4808e-08	3.0000

Table: Size of linear system, conjugate gradient iteration, condition number κ of the matrices B, errors, and convergence rates.

Example: 1D Biharmonic Equations

- Consider the following differential equation:

$$\begin{cases} u^{(4)} = f \quad \text{on} \quad (0,1), \\ u(0) = 16, \quad u'(0) = -64, \quad u(1) = 0, \quad u'(1) = 0, \end{cases}$$

  where $f(x) = 24(15x^2 - 50x + 41)$. The exact solution is $u(x) = (x-2)^4(x-1)^2$.

- Its corresponding Galerkin formulation is

$$\sum_{k=1}^{2^{n+2}} B_{j,k} c_k = \langle g_j, f\rangle - u(0)\chi_{\{2,3\}}(j)\langle g_j'', g_1''\rangle - u'(0)\chi_{\{2,3\}}(j)\langle g_j'', g_4''\rangle,$$

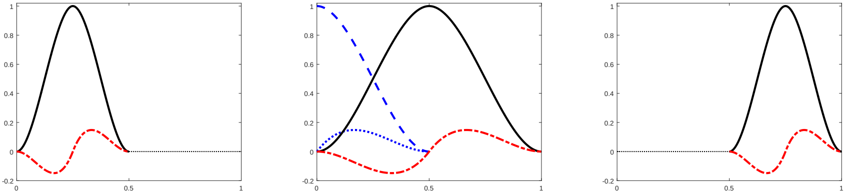
  with the coefficient matrix $B_{j,k} := \langle g_j'', g_k''\rangle$.

- According to boundary conditions, the boundary elements are

$$\phi^L := \phi_1|_{[0,1]}, \qquad \phi^R := \phi_2|_{[0,1]},$$
$$\psi^L := \emptyset, \qquad \psi^R := \emptyset.$$

Boundary Elements and Numerical Performance

Use the second-order derivative-orthogonal Riesz wavelet derived from Hermite cubic splines with $m = 2$.



Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H_1}$	$\log_2 \frac{\|e_{n-1}\|_{H^1}}{\|e_n\|_{H_1}}$
5	126	1	1.5129e-07	5.5402e-08	—	1.2283e-05	—
6	254	1	9.5006e-09	3.4413e-09	4.0089	1.5354e-06	3.0000
7	510	1	5.9521e-10	2.1642e-10	3.9910	1.9193e-07	3.0000
8	1022	1	3.7247e-11	1.3528e-11	3.9998	2.4004e-08	2.9992
9	2046	1	2.3306e-12	8.4696e-13	3.9975	3.1070e-09	2.9497

Table: Size of linear system, no iteration is needed, condition number κ of the matrices B (B is the identity matrix), errors, and convergence rates.

General Construction of Wavelets on $[0,1]$

Theorem

Let $\Phi = \{\phi^1, \dots, \phi^r\}$, $\Psi = \{\psi^1, \dots, \psi^s\}$, *and* $\tilde{\Phi} = \{\tilde{\phi}^1, \dots, \tilde{\phi}^r\}$, $\tilde{\Psi} = \{\tilde{\psi}^1, \dots, \tilde{\psi}^s\}$ *be sets of compactly supported functions in* $L_2(\mathbb{R})$*:*

$$\phi^\ell(c_\ell^\phi - \cdot) = \epsilon_\ell^\phi \phi^\ell, \quad \tilde{\phi}^\ell(c_\ell^\phi - \cdot) = \epsilon_\ell^\phi \tilde{\phi}^\ell \quad \textit{with} \quad c_\ell^\phi \in \mathbb{Z}, \epsilon_\ell^\phi \in \{-1, 1\},$$

$$\psi^\ell(c_\ell^\psi - \cdot) = \epsilon_\ell^\psi \psi^\ell, \; \tilde{\psi}^\ell(c_\ell^\psi - \cdot) = \epsilon_\ell^\psi \tilde{\psi}^\ell \quad \textit{with} \quad c_\ell^\psi \in \mathbb{Z}, \epsilon_\ell^\psi \in \{-1, 1\}.$$

For $\epsilon \in \{-1, 1\}$ *and* $c \in \mathbb{Z}$, *define* $\mathcal{I} := [\frac{c}{2}, \frac{c}{2} + 1]$. *For* $j \in \mathbb{N}_0$,

$$d_{j,\ell}^\phi := \lfloor 2^{j-1} c - 2^{-1} c_\ell^\phi \rfloor, \quad d_{j,\ell}^\psi := \lfloor 2^{j-1} c - 2^{-1} c_\ell^\psi \rfloor,$$
$$o_{j,\ell}^\phi := \operatorname{odd}(2^j c - c_\ell^\phi), \qquad o_{j,\ell}^\psi := \operatorname{odd}(2^j c - c_\ell^\psi),$$

where $\operatorname{odd}(m) := 1$ *if* m *is odd and* $\operatorname{odd}(m) := 0$ *if* m *is even. Let* $\chi_{\mathcal{I}}$ *denote the characteristic function of the interval* $\mathcal{I}$.

General Construction of Wavelets on $[0,1]$...

Theorem

For $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, define Ψ_j^ℓ to be

$$\begin{cases} \{F_{c,\epsilon}(\psi^\ell_{2^j;k})\chi_{\mathcal{I}} \;:\; k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j\}, & o^\psi_{j,\ell}=1,\\ \{F_{c,\epsilon}(\psi^\ell_{2^j;k})\chi_{\mathcal{I}} \;:\; k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\}, & o^\psi_{j,\ell}=0,\ \epsilon^\psi_\ell=-\epsilon,\\ \{F_{c,\epsilon}(\psi^\ell_{2^j;k})\chi_{\mathcal{I}} \;:\; k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\} & \\ \qquad \cup\{\frac{1}{\sqrt{2}}F_{c,\epsilon}(\psi^\ell_{2^j;d^\psi_{j,\ell}})\chi_{\mathcal{I}}, \frac{1}{\sqrt{2}}F_{c,\epsilon}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^j})\chi_{\mathcal{I}}\}, & o^\psi_{j,\ell}=0,\ \epsilon^\psi_\ell=\epsilon, \end{cases}$$

where $F_{c,\epsilon}(f) := \sum_{k\in\mathbb{Z}}(f(\cdot-2k)+\epsilon f(c+2k-\cdot))$, and define $\tilde{\Psi}_j^\ell$, $\Phi_j^\ell, \tilde{\Phi}_j^\ell$ similarly. For $J \in \mathbb{N}_0$, define

$$\mathscr{B}_J := (\cup_{\ell=1}^r \Phi_J^\ell) \cup \cup_{j=J}^\infty (\cup_{\ell=1}^s \Psi_j^\ell), \quad \tilde{\mathscr{B}}_J := (\cup_{\ell=1}^r \tilde{\Phi}_J^\ell) \cup \cup_{j=J}^\infty (\cup_{\ell=1}^s \tilde{\Psi}_j^\ell).$$

If $(\{\tilde{\Phi};\tilde{\Psi}\},\{\Phi;\Psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$, then $(\tilde{\mathscr{B}}_J, \mathscr{B}_J)$ is a pair of biorthogonal bases for $L_2(\mathcal{I})$ for all $J \in \mathbb{N}_0$. Moreover, $\mathscr{B}_J$ and $\tilde{\mathscr{B}}_J$ are Riesz bases for $L_2(\mathcal{I})$.

Wavelets on $[0, 1]$ with Simple Structure

Let ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ be compactly supported vector functions in $L_2(\mathbb{R})$ with symmetry and all the symmetry centers satisfy

$$c_1^{\phi} = \cdots = c_r^{\phi} = c_1^{\psi} = \cdots = c_s^{\psi} = 0.$$

In addition we assume that

all the elements/entries in ϕ and ψ vanish outside $[-1, 1]$.

Take $c = 0$. Since $d_{j,\ell}^{\phi} = d_{j,\ell}^{\psi} = o_{j,\ell}^{\phi} = o_{j,\ell}^{\psi} = 0$, Ψ_j^{ℓ} becomes

$$\begin{cases} \{\psi_{2^j;k}^{\ell} \ : \ k = 1, \ldots, 2^j - 1\}, & \epsilon_\ell^{\psi} = -\epsilon, \\ \{\psi_{2^j;k}^{\ell} \ : \ k = 1, \ldots, 2^j - 1\} \cup \{\sqrt{2}\psi_{2^j;0}^{\ell}\chi_{[0,1]}, \sqrt{2}\psi_{2^j;2^j}^{\ell}\chi_{[0,1]}\}, & \epsilon_\ell^{\psi} = \epsilon. \end{cases}$$

(i) If $\epsilon = -1$ and all entries in ϕ, ψ are continuous, then $h(0) = h(1) = 0$ for all $h \in \mathscr{B}_J$ (Dirichlet boundary condition).

(ii) If $\epsilon = 1$ and all entries in ϕ, ψ are in $\mathscr{C}^1(\mathbb{R})$, then $h'(0) = h'(1) = 0$ for all $h \in \mathscr{B}_J$ (von Neumann boundary).

Tight Framelet from Hermite Linear Splines

A tight framelet $\{\phi; \psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

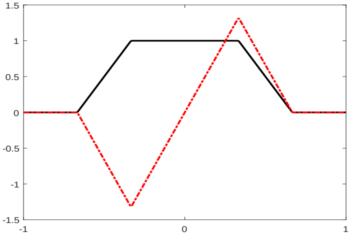
through a tight framelet filter bank $\{a; b\}$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{7}}{14} \\ -\frac{\sqrt{7}}{8} & -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{7}}{14} \\ \frac{\sqrt{7}}{8} & -\frac{1}{4} \end{bmatrix} \right\}_{[-1,1]},$$

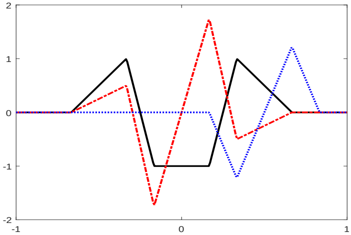
$$b = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{7}}{14} \\ \frac{1}{8} & \frac{\sqrt{7}}{28} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{7}}{4} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{7}}{14} \\ -\frac{1}{8} & \frac{\sqrt{7}}{28} \\ 0 & \frac{\sqrt{42}}{14} \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\{\phi; \psi\}$ generates a tight frame in $L_2(\mathbb{R})$ and has symmetry property.

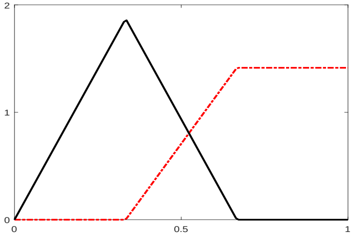
Tight Framelet in $L_2([0,1])$ from Linear Splines



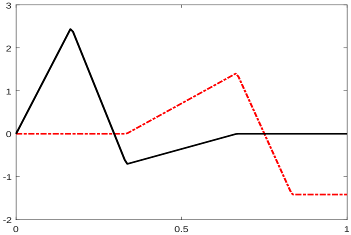
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2, \psi^3)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Tight Framelet from Hermite Quadratic Splines

A tight framelet $\{\phi;\psi\}$ is given by

$$\widehat{\phi}(2\xi)=\widehat{a}(\xi)\widehat{\phi}(\xi),\qquad \widehat{\psi}(2\xi)=\widehat{b}(\xi)\widehat{\phi}(\xi)$$

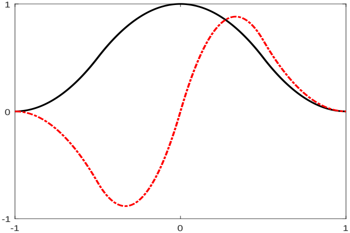
through a tight framelet filter bank $\{a;b\}$:

$$a=\left\{\begin{bmatrix}\frac{1}{4} & \frac{\sqrt{7}}{28}\\ -\frac{\sqrt{7}}{8} & -\frac{1}{8}\end{bmatrix},\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{4}\end{bmatrix},\begin{bmatrix}\frac{1}{4} & -\frac{\sqrt{7}}{28}\\ \frac{\sqrt{7}}{8} & -\frac{1}{8}\end{bmatrix}\right\}_{[-1,1]},$$

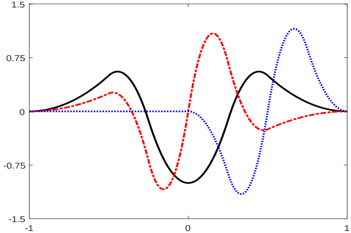
$$b=\left\{\begin{bmatrix}\frac{1}{4} & \frac{\sqrt{7}}{28}\\ \frac{1}{8} & \frac{\sqrt{7}}{56}\\ 0 & 0\end{bmatrix},\begin{bmatrix}-\frac{1}{2} & 0\\ 0 & \frac{\sqrt{7}}{4}\\ 0 & 0\end{bmatrix},\begin{bmatrix}\frac{1}{4} & -\frac{\sqrt{7}}{28}\\ -\frac{1}{8} & \frac{\sqrt{7}}{56}\\ 0 & \frac{\sqrt{21}}{7}\end{bmatrix}\right\}_{[-1,1]}.$$

Then $\{\phi;\psi\}$ generates a tight frame in $L_2(\mathbb{R})$ and has symmetry property.

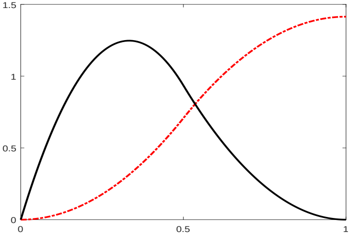
Tight Framelet in $L_2([0,1])$ from Quadratic Splines



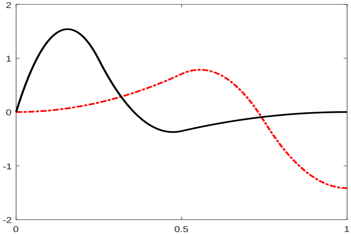
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2, \psi^3)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Riesz Wavelet from Hermite Cubic Splines

A Riesz wavelet $\{\phi; \psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

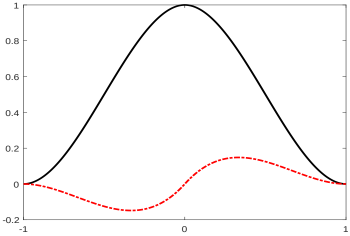
through a filter bank $\{a; b\}$ (part of a biorthogonal wavelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{16} & -\frac{1}{16} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ \frac{1}{16} & -\frac{1}{16} \end{bmatrix} \right\}_{[-1,1]},$$

$$b = \left\{ \begin{bmatrix} -\frac{1}{4} & -\frac{23}{24} \\ \frac{1}{16} & \frac{91}{176} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{37}{44} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} & \frac{23}{24} \\ -\frac{1}{16} & \frac{91}{176} \end{bmatrix} \right\}_{[-1,1]},$$

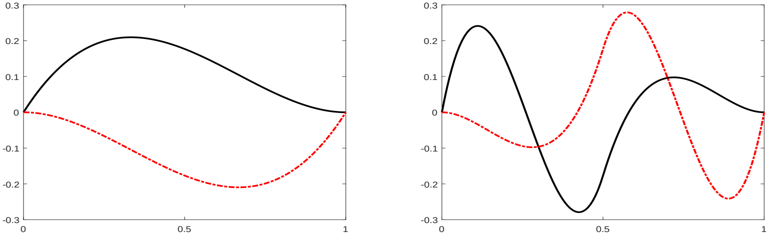
Then $\{\phi; \psi\}$ generates a Riesz wavelet in $L_2(\mathbb{R})$ and has symmetry property.

Riesz Wavelet in $L_2([0,1])$ from Cubic Splines



(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$

(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R

(d) ψ^L, ψ^R

Riesz Wavelet from B_2 Spline

A Riesz wavelet $\{\phi;\psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

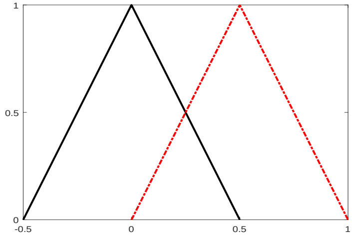
through a filter bank $\{a; b\}$ (part of a biorthogonal wavelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})$:

$$a = \left\{ \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \right\}_{[-1,1]},$$

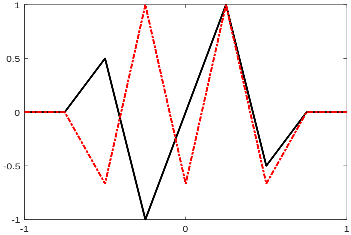
$$b = \left\{ \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} & 0 \\ -\frac{1}{3} & 0 \end{bmatrix} \right\}_{[-1,1]},$$

Then $\{\phi;\psi\}$ generates a Riesz wavelet in $L_2(\mathbb{R})$ and has symmetry property.

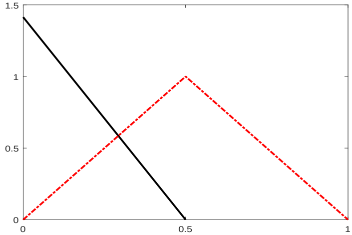
Riesz Wavelet in $L_2([0,1])$ from B_2 Spline



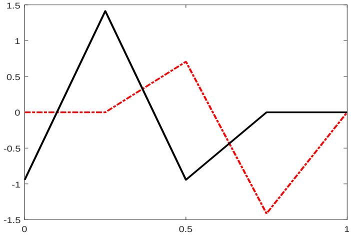
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Orthogonal Multiwavelet

An orthogonal wavelet $\{\phi; \psi\}$ (Gernimo-Hardin-Massopust) is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

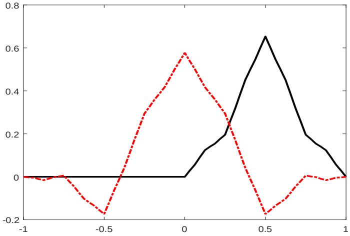
through a filter bank $\{a; b\}$ (part of a biorthogonal wavelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})$:

$$a = \left\{ \begin{bmatrix} \frac{3}{5} & \frac{4\sqrt{2}}{5} \\ -\frac{\sqrt{2}}{20} & -\frac{3}{10} \end{bmatrix}, \begin{bmatrix} \frac{3}{5} & 0 \\ \frac{9\sqrt{2}}{20} & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{9\sqrt{2}}{20} & -\frac{3}{10} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{20} & 0 \end{bmatrix} \right\}_{[0,3]},$$

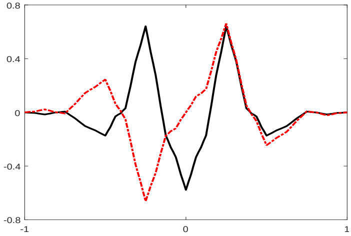
$$b = \left\{ \begin{bmatrix} -\frac{\sqrt{2}}{20} & -\frac{3}{10} \\ \frac{1}{10} & \frac{3\sqrt{2}}{10} \end{bmatrix}, \begin{bmatrix} \frac{9\sqrt{2}}{20} & -1 \\ -\frac{9}{10} & 0 \end{bmatrix}, \begin{bmatrix} \frac{9\sqrt{2}}{20} & -\frac{3}{10} \\ \frac{9}{10} & -\frac{3\sqrt{2}}{10} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{20} & 0 \\ -\frac{1}{10} & 0 \end{bmatrix} \right\}_{[0,3]}.$$

Then $\{\phi; \psi\}$ generates an orthogonal wavelet in $L_2(\mathbb{R})$ and has symmetry property.

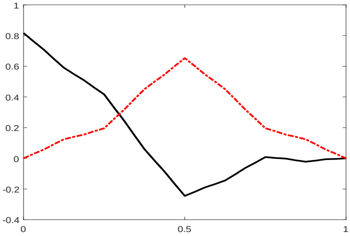
Orthogonal Multiwavelet in $L_2([0,1])$



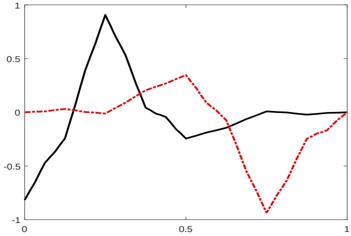
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Orthogonal Multiwavelet

An orthogonal wavelet $\{\phi; \psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

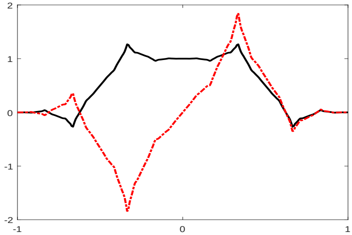
through a filter bank $\{a; b\}$ (part of a biorthogonal wavelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{-\sqrt{7}}{8} & \frac{-\sqrt{7}}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{\sqrt{7}}{8} & -\frac{\sqrt{7}}{8} \end{bmatrix} \right\}_{[-1,1]},$$

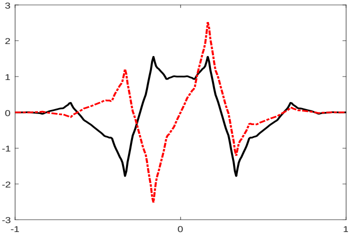
$$b = \left\{ \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{7}}{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{8} \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\{\phi; \psi\}$ generates an orthogonal wavelet in $L_2(\mathbb{R})$ and has symmetry property.

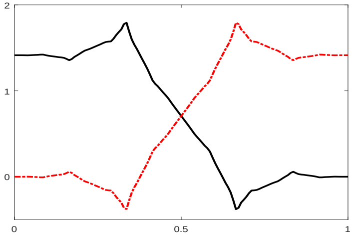
Orthogonal Multiwavelet in $L_2([0,1])$



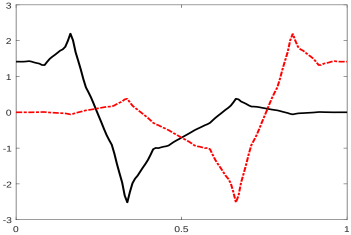
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Summary

- Riesz wavelets with short support and high vanishing moments are often used in wavelet application to numerical PDEs.
- To have short support and high vanishing moments, people often adopt Riesz multiwavelets derived from biorthogonal multiwavelets using matrix-valued filter banks.
- Wavelets on the real line have to be adapted into bounded intervals with prescribed boundary conditions.
- Then either Galerkin scheme or collocation method is used by using wavelet bases.
- Advantages of wavelet applications to PDEs:
  1. Uniformly bounded condition numbers.
  2. Sparse coefficient matrices for efficient computing.
  3. Adaptive wavelet numerical method can handle singularities in solutions of PDEs.
- Shortcomings: Not that easy to design wavelets with prescribed boundary conditions for general domains.