Basics on Wavelets and Framelets

Bin Han

Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, Canada

Present at 2021 International Undergraduate

Summer Enrichment Program at UofA

July 12–August 13, 2021

Outline of Mini-course Talks

- Part I: Multi-level fast wavelet/framelet transform:
  1. Basic properties: perfect reconstruction, sparsity, stability.
  2. Multi-level fast wavelet transform
  3. Different types of wavelets and framelets.
  4. Advantages of wavelets and framelets.
- Part II: Wavelet applications to signal and image processing:
  1. Wavelet/Framelet transform for signals on bounded intervals.
  2. Processing of wavelet coefficients.
  3. Wavelet applications to signal and image processing.

Declaration: Some figures and graphs in this talk are from the book [Bin Han, Framelets and Wavelets: Algorithms, Analysis and Applications, Birkhäuser/Springer, 2017] and various other sources from Internet, or from published papers, or produced by `matlab`, `maple`, or C programming. [Details and sources of all graphs can be provided upon request of the audience.]

Part I: Multi-level Fast wavelet/framelet transform

- $l(\mathbb{Z})$ for signals: all $v = \{v(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$.
- $l_0(\mathbb{Z})$ for filters: all finitely supported sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ on $\mathbb{Z}$.
- For $v = \{v(k)\}_{k\in\mathbb{Z}} \in l(\mathbb{Z})$, define $v^\star(k) := \overline{v(-k)}$ for $k \in \mathbb{Z}$ and

$$\widehat{v}(\xi) := \sum_{k\in\mathbb{Z}} v(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}.$$

- Convolution $u * v$ and inner product:

$$[u * v](n) := \sum_{k\in\mathbb{Z}} u(k)v(n-k), \quad n \in \mathbb{Z},$$

$$\langle v, w\rangle := \sum_{k\in\mathbb{Z}} v(k)\overline{w(k)}, \quad v, w \in l_2(\mathbb{Z})$$

- $\widehat{v^\star}(\xi) = \overline{\widehat{v}(\xi)}$ and $\widehat{u * v}(\xi) = \widehat{u}(\xi)\widehat{v}(\xi)$.

Subdivision and Transition Operators

- The subdivision operator $\mathcal{S}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$:

$$[\mathcal{S}_u v](n) := 2\sum_{k\in\mathbb{Z}} v(k)u(n-2k), \quad n \in \mathbb{Z}$$

- The transition operator $\mathcal{T}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$ is

$$[\mathcal{T}_u v](n) := 2\sum_{k\in\mathbb{Z}} v(k)\overline{u(k-2n)}, \quad n \in \mathbb{Z}.$$

- Upsampling operator $\uparrow \mathrm{d} : l(\mathbb{Z}) \to l(\mathbb{Z})$:

$$[v\uparrow \mathrm{d}](n) := v(n/\mathrm{d}) \text{ if } n/\mathrm{d} \in \mathbb{Z} \quad \text{and} \quad [v\uparrow \mathrm{d}](n) = 0 \text{ otherwise.}$$

- Downsampling (or decimation): $[v\downarrow \mathrm{d}](n) := v(\mathrm{d}n)$, $n \in \mathbb{Z}$.
- Subdivision and transition operators:

$$\mathcal{S}_u v = 2u * (v\uparrow 2) \quad \text{and} \quad \mathcal{T}_u v = 2(u^\star * v)\downarrow 2.$$

Discrete Framelet Transform (DFrT)

- Let $u_0, \dots, u_s, \tilde{u}_0, \dots, \tilde{u}_s \in l_0(\mathbb{Z})$ be filters.
- For data $v \in l(\mathbb{Z})$, a 1-level framelet decomposition:

$$w_\ell := \frac{\sqrt{2}}{2} \mathcal{T}_{u_\ell} v = \sqrt{2}(u_\ell^\star * v)\downarrow 2, \qquad \ell = 0, \dots, s,$$

or using a framelet decomposition operator:

$$\mathcal{W} v = \frac{\sqrt{2}}{2}(\mathcal{T}_{u_0} v, \dots, \mathcal{T}_{u_s} v) = \sqrt{2}(u_0^\star * v, \dots, u_s^\star * v)\downarrow 2.$$

- The total number of nonzero coefficients of w_ℓ is approximately half of that of v. Hence, $\mathcal{W} v$ is roughly $\frac{1}{2}(s+1)$ of that of v.
- A one-level framelet reconstruction by $\mathcal{V} : (l(\mathbb{Z}))^{1\times(s+1)} \to l(\mathbb{Z})$:

$$\mathcal{V}(w_0, \dots, w_s) = \frac{\sqrt{2}}{2} \sum_{\ell=0}^{s} \mathcal{S}_{\tilde{u}_\ell} w_\ell = \sqrt{2}\tilde{u}_0 * (w_0 \uparrow 2) + \cdots + \sqrt{2}\tilde{u}_s * (w_s \uparrow 2).$$

- A filter bank $(\{u_0, \dots, u_s\}, \{\tilde{u}_0, \dots \tilde{u}_s\})$ has the perfect reconstruction (PR) if $\mathcal{V}\mathcal{W} v = v$ for all data v.

Diagram of 1-level DFrTs

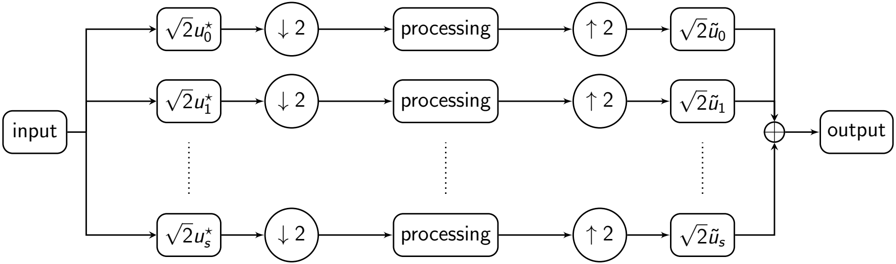


Figure: Diagram of a one-level discrete framelet transform using a dual framelet filter bank $(\{u_0, \dots, u_s\}, (\tilde{u}_0, \dots, \tilde{u}_s\})$. Decomposition: $\sqrt{2}(u_\ell^\star * v){\downarrow}2$. Reconstruction: $\sqrt{2}\tilde{u}_0 * (w_0{\uparrow}2) + \cdots + \sqrt{2}\tilde{u}_s * (w_s{\uparrow}2)$.

It is called a wavelet filter bank for $s = 1$, and a framelet filter bank if $s > 1$.

Property of DFrT: Perfect Reconstruction (PR)

Theorem: A filter bank $(\{u_0, \dots, u_s\}, \{\tilde{u}_0, \dots, \tilde{u}_s\})$ has

Perfect Reconstruction (PR): $\quad v = \mathcal{V}\mathcal{W}v = \frac{1}{2}\sum_{\ell=0}^{s} \mathcal{S}_{\tilde{u}_\ell}\mathcal{T}_{u_\ell} v, \quad \forall\, v \in l(\mathbb{Z})$

$\iff$ $(\{u_0, \dots, u_s\}, \{\tilde{u}_0, \dots, \tilde{u}_s\})$ is a dual framelet filter bank:

$$\overline{\widehat{u_0}(\xi)}\widehat{\tilde{u}_0}(\xi) + \overline{\widehat{u_1}(\xi)}\widehat{\tilde{u}_1}(\xi) + \cdots + \overline{\widehat{u_s}(\xi)}\widehat{\tilde{u}_s}(\xi) = 1,$$
$$\overline{\widehat{u_0}(\xi+\pi)}\widehat{\tilde{u}_0}(\xi) + \overline{\widehat{u_1}(\xi+\pi)}\widehat{\tilde{u}_1}(\xi) + \cdots + \overline{\widehat{u_s}(\xi+\pi)}\widehat{\tilde{u}_s}(\xi) = 0.$$

That is,

$$\begin{bmatrix} \widehat{\tilde{u}_0}(\xi) & \cdots & \widehat{\tilde{u}_s}(\xi) \\ \widehat{\tilde{u}_0}(\xi+\pi) & \cdots & \widehat{\tilde{u}_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{u_0}(\xi) & \cdots & \widehat{u_s}(\xi) \\ \widehat{u_0}(\xi+\pi) & \cdots & \widehat{u_s}(\xi+\pi) \end{bmatrix}^{\star} = I_2,$$

where I_2 denotes the 2×2 identity matrix and $A^{\star} := \overline{A}^{\mathsf{T}}$.

Biorthogonal Wavelet Filter Bank

A dual framelet filter bank with $s = 1$ is called a biorthogonal wavelet filter bank, a nonredundant filter bank.

Proposition

Let $(\{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots, \tilde{u}_s\})$ *be a dual framelet filter bank. Then the following statements are equivalent:*

(i) $\mathcal{W}$ *is onto or* $\mathcal{V}$ *is one-one;*

(ii) $\mathcal{V}\mathcal{W} = \mathrm{Id}_{l(\mathbb{Z})}$ *and* $\mathcal{W}\mathcal{V} = \mathrm{Id}_{(l(\mathbb{Z}))^{1\times(s+1)}}$, *that is,* $\mathcal{V}$ *and* $\mathcal{W}$ *are inverse operators to each other;*

(iii) $s = 1$.

$$\begin{bmatrix} \widehat{\tilde{u}_0}(\xi) & \widehat{\tilde{u}_1}(\xi) \\ \widehat{\tilde{u}_0}(\xi+\pi) & \widehat{\tilde{u}_1}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{u_0}(\xi) & \widehat{u_1}(\xi) \\ \widehat{u_0}(\xi+\pi) & \widehat{u_1}(\xi+\pi) \end{bmatrix}^{\star} = I_2.$$

Role of $\frac{\sqrt{2}}{2}$ in DFrT

Theorem

Let $u_0, \dots, u_s \in l_0(\mathbb{Z})$. Then the following are equivalent:

(i) $\|\mathcal{W}v\|^2_{(l_2(\mathbb{Z}))^{1\times(s+1)}} = \|v\|^2_{l_2(\mathbb{Z})}$ *for all* $v \in l_2(\mathbb{Z})$, *that is,*

$$\|\tfrac{\sqrt{2}}{2}\mathcal{T}_{u_0}v\|^2_{l_2(\mathbb{Z})} + \cdots + \|\tfrac{\sqrt{2}}{2}\mathcal{T}_{u_s}v\|^2_{l_2(\mathbb{Z})} = \|v\|^2_{l_2(\mathbb{Z})}, \quad \forall\, v \in l_2(\mathbb{Z});$$

(ii) $\langle \mathcal{W}v, \mathcal{W}\tilde{v}\rangle = \langle v, \tilde{v}\rangle$ *for all* $v, \tilde{v} \in l_2(\mathbb{Z})$;

(iii) *the filter bank* $(\{u_0, \dots, u_s\}, \{u_0, \dots, u_s\})$ *has PR:*

$$\begin{bmatrix} \widehat{u_0}(\xi) & \cdots & \widehat{u_s}(\xi) \\ \widehat{u_0}(\xi+\pi) & \cdots & \widehat{u_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{u_0}(\xi) & \cdots & \widehat{u_s}(\xi) \\ \widehat{u_0}(\xi+\pi) & \cdots & \widehat{u_s}(\xi+\pi) \end{bmatrix}^\star = I_2,$$

Definition: $\{u_0, \dots, u_s\}$ with PR is called a tight framelet filter bank. If in addition $s = 1$, it is called an orthogonal wavelet filter bank.

Examples of Wavelet Filter Banks

To list a filter $u = \{u(k)\}_{k\in\mathbb{Z}}$ with support $[m, n]$,

$$u = \{u(m), \ldots, u(-1), \underline{\mathbf{u(0)}}, u(1), \ldots, u(n)\}_{[m,n]},$$

- $\{u_0, u_1\}$ is the Haar orthogonal wavelet filter bank:

$$u_0 = \{\underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{2}\}_{[0,1]}, \qquad u_1 = \{\underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, -\tfrac{1}{2}\}_{[0,1]}.$$

- $(\{u_0, u_1\}, \{\tilde{u}_0, \tilde{u}_1\})$ is a biorthogonal wavelet filter bank, where

$$u_0 = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\tfrac{\mathbf{3}}{\mathbf{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}, \qquad u_1 = \{\underline{\tfrac{\mathbf{1}}{\mathbf{4}}}, -\tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]},$$
$$\tilde{u}_0 = \{\tfrac{1}{4}, \underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \qquad \tilde{u}_1 = \{\tfrac{1}{8}, \underline{\tfrac{\mathbf{1}}{\mathbf{4}}}, -\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{8}\}_{[-1,3]}.$$

Called the LeGall wavelet filter bank, which is often used for lossless and lossy image compression.

Illustration: I

Apply the Haar orthogonal filter bank to

$$v = \{-21, -22, -23, -23, -25, 38, 36, 34\}_{[0,7]}$$

Note that

$$[\mathcal{T}_{u_0} v](n) = v(2n) + v(2n+1), \quad [\mathcal{T}_{u_1} v](n) = v(2n) - v(2n+1), \ n \in \mathbb{Z}.$$

We have the wavelet coefficients:

$$w_0 = \tfrac{\sqrt{2}}{2}\{-42, -46, 13, 70\}_{[0,3]}, \quad w_1 = \tfrac{\sqrt{2}}{2}\{-1, 0, -63, -2\}_{[0,3]}.$$

Note that

$$[\mathcal{S}_{u_0} w_0](2n) = w_0(n), \ [\mathcal{S}_{u_0} w_0](2n+1) = w_0(n), \ n \in \mathbb{Z}$$
$$[\mathcal{S}_{u_1} w_1](2n) = w_1(n), \ [\mathcal{S}_{u_1} w_1](2n+1) = -w_1(n), \ n \in \mathbb{Z}.$$

Illustration: II

Hence, we have

$$\frac{\sqrt{2}}{2}\mathcal{S}_{u_0} w_0 = \frac{1}{2}\{-43, -43, -46, -46, 13, 13, 70, 70\}_{[0,7]},$$
$$\frac{\sqrt{2}}{2}\mathcal{S}_{u_1} w_1 = \frac{1}{2}\{1, -1, 0, 0, -63, 63, 2, -2\}_{[0,7]}.$$

Clearly, we have the perfect reconstruction of v:

$$\frac{\sqrt{2}}{2}\mathcal{S}_{u_0} w_0 + \frac{\sqrt{2}}{2}\mathcal{S}_{u_1} w_1 = \{-21, -22, -23, -23, -25, 38, 36, 34\}_{[0,7]} = v$$

and the following energy-preserving identity

$$\|w_0\|^2_{l_2(\mathbb{Z})} + \|w_1\|^2_{l_2(\mathbb{Z})} = 4517 + 1987 = 6504 = \|v\|^2_{l_2(\mathbb{Z})}.$$

Property of DFrT: Sparsity

- One key feature of DFrT is its sparse representation for smooth or piecewise smooth signals.
- It is desirable to have as many as possible negligible framelet coefficients for smooth signals.
- Smooth signals are modeled by polynomials. Let $\mathsf{p} : \mathbb{R} \to \mathbb{C}$ be a polynomial: $\mathsf{p}(x) = \sum_{n=0}^{m} p_n x^n$.
- a polynomial sequence $\mathsf{p}|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ by $[\mathsf{p}|_{\mathbb{Z}}](k) = \mathsf{p}(k), k \in \mathbb{Z}$.
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- Π_{m-1} is the set of all polynomials of degree less than m.

Transition Operator Acting on Polynomials

Theorem

Let $u \in l_0(\mathbb{Z})$. *Then for any polynomial* $\mathsf{p} \in \Pi$,

$$\mathcal{T}_u \mathsf{p} = 2[\mathsf{p} * u^\star](2\cdot) = \mathsf{p}(2\cdot) * \mathring{u} = \sum_{n=0}^{\infty} \frac{2(-i)^n}{n!} \mathsf{p}^{(n)}(2\cdot)\overline{\widehat{u}^{(n)}(0)},$$

where $\mathring{u}$ *is a finitely supported sequence on* $\mathbb{Z}$ *such that*

$$\widehat{\mathring{u}}(\xi) = 2\overline{\widehat{u}(\xi/2)} + \mathscr{O}(|\xi|^{\deg(\mathsf{p})+1}), \qquad \xi \to 0.$$

In particular, for any $m \in \mathbb{N}$, *the following are equivalent:*

1. $\mathcal{T}_u \mathsf{p} = 0$ *for all polynomial sequences* $\mathsf{p} \in \Pi_{m-1}$;
2. $\widehat{u}(\xi) = \mathscr{O}(|\xi|^m)$ *as* $\xi \to 0$: $\widehat{u}(0) = \widehat{u}'(0) = \cdots = \widehat{u}^{(m-1)}(0) = 0$.
3. $\widehat{u}(\xi) = (1 - e^{-i\xi})^m \mathbf{Q}(\xi)$ *for some* 2π*-periodic trigonometric polynomial* $\mathbf{Q}$.

Vanishing Moments

- We say that a filter u has m vanishing moments if any of items (1)–(4) in Theorem holds, that is,

$$\widehat{u}(0) = \widehat{u}'(0) = \cdots = \widehat{u}^{(m-1)}(0) = 0$$

or equivalently,

$$\langle u, (\cdot)^j \rangle = \sum_{k \in \mathbb{Z}} u(k) k^j = 0, \qquad j = 0, \ldots, m-1.$$

- Most framelet coefficients are zero for any input signal which is a polynomial to certain degree.
- If u has m vanishing moments. For a signal v, if v agrees with some polynomial of degree less than m on the support of $u(\cdot - 2n)$, then $[\mathcal{T}_u v](n) = \langle v, u(\cdot - 2n) \rangle = 0$.

Subdivision Operator on Polynomials

Theorem

Let $u = \{u(k)\}_{k\in\mathbb{Z}}$*. For* $m \in \mathbb{N}$*, the following are equivalent:*

1. $\mathcal{S}_u \Pi_{m-1} \subseteq \Pi_{m-1}$;
2. u *has m sum rules:* $\widehat{u}(\xi+\pi) = \mathscr{O}(|\xi|^m)$ *as* $\xi \to 0$*, i.e.,*

$$\widehat{u}(\pi) = \widehat{u}'(\pi) = \cdots = \widehat{u}^{(m-1)}(\pi) = 0.$$

3. $\widehat{u}(\xi) = (1+e^{-i\xi})^m \mathbf{Q}(\xi)$ *for some* 2π*-periodic* $\mathbf{Q}$;
4. u *has order m sum rules:*

$$\sum_{k\in\mathbb{Z}} u(2k)(2k)^j = \sum_{k\in\mathbb{Z}} u(1+2k)(1+2k)^j, \quad j = 0, \ldots, m-1.$$

Moreover, $\mathcal{S}_u \mathsf{p} = 2^{-1}\mathsf{p}(2^{-1}\cdot) * u$.

Multi-level Fast Framelet Transform (FFrT)

- Let $\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}$ and $\{a; b_1, \ldots, b_s\}$ be filters in $l_0(\mathbb{Z})$.
- For a positive integer J, a J-level discrete framelet decomposition is given by: For $j = J, \ldots, 1$,

$$v_{j-1} := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{a}} v_j, \qquad w_{j-1;\ell} := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{b}_\ell} v_j, \quad \ell = 1, \ldots, s.$$

  where $v_J : \mathbb{Z} \to \mathbb{C}$ is an input signal.
- $\mathcal{W}_J v_J := (w_{J-1;1}, \ldots, w_{J-1;s}, \ldots, w_{0;1}, \ldots, w_{0;s}, v_0)$.
- a J-level discrete framelet reconstruction is

$$v_j := \frac{\sqrt{2}}{2}\mathcal{S}_a v_{j-1} + \frac{\sqrt{2}}{2}\sum_{\ell=1}^{s}\mathcal{S}_{b_\ell} w_{j-1;\ell}, \quad j = 1, \ldots, J.$$

- $\mathcal{V}_J(w_{J-1;1}, \ldots, w_{J-1;s}, \ldots, w_{0;1}, \ldots, w_{0;s}, v_0) = v_J$.
- A fast framelet transform with $s = 1$ is called a fast wavelet transform.

Diagram of Multi-level FFrT

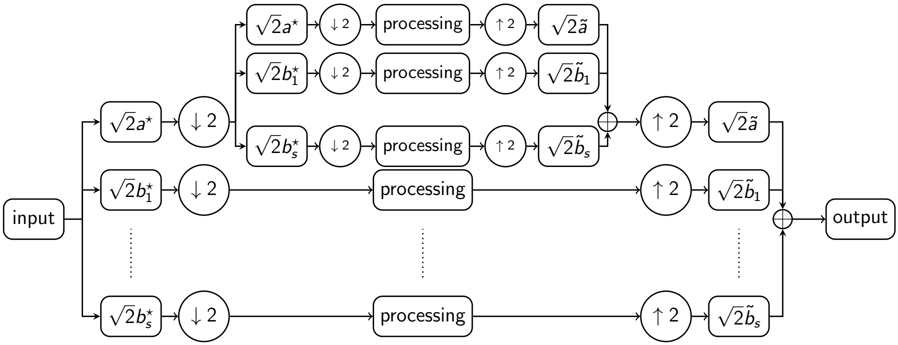


Figure: Diagram of a two-level discrete framelet transform using a dual framelet filter bank $(\{a; b_1, \ldots, b_s\}, (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})$.

Property of FFrT: Stability

Definition: A multi-level discrete framelet transform employing a dual framelet filter bank $\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})$ has stability in the space $l_2(\mathbb{Z})$ if there exists $C > 0$ such that for $J \in \mathbb{N}_0$,

$$\|\mathcal{W}_J v\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}} \leqslant C\|v\|_{l_2(\mathbb{Z})}, \qquad \forall\, v \in l_2(\mathbb{Z}),$$
$$\|\mathcal{V}_J \vec{w}\|_{l_2(\mathbb{Z})} \leqslant C\|\vec{w}\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}}, \qquad \forall\, \vec{w} \in (l_2(\mathbb{Z}))^{1\times(sJ+1)}.$$

Theorem: Let $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})$ be a dual framelet filter bank with $\widehat{a}(0) = \widehat{\tilde{a}}(0) = 1$. Define

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \quad \widehat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\tilde{a}}(2^{-j}\xi), \qquad \xi \in \mathbb{R}.$$

Then a multi-level discrete framelet transform has stability in the space $l_2(\mathbb{Z}) \iff \phi, \tilde{\phi} \in L_2(\mathbb{R})$ and

$$\widehat{b_1}(0) = \cdots = \widehat{b_s}(0) = \widehat{\tilde{b}_1}(0) = \cdots = \widehat{\tilde{b}_s}(0) = 0.$$

Theory of Discrete Wavelets and Framelets

- J-level discrete affine system $\mathrm{DAS}_J(\{a; b_1, \ldots, b_s\}) :=:$

  $$\{a_{J;k} \ : \ k \in \mathbb{Z}\} \cup \{b_{\ell,j;k} \ : \ k \in \mathbb{Z}, \ell = 1, \ldots, s, j = 1, \ldots, J\},$$

  where $a_{j;k} := a_j(\cdot - 2^j k)$ and $b_{\ell,j;k} := b_{\ell,j}(\cdot - 2^j k)$ with

  $$\widehat{a_j} := 2^{j/2}\widehat{a}(\cdot) \cdots \widehat{a}(2^{j-1}\cdot), \ \widehat{b_{\ell,j}} := 2^{j/2}\widehat{a}(\cdot) \cdots \widehat{a}(2^{j-2}\cdot)\widehat{b_\ell}(2^{j-1}\cdot).$$

- $a_j = (\frac{\sqrt{2}}{2})^j \mathcal{S}_a^j \boldsymbol{\delta}$ and $b_{\ell,j} = (\frac{\sqrt{2}}{2})^{j-1} \mathcal{S}_a^{j-1} \mathcal{S}_{b_\ell} \boldsymbol{\delta}$ with $\widehat{\boldsymbol{\delta}}(\xi) = 1$.
- A J-level fast framelet transform for $v \in l_2(\mathbb{Z})$ becomes

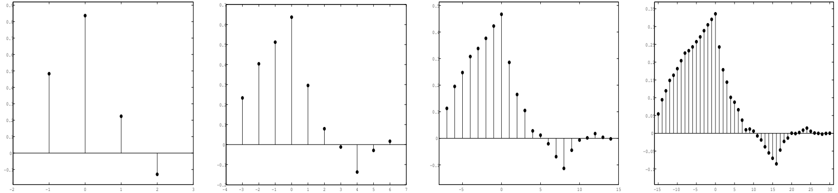
  $$v = \sum_{k \in \mathbb{Z}} \langle v, a_{J;k} \rangle a_{J;k} + \sum_{j=1}^{J} \sum_{\ell=1}^{s} \sum_{k \in \mathbb{Z}} \langle v, b_{\ell,j;k} \rangle b_{\ell,j;k}.$$

- $\mathrm{DAS}_J(\{a; b_1, \ldots, b_s\})$ is a tight frame for $l_2(\mathbb{Z})$ $\iff$ $\{a; b_1, \ldots, b_s\}$ is a tight framelet filter bank.

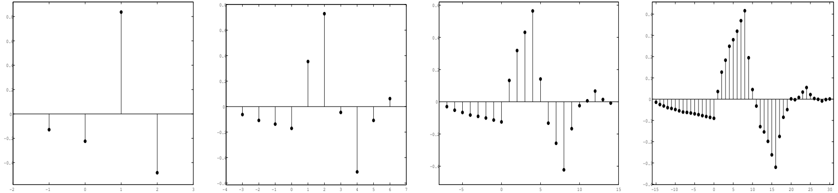
- $\mathrm{DAS}_J(\{a; b\})$ is an orthonormal basis for $l_2(\mathbb{Z})$ $\iff$ $\{a; b\}$ is an orthogonal wavelet filter bank.

An Example: Daubechies Orthogonal Wavelets

$$a = \{\tfrac{1+\sqrt{3}}{8}, \underline{\tfrac{3+\sqrt{3}}{8}}, \tfrac{3-\sqrt{3}}{8}, \tfrac{1-\sqrt{3}}{8}\}, \qquad b = \{-\tfrac{1-\sqrt{3}}{8}, \underline{\tfrac{3-\sqrt{3}}{8}}, -\tfrac{3+\sqrt{3}}{8}, \tfrac{1+\sqrt{3}}{8}\}.$$



(a) a_1 (b) a_2 (c) a_3 (d) a_4



(e) b_1 (f) b_2 (g) b_3 (h) b_4

Links between Discrete and Continuum Settings

- Let $a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$ with $\widehat{a}(0) = \sum_{k\in\mathbb{Z}} a(k) = 1$.
- The refinable function $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ satisfying

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2x-k) \quad \text{i.e.,} \quad \widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi).$$

- Wavelet functions $\widehat{\psi_\ell}(\xi) := \widehat{b_\ell}(\xi/2)\widehat{\phi}(\xi/2)$, i.e., $\psi_\ell = 2\sum_{k\in\mathbb{Z}} b_\ell(k)\phi(2\cdot -k)$.
- Links between discrete and continuum settings:

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |(\mathcal{S}_a^n \boldsymbol{\delta})(k) - \phi(2^{-j}k)| = 0,$$

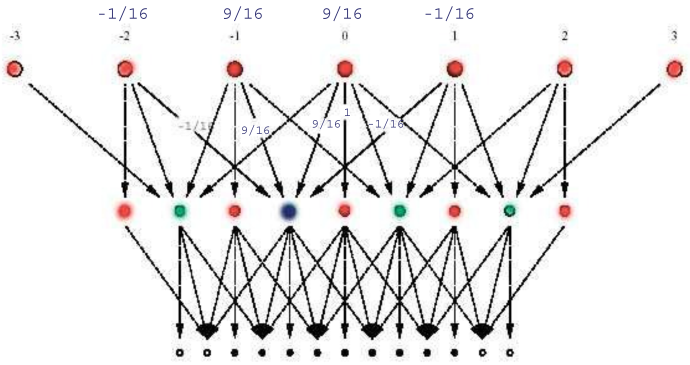
$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |(\mathcal{S}_a^{n-1}\mathcal{S}_{b_\ell} \boldsymbol{\delta})(k) - \psi_\ell(2^{-j}k)| = 0.$$

- $\{a; b_1, \ldots, b_s\}$ is a tight framelet filter bank $\Longleftrightarrow$ $\mathrm{DAS}_J(\{a; b_1, \ldots, b_s\})$ is a tight frame for $l_2(\mathbb{Z})$ $\Longleftrightarrow$ $\{\phi; \psi_1, \ldots, \psi_s\}$ is a tight framelet in $L_2(\mathbb{R})$.

Subdivision Curves in Computer Graphics

- The subdivision operator is used in the reconstruction algorithm of the fast wavelet/framelet transform.
- The effectiveness of wavelets and framelets can be explained through their prediction power of subdivision schemes.
- Let $v : \mathbb{Z} \to \mathbb{R}^2$ or $\mathbb{R}^3$ be given initial 2D or 3D curves outlining the rough shape of the curve.
- Write $v = (v^1, v^2, v^3)$ with sequences $v^1, v^2, v^3 : \mathbb{Z} \to \mathbb{R}$.
- Apply the subdivision operator to each entry of v to obtain $\mathcal{S}_a^n v := (\mathcal{S}_a^n v^1, \mathcal{S}_a^n v^2, \mathcal{S}_a^n v^3)$. Then plot the curve $\mathcal{S}_a^n v$.
- When n is large (usually, the choice of $n = 4$ to 8 is sufficient), it gives a subdivision curve.
- Different choices of filters (called masks in computer aided geometric design) affect the shapes of subdivision curves.

The 4-point Interpolatory Subdivision Scheme

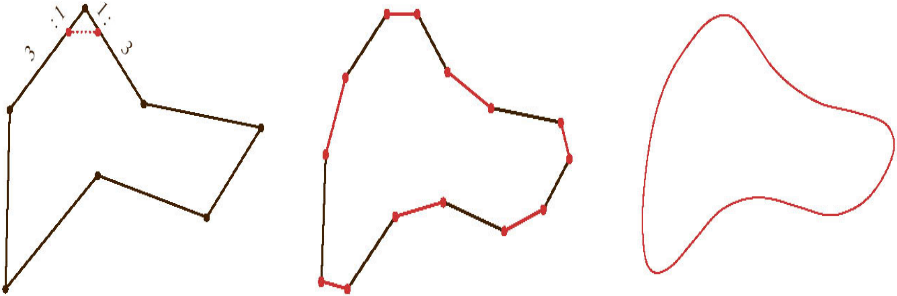


- The mask is the Deslauriers-Dubuc interpolatory mask

$a^I_4 = [-\frac{1}{32}, 0, \frac{9}{32}, \frac{1}{2}, \frac{9}{32}, 0, -\frac{1}{32}]$.

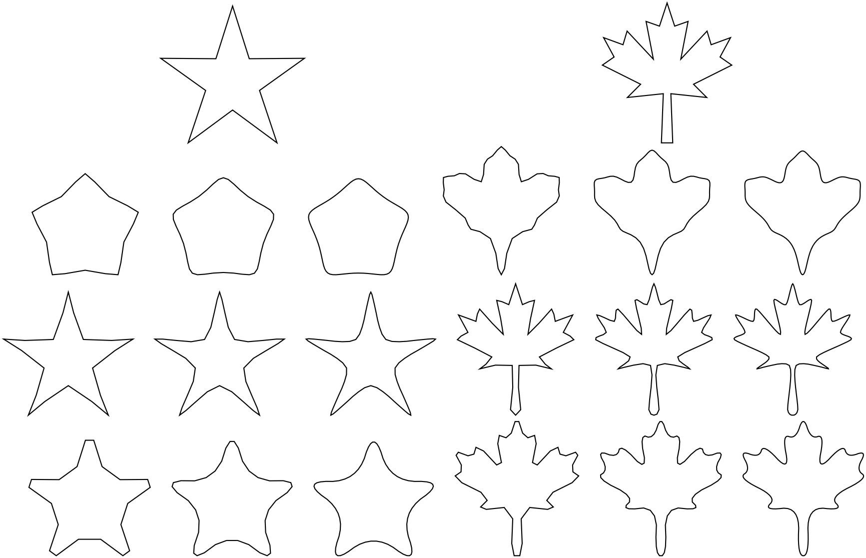
Even stencil $[1]$ and odd stencil $[-\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, -\frac{1}{16}]$.

The Corner Cutting Scheme



- The mask is the cubic B-spline filter of order 4 $a_4^B = \{\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\}_{[-1,2]}$.

Example of Subdivision Curves



Some Basics on Wavelet Theory in $L_2(\mathbb{R})$

- For $\phi, \psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$, define an affine system as

$$\mathrm{AS}(\phi; \psi^1, \ldots, \psi^s) := \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi^\ell_{2^j;k} := 2^{j/2}\psi^\ell(2^j \cdot - k) \ : \ j \geqslant 0, k \in \mathbb{Z}, \ell = 1, \ldots, s\}.$$

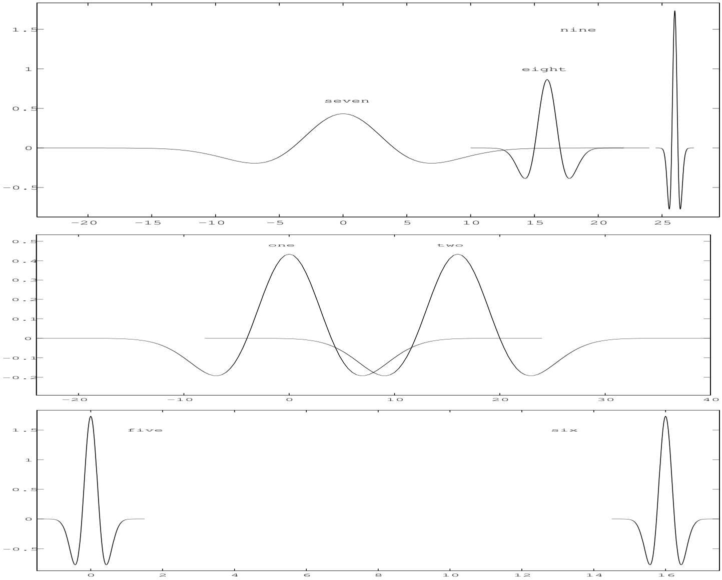
- $\{\phi; \psi^1, \ldots, \psi^s\}$ is called an orthonormal wavelet in $L_2(\mathbb{R})$ if $\mathrm{AS}(\phi; \psi^1, \ldots, \psi^s)$ is an orthonormal basis of $L_2(\mathbb{R})$.
- $\{\phi; \psi^1, \ldots, \psi^s\}$ is called a tight framelet in $L_2(\mathbb{R})$ if $\mathrm{AS}(\phi; \psi^1, \ldots, \psi^s)$ is a tight frame for $L_2(\mathbb{R})$:

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{2^j;k}\rangle|^2.$$

- Wavelet representation of functions:

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{2^j;k}\rangle \psi^\ell_{j;k}.$$

Dilates and Shifts of Affine Systems



Why Wavelets?

A wavelet ψ often has

1. compact support $\Rightarrow$ good spatial localization.
2. high smoothness/regularity $\Rightarrow$ good frequency localization.
3. high vanishing moments $\Rightarrow$ multiscale sparse representation. That is, most wavelet coefficients are small for smooth functions/signals.
4. associated filter banks $\Rightarrow$ fast wavelet transform to compute coefficients $\langle f, \psi_{2^j;k}\rangle$ through filter banks.
5. singularities of signals and their locations can be captured in large wavelet coefficients.
6. function spaces (Sobolev and Besov spaces) can be characterized by wavelets. This is important in harmonic analysis and numerical PDEs.

Explanation for Sparse Representation

- A wavelet function ψ has m vanishing moments if

$$\int_{\mathbb{R}} x^n \psi(x) dx = 0, \qquad n = 0, \dots, m-1.$$

- The multiscale wavelet representation of $f \in L_2(\mathbb{R})$ is

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \langle f, \psi_{2^j;k}\rangle \psi_{2^j;k} \quad \text{with} \quad \psi_{2^j;k}(x) := 2^{j/2}\psi(2^j x - k).$$

- $\operatorname{supp}\psi_{2^j;k} = 2^{-j}k + 2^{-j}\operatorname{supp}\psi \approx 2^{-j}k$ when $j \to \infty$.
- Wavelet coefficient $\langle f, \psi_{2^j;k}\rangle$ only depends f in the support of $\psi_{2^j;k}$. If f is smooth and can be well approximated by a polynomial P of degree $< m$, then

$$\langle f, \psi_{2^j;k}\rangle = \langle f - P, \psi_{2^j;k}\rangle \approx 0.$$

- If $\langle f, \psi_{2^j;k}\rangle$ is large for large j, we know the position of singularity, since $\operatorname{supp}\psi_{2^j\ k} = 2^{-j}\operatorname{supp}\psi + 2^{-j}k \approx 2^{-j}k$.

Wavelets/Framelets and Filter Banks

Theorem: Let $a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$ with $\widehat{a}(0) = 1$. Define

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi), \quad \widehat{\psi^{\ell}}(\xi) := \widehat{b_\ell}(\xi/2)\widehat{\phi}(\xi/2), \qquad \ell = 1, \ldots, s.$$

- Then $\{\phi; \psi^1, \ldots, \psi^s\}$ is a tight framelet in $L_2(\mathbb{R})$ $\iff$ $\{a; b_1, \ldots, b_s\}$ is a tight framelet filter bank:

$$|\widehat{a}(\xi)|^2 + \sum_{\ell=1}^{s} |\widehat{b_\ell}(\xi)|^2 = 1, \quad \widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} + \sum_{\ell=1}^{s} \widehat{b_\ell}(\xi)\overline{\widehat{b_\ell}(\xi+\pi)} = 0.$$

- Then $\{\phi; \psi^1\}$ is an orthogonal wavelet in $L_2(\mathbb{R})$ $\iff$ $s = 1$, $\{a; b_1\}$ is an orthogonal wavelet filter bank, and

$$[\widehat{\phi}, \widehat{\phi}](\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + 2\pi k)|^2 = 1.$$

Types of Wavelet and Framelet Filter Banks

Proposition: Let $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ be a dual framelet filter bank. Then the following statements are equivalent:

(i) Either $\mathcal{W}$ or $\mathcal{V}$ is bijective with $\mathcal{W} := \mathcal{W}_1$ and $\mathcal{V} := \mathcal{V}_1$;

(ii) $\mathcal{V}\mathcal{W} = \mathrm{Id}_{l_2(\mathbb{Z})}$ and $\mathcal{W}\mathcal{V} = \mathrm{Id}_{(l_2(\mathbb{Z}))^{1\times(s+1)}}$, that is, $\mathcal{V}^{-1} = \mathcal{W}$;

(iii) $s = 1$. [This is called a biorthogonal wavelet filter bank.]

Theorem: Let $a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$. Then the following statements are equivalent:

(i) $\|\mathcal{W}v\|^2_{(l_2(\mathbb{Z}))^{1\times(s+1)}} = \|v\|^2_{l_2(\mathbb{Z})}$ for all $v \in l_2(\mathbb{Z})$.

(ii) $\{a; b_1, \ldots, b_s\}$ is a tight framelet filter bank:

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix}^{\star} = I_2,$$

A tight framelet filter bank with $s = 1$ is called an orthogonal wavelet filter bank. That is, a framelet filter bank has redundancy with $s > 1$.

Wavelet and Framelets Filter Banks

- Perfect reconstruction for a dual framelet filter bank $(\{a; b_1, \dots, b_s\}, \{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\})$:

$$\overline{\widehat{a}(\xi)}\widehat{\tilde{a}}(\xi) + \overline{\widehat{b_1}(\xi)}\widehat{\tilde{b}_1}(\xi) + \cdots + \overline{\widehat{b_s}(\xi)}\widehat{\tilde{b}_s}(\xi) = 1,$$
$$\overline{\widehat{a}(\xi+\pi)}\widehat{\tilde{a}}(\xi) + \overline{\widehat{b_1}(\xi+\pi)}\widehat{\tilde{b}_1}(\xi) + \cdots + \overline{\widehat{b_s}(\xi+\pi)}\widehat{\tilde{b}_s}(\xi) = 0.$$

- Vanishing moments for high-pass filters: $(1 - e^{-i\xi})^n \mid \widehat{b_\ell}(\xi)$:

$$\widehat{b_\ell}(0) = \widehat{b_\ell}'(0) = \cdots = \widehat{b_s}^{(n-1)}(0) = 0, \qquad \ell = 1, \cdots, s.$$

- Sum rules for low-pass filters a: $(1 + e^{-i\xi})^m \mid \widehat{a}(\xi)$:

$$\widehat{a}(\pi) = \widehat{a}'(\pi) = \cdots = \widehat{a}^{(m-1)}(\pi) = 0.$$

- Symmetry is often required for wavelet-based compression.

Construction of Orthogonal Wavelet Filter Bank

- An orthogonal wavelet filter bank $\{a; b\}$ satisfies

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{a}(\xi)} & \overline{\widehat{a}(\xi+\pi)} \\ \overline{\widehat{b}(\xi)} & \overline{\widehat{b}(\xi+\pi)} \end{bmatrix} = I_2.$$

$$\text{i.e.,} \quad \begin{bmatrix} \overline{\widehat{a}(\xi)} & \overline{\widehat{a}(\xi+\pi)} \\ \overline{\widehat{b}(\xi)} & \overline{\widehat{b}(\xi+\pi)} \end{bmatrix} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix} = I_2,$$

- which is further equivalent to

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 = 1, \qquad \overline{\widehat{a}(\xi)}\widehat{b}(\xi) + \overline{\widehat{a}(\xi+\pi)}\widehat{b}(\xi+\pi) = 0.$$

- The second identity holds if $\widehat{b}(\xi) = e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}$.
- A filter c is interpolatory if $\widehat{c}(\xi) + \widehat{c}(\xi+\pi) = 1$.

Interpolatory Filters

- A filter $a \in l_0(\mathbb{Z})$ is interpolatory if $\widehat{a}(\xi) + \widehat{a}(\xi + \pi) = 1$, i.e.,

$$a(0) = \frac{1}{2} \quad \text{and} \quad a(2k) = 0, \quad \forall\, k \in \mathbb{Z}\backslash\{0\}.$$

- For $m, n \in \mathbb{N}$, the polynomial $\mathsf{P}_{m,n}$ is

$$\mathsf{P}_{m,n}(x) := \sum_{j=0}^{n-1} \binom{m+j-1}{j} x^j$$

- $(1-x)^m \mathsf{P}_{m,m}(x) + x^m \mathsf{P}_{m,m}(1-x) = 1$ for all $x \in \mathbb{R}, m \in \mathbb{N}$.
- A family of interpolatory filters a^I_{2m} is given by

$$\widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2) \mathsf{P}_{m,m}(\sin^2(\xi/2)).$$

- The filters are called Deslauriers-Dubuc interpolatory filters.
- $\widehat{a^I_{2m}}(0) = 1$, $\mathrm{sr}(a^I_{2m}) = 2m$, and $\widehat{a^I_{2m}}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$.

Interpolatory Filters a^I_{2m}

$$a^I_2 = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]},$$

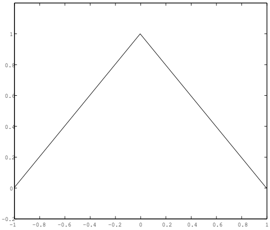
$$a^I_4 = \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]},$$

$$a^I_6 = \{\tfrac{3}{512}, 0, -\tfrac{25}{512}, 0, \tfrac{75}{256}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{75}{256}, 0, -\tfrac{25}{512}, 0, \tfrac{3}{512}\}_{[-5,5]},$$

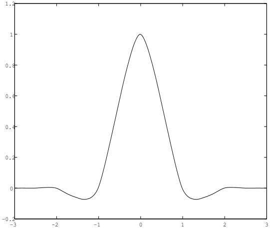
$$a^I_8 = \{-\tfrac{5}{4096}, 0, \tfrac{49}{4096}, 0, -\tfrac{245}{4096}, 0, \tfrac{1225}{4096}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1225}{4096}, 0, -\tfrac{245}{4096}, 0, \tfrac{49}{4096}, 0, -\tfrac{5}{4096}\}_{[-7,7]},$$

$$a^I_{10} = \{\tfrac{35}{131072}, 0, -\tfrac{405}{131072}, 0, \tfrac{567}{32768}, 0, -\tfrac{2205}{32768}, 0, \tfrac{19845}{65536}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{19845}{65536}, 0, -\tfrac{2205}{32768}, 0, \tfrac{567}{32768}, 0, -\tfrac{405}{131072}, 0, \tfrac{35}{131072}\}_{[0,9]}.$$

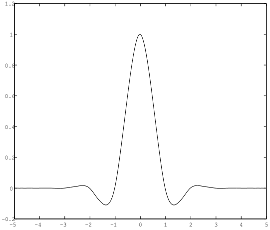
Compactly Supported Interpolating Function



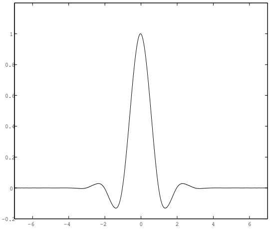
(a) $\phi^{a^I_2}$



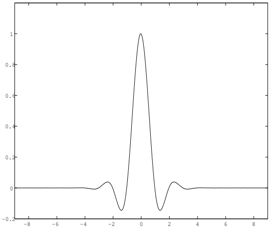
(b) $\phi^{a^I_4}$



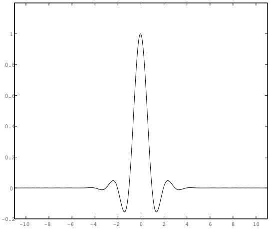
(c) $\phi^{a^I_6}$



(d) $\phi^{a^I_8}$



(e) $\phi^{a^I_{10}}$



(f) $\phi^{a^I_{12}}$

Daubechies Orthogonal Wavelets

Let a^I_{2m} be the interpolatory filter. Since $\widehat{a^I_{2m}}(\xi) \geqslant 0$, by Fejér-Riesz lemma, there exists $a^D_m \in l_0(\mathbb{Z})$ such that $\widehat{a^D_m}(0) = 1$.

$$|\widehat{a^D_m}(\xi)|^2 = \widehat{a^I_{2m}}(\xi) := \widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2)\mathrm{P}_{m,m}(\sin^2(\xi/2)).$$

Define ϕ through $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a^D_m}(2^{-j}\xi)$. Then $[\widehat{\phi}, \widehat{\phi}] = 1$ and $\{a^D_m; b^D_m\}$ is an orthogonal wavelet filter bank with

$$\widehat{b^D_m}(\xi) := e^{-i\xi}\overline{\widehat{a^D_m}(\xi+\pi)}, \quad \widehat{\psi}(\xi) := \widehat{b^D_m}(\xi/2)\widehat{\phi}(\xi/2).$$

Then $\{\phi; \psi\}$ is a compactly supported orthogonal wavelet such that the low-pass filter a^D_m has order m sum rules and the high-pass filter b^D_m has m vanishing moments, called the Daubechies orthogonal wavelet of order m.

Daubechies Orthogonal Filters

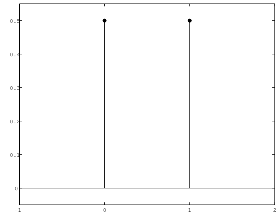
$$a_1^D = \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]},$$

$$a_2^D = \{\tfrac{1+\sqrt{3}}{8}, \underline{\mathbf{\tfrac{3+\sqrt{3}}{8}}}, \tfrac{3-\sqrt{3}}{8}, \tfrac{1-\sqrt{3}}{8}\}_{[-1,2]}$$

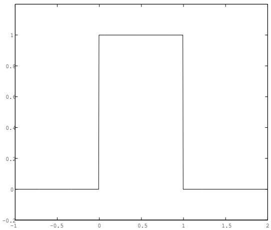
$$a_3^D = \{\tfrac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{32}, \tfrac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{32}, \underline{\mathbf{\tfrac{5-\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}}}, \tfrac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{32}, \tfrac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{32}\}_{[-2,3]},$$

$$a_4^D = \{-0.0535744507091, -0.0209554825625, 0.351869534328, \underline{\mathbf{0.568329121704}}, 0.210617267102, -0.0701588120893, -0.00891235072084, 0.0227851729480\}_{[-3,4]}.$$

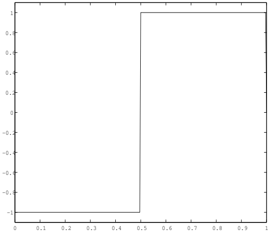
Daubechies Orthogonal Wavelets



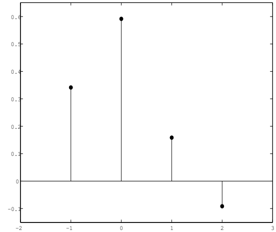
(a) Filter a_1^D



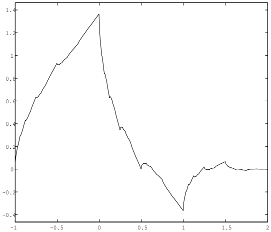
(b) $\phi^{a_1^D}$



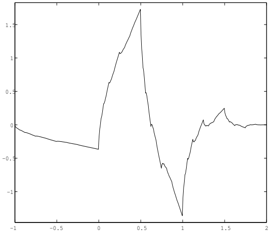
(c) $\psi^{a_1^D}$



(d) Filter a_2^D



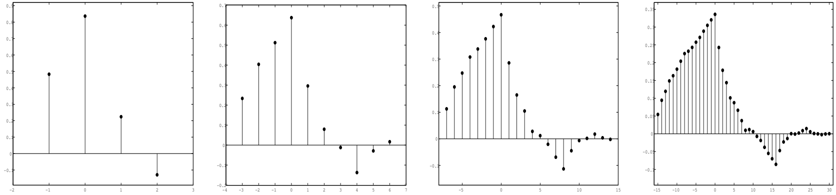
(e) $\phi^{a_2^D}$



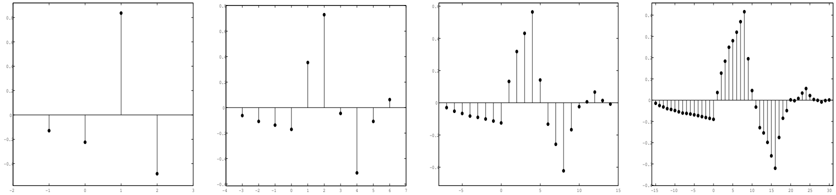
(f) $\psi^{a_2^D}$

An Example: Daubechies Orthogonal Wavelets

$$a = \left\{\frac{1+\sqrt{3}}{8}, \underline{\frac{3+\sqrt{3}}{8}}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8}\right\}, \qquad b = \left\{-\frac{1-\sqrt{3}}{8}, \underline{\frac{3-\sqrt{3}}{8}}, -\frac{3+\sqrt{3}}{8}, \frac{1+\sqrt{3}}{8}\right\}.$$

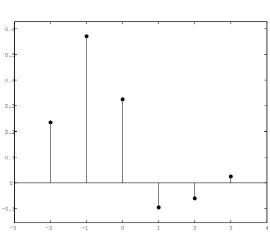


(g) a_1 (h) a_2 (i) a_3 (j) a_4

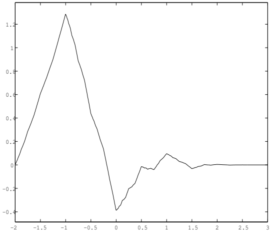


(k) b_1 (l) b_2 (m) b_3 (n) b_4

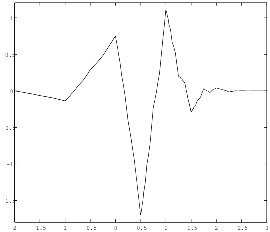
Daubechies Orthogonal Wavelets



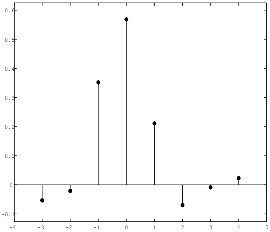
(a) Filter a_3^D



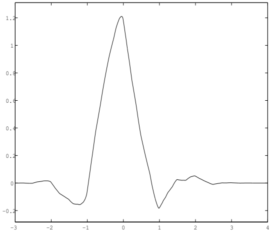
(b) $\phi^{a_3^D}$



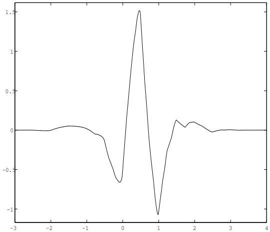
(c) $\psi^{a_3^D}$



(d) Filter a_4^D



(e) $\phi^{a_4^D}$



(f) $\psi^{a_4^D}$

Biorthogonal Wavelets

- Let $\phi, \psi \in L_2(\mathbb{R})$ and $\tilde{\phi}, \tilde{\psi} \in L_2(\mathbb{R})$.
- $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$ if
  1. Both $\{\tilde{\phi}; \tilde{\psi}\}$ and $\{\phi; \psi\}$ are Riesz wavelets in $L_2(\mathbb{R})$, i.e.,

$$C_3 \sum_{h \in \mathrm{AS}} |c_h|^2 \leqslant \Big\| \sum_{h \in \mathrm{AS}} c_h h \Big\|^2_{L_2(\mathbb{R})} \leqslant C_4 \sum_{h \in \mathrm{AS}} |c_h|^2,$$

where

$$\mathrm{AS}(\phi; \psi) := \{\phi(\cdot - k) \; : \; k \in \mathbb{Z}\} \cup \{\psi_{2^j;k} := 2^{j/2}\psi(2^j \cdot - k) \; : \; j \geqslant 0, k \in \mathbb{Z}\}.$$

  2. $\mathrm{AS}(\tilde{\phi}; \tilde{\psi})$ and $\mathrm{AS}(\phi; \psi)$ are biorthogonal to each other:

$$\langle h, \tilde{h} \rangle = 1 \quad \text{and} \quad \langle h, g \rangle = 0, \quad \forall\, g \in \mathrm{AS}(\phi; \psi) \backslash \{h\}.$$

Characterization of Biorthogonal Wavelets

Theorem: Let $\phi, \psi \in L_2(\mathbb{R})$ and $\tilde{\phi}, \tilde{\psi} \in L_2(\mathbb{R})$. Then $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$ if and only if

1. $[\widehat{\phi}, \widehat{\phi}] \in L_\infty(\mathbb{R})$, $[\widehat{\tilde{\phi}}, \widehat{\tilde{\phi}}] \in L_\infty(\mathbb{R})$, and $[\widehat{\tilde{\phi}}, \widehat{\phi}] = 1$.
2. There exist $a, b, \tilde{a}, \tilde{b} \in l_2(\mathbb{Z})$ such that

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi),$$
$$\widehat{\tilde{\phi}}(2\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi), \qquad \widehat{\tilde{\psi}}(2\xi) = \widehat{\tilde{b}}(\xi)\widehat{\tilde{\phi}}(\xi).$$

3. $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ is a biorthogonal wavelet filter bank:

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} = I_2.$$

Construction of Biorthogonal Wavelet Filter Bank

Proposition

Let $a, b, \tilde{a}, \tilde{b} \in l_0(\mathbb{Z})$. *Then* $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ *is a biorthogonal wavelet filter bank:*

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{b}}(\xi) \\ \widehat{\tilde{a}}(\xi+\pi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b}(\xi+\pi) \end{bmatrix}}^{\mathsf{T}} = I_2$$

if and only if $(\tilde{a}, a)$ *is a biorthogonal low-pass filter:*

$$\widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi)} + \widehat{\tilde{a}}(\xi+\pi)\overline{\widehat{a}(\xi+\pi)} = 1$$

and there exist $c, \tilde{c} \in \mathbb{T}$ *and* $n, \tilde{n} \in \mathbb{Z}$ *such that*

$$\widehat{\tilde{b}}(\xi) = c e^{i(2n-1)\xi}\overline{\widehat{a}(\xi+\pi)}, \qquad \widehat{b}(\xi) = \tilde{c} e^{i(2\tilde{n}-1)\xi}\overline{\widehat{\tilde{a}}(\xi+\pi)}.$$

Example of Biorthogonal Wavelets

We can obtain a pair of biorthogonal wavelet filters by splitting the interplatory filter

$$\overline{\widehat{\tilde{a}_m}(\xi)}\widehat{a_m}(\xi) := \widehat{a^I_{2m}}(\xi) = \cos^{2m}(\xi/2)\mathsf{P}_{m,m}(\sin^2(\xi/2))$$

as follows: $\mathsf{P}(x)\tilde{\mathsf{P}}(x) = \mathsf{P}_{m,m}(x)$ and

$$\widehat{a_m}(\xi) = 2^{-m}(1+e^{-i\xi})^m\mathsf{P}(\sin^2(\xi/2)), \qquad \widehat{b_m}(\xi) := e^{-i\xi}\overline{\widehat{\tilde{a}_m}(\xi+\pi)},$$

$$\widehat{\tilde{a}_m}(\xi) = 2^{-m}(1+e^{-i\xi})^m\tilde{\mathsf{P}}(\sin^2(\xi/2)), \qquad \widehat{\tilde{b}_m}(\xi) := e^{-i\xi}\overline{\widehat{a_m}(\xi+\pi)}.$$

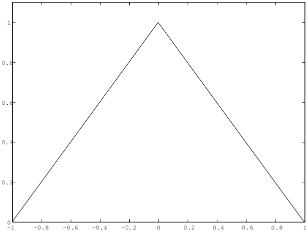
For $m=2$, we have the LeGall biorthogonal wavelet filter bank:

$$a_2 = \{\tfrac{1}{4}, \underline{\boldsymbol{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}$$

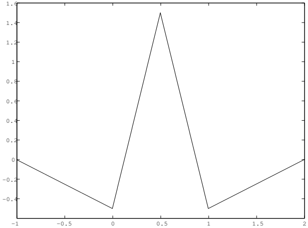
and

$$\tilde{a}_2 = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\boldsymbol{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}.$$

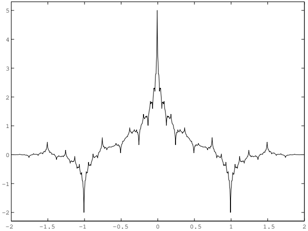
Examples of Biorthogonal Wavelets



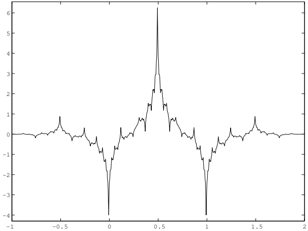
(g) ϕ^{a_2}



(h) ψ^{a_2,b_2}



(i) $\phi^{\tilde{a}_2}$



(j) $\psi^{\tilde{a}_2,\tilde{b}_2}$

The Most Famous Biorthogonal Wavelet

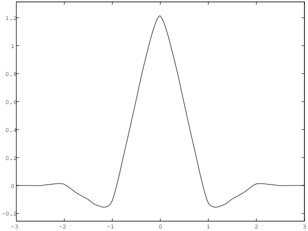
For $m = 4$,

$$a_4 = \{-\frac{t}{64}, \frac{2-t}{32}, \frac{16+t}{64}, \underline{\mathbf{\frac{6+t}{16}}}, \frac{16+t}{64}, \frac{2-t}{32}, -\frac{t}{64}\}_{[-3,3]},$$

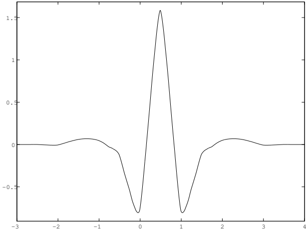
$$\tilde{a}_4 = \{\frac{t^2-4t+10}{256}, \frac{t-4}{64}, \frac{-t^2+6t-14}{64}, \frac{20-t}{64}, \underline{\mathbf{\frac{3t^2-20t+110}{128}}}, \frac{20-t}{64}, \frac{-t^2+6t-14}{64}, \frac{t-4}{64}, \frac{t^2-4t+10}{256}\}_{[-4,4]},$$

where $t \approx 2.92069$. The derived biorthogonal wavelet is called Daubechies 7/9 filter and has very impressive performance in many applications.

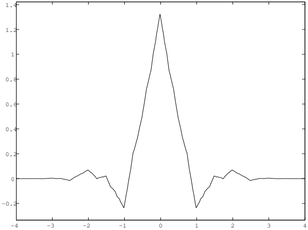
Example of Biorthogonal Wavelets



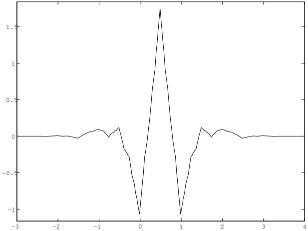
(k) ϕ^{a_4}



(l) ψ^{a_4, b_4}



(m) $\phi^{\tilde{a}_4}$



(n) $\psi^{\tilde{a}_4, \tilde{b}_4}$

B-spline Functions

- For $m \in \mathbb{N}$, the B-spline function B_m of order m is defined to be

$$B_1 := \chi_{(0,1]} \quad \text{and} \quad B_m := B_{m-1} * B_1 = \int_0^1 B_{m-1}(\cdot - t)dt.$$

- $\mathrm{supp}(B_m) = [0, m]$ and $B_m(x) > 0$ for all $x \in (0, m)$.
- $B_m = B_m(m - \cdot)$ and $B_m \in \mathscr{C}^{m-2}(\mathbb{R})$.
- $B_m|_{(k,k+1)} \in \mathbb{P}_{m-1}$ for all $k \in \mathbb{Z}$.
- $\widehat{B_m}(\xi) = (\frac{1-e^{-i\xi}}{i\xi})^m$ and B_m is refinable:

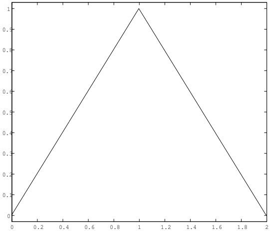
$$B_m = 2 \sum_{k\in\mathbb{Z}} a_m^B(k) B_m(2 \cdot -k),$$

where a_m^B is the B-spline filter of order m:

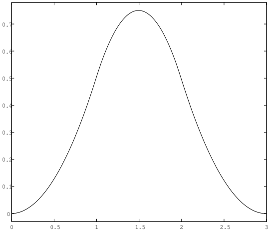
$$\widehat{a_m^B}(\xi) := 2^{-m}(1 + e^{-i\xi})^m.$$

Graphs of B-spline Functions

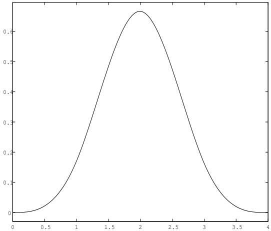
(a) B_1



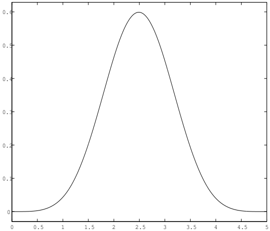
(b) B_2



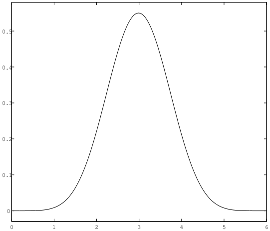
(c) B_3



(d) B_4



(e) B_5



(f) B_6

B-spline Filters a_m^B

$$a_1^B = \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]},$$
$$a_2^B = \{\underline{\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]},$$
$$a_3^B = \{\underline{\mathbf{\tfrac{1}{8}}}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]},$$
$$a_4^B = \{\underline{\mathbf{\tfrac{1}{16}}}, \tfrac{1}{4}, \tfrac{3}{8}, \tfrac{1}{4}, \tfrac{1}{16}\}_{[0,4]},$$
$$a_5^B = \{\underline{\mathbf{\tfrac{1}{32}}}, \tfrac{5}{32}, \tfrac{5}{16}, \tfrac{5}{16}, \tfrac{15}{32}, \tfrac{1}{32}\}_{[0,5]},$$
$$a_6^B = \{\underline{\mathbf{\tfrac{1}{64}}}, \tfrac{3}{32}, \tfrac{15}{64}, \tfrac{5}{16}, \tfrac{15}{64}, \tfrac{3}{32}, \tfrac{1}{64}\}_{[0,6]}.$$

Tight Framelet Filter Bank

- The definition of a tight framelet filter bank $\{a; b_1, \ldots, b_s\}$ can be given in the matrix form: $A(\xi)\overline{A(\xi)}^\mathsf{T} = I_2$, where

$$A(\xi) := \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix}$$

- If $s = 1$, $\{a; b_1\}$ is called an orthogonal wavelet filter bank:

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix} \overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix}}^\mathsf{T} = I_2$$

and

$$\overline{\begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix}}^\mathsf{T} \begin{bmatrix} \widehat{a}(\xi) & \widehat{b_1}(\xi) \\ \widehat{a}(\xi+\pi) & \widehat{b_1}(\xi+\pi) \end{bmatrix} = I_2.$$

- A tight framelet filter bank can have $s \geqslant 1$ high-pass filters.

Matrix Form of Fejér-Riesz Lemma

Theorem

Let $U(\xi)$ be an $r \times r$ matrix of 2π-periodic trigonometric polynomials such that $U(\xi) \geqslant 0$, i.e., $\overline{U(\xi)}^{\mathsf{T}} = U(\xi)$ and $\overline{x}^{\mathsf{T}} U(\xi) x \geqslant 0$ for all $x \in \mathbb{C}^r$ and $\xi \in \mathbb{R}$. Then there exists an $r \times r$ matrix $V(\xi)$ of 2π-periodic trigonometric polynomials such that

$$V(\xi)\overline{V(\xi)}^{\mathsf{T}} = U(\xi).$$

- Note that $U(\xi) \geqslant 0$ particularly implies $\det(U(\xi)) \geqslant 0$.
- The choice of $V(\xi)$ is not unique.
- Effective algorithms exist and are still under development.

Construction of Tight Framelet Filter Banks

- If $\{a; b_1, \ldots, b_s\}$ is a tight framelet filter bank, then

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 \leqslant 1, \quad \forall \xi \in \mathbb{R}.$$

- Conversely, if a satisfies the above inequality, one can obtain through Matrix Form of Fejér-Riesz lemma a tight framelet filter bank $\{a; b_1, b_2\}$.
- Recall that $a \in l_0(\mathbb{Z})$ is called an orthogonal low-pass filter if $|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi+\pi)|^2 = 1$. Hence, the requirement for constructing a tight framelet filter bank is much weaker than for constructing an orthogonal wavelet filter bank.
- For example, all B-spline filters a_m^B and all interpolatory filters a_{2m}^I satisfy this condition.

Example from a_2^B

Let

$$a_2^B = \{\underline{\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]}$$

be the B-spline filter of order 2. Let

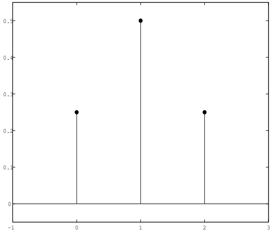
$$b_1 = \{\underline{-\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, -\tfrac{1}{4}\}_{[0,2]},$$

$$b_2 = \{\underline{-\mathbf{\tfrac{\sqrt{2}}{4}}}, 0, \tfrac{\sqrt{2}}{4}\}_{[0,2]}.$$

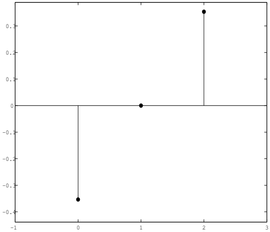
Then $\{a_2^B; b_1, b_2\}$ is a tight framelet filter bank such that a_2^B has order 2 sum rules and both b_1, b_2 have 1 vanishing moments.

This example is obtained by Ron and Shen in JFA, 1997.

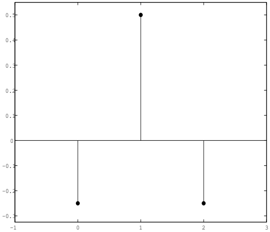
Tight Framelet from B_2



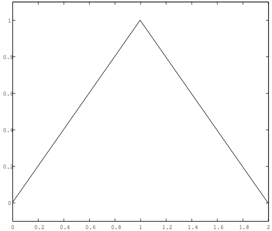
(a) Filter a_2^B



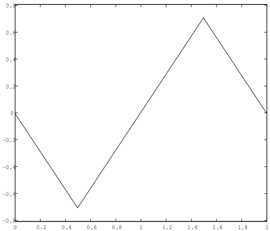
(b) Filter b_1



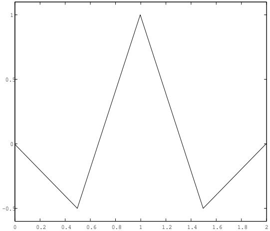
(c) Filter b_2



(d) B_2



(e) ψ^1



(f) ψ^2

Example from B_3

Let

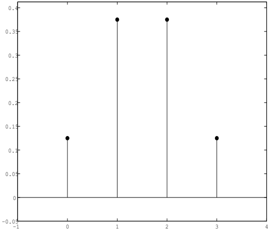
$$a_3^B = \{\underline{\mathbf{\tfrac{1}{8}}}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]}$$

be the B-spline filter of order 3. Let

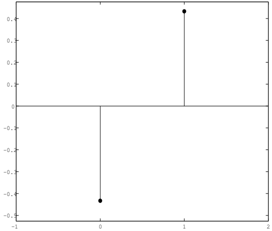
$$b_1 = \tfrac{\sqrt{3}}{4}\{\underline{-\mathbf{1}}, 1\}_{[0,1]},$$
$$b_2 = \{\underline{-\mathbf{\tfrac{1}{8}}}, -\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]}$$

Then $\{a; b_1, b_2\}$ is a tight framelet filter bank such that a_2^B has order 3 sum rules and both b_1, b_2 have 1 vanishing moments.

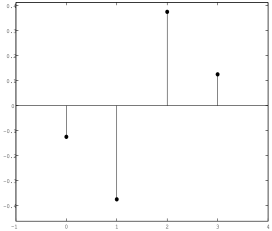
Tight Framelet from B_3



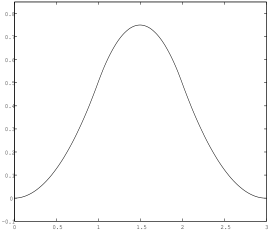
(a) Filter a_3^B



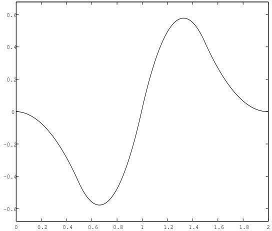
(b) Filter b_1



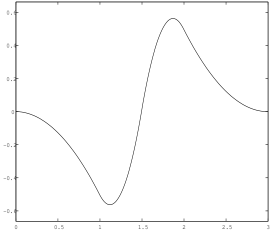
(c) Filter b_2



(d) B_3



(e) ψ^1



(f) ψ^2

Example from a_4^I

Let

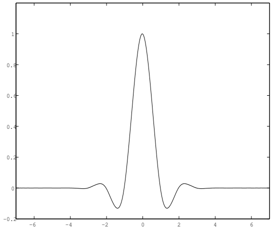
$$a_4^I = \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]}$$

be the interpolatory filter. Let

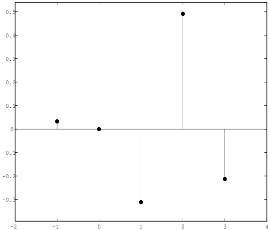
$$\begin{aligned}
\mathrm{b}_1(z) &= \tfrac{\sqrt{2}}{8\sqrt{9-4\sqrt{3}}} z^2(1-z^{-1})^2(z^{-1}-\sqrt{3})(z+2-\sqrt{3}),\\
\mathrm{b}_2(z) &= \tfrac{2\sqrt{3}+1}{352\sqrt{9-4\sqrt{3}}}(1-z^{-1})^2(x+2-\sqrt{3})[(1-2\sqrt{3})z^{-1}\\
&\qquad + (6-\sqrt{3}) + 33\,z + 11\sqrt{3}z^2],
\end{aligned}$$

where $\mathrm{b}(z) := \sum_{k\in\mathbb{Z}} b(k)z^k$. Then $\{a; b_1, b_2\}$ is a tight framelet filter bank such that a_4^I has order 4 sum rules and both b_1, b_2 have 2 vanishing moments.

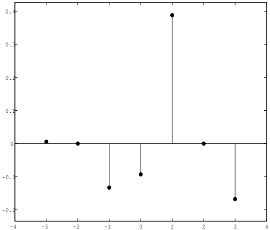
Tight Framelet from a_4^I



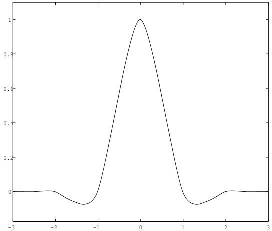
(a) Filter a_4^I



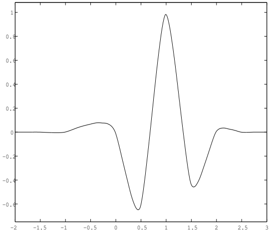
(b) Filter b_1



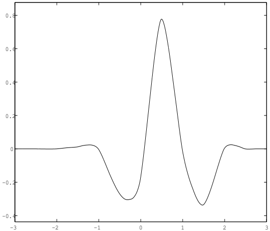
(c) Filter b_2



(d) $\phi^{a_4^I}$



(e) ψ^1



(f) ψ^2

Example from a_4^I

Let

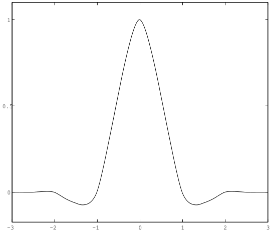
$$a_4^I = \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]}$$

be the interpolatory filter. Let

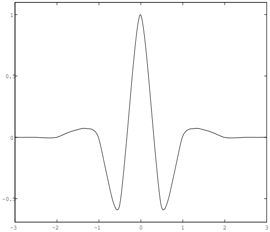
$$\begin{aligned} b_1 &= \tfrac{1}{32}\{1, 0, -9, \underline{\mathbf{16}}, -9, 0, 1\}_{[-3,3]}, \\ b_2 &= \tfrac{\sqrt{6}}{32}\{-1, 0, 1, \underline{\mathbf{0}}, 1, 0, -1\}_{[-3,3]}, \\ b_3 &= \tfrac{\sqrt{2}}{16}\{-1, 0, 3, \underline{\mathbf{0}}, -3, 0, 1\}_{[-3,3]}. \end{aligned}$$

Then $\{a; b_1, b_2, b_3\}$ is a real-valued interpolatory tight framelet filter bank such that a_4^I has order 4 sum rules and both b_1, b_2, b_3 have vanishing moments $4, 2, 3$, respectively.

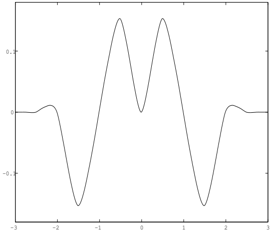
Tight Framelet from a_4^I



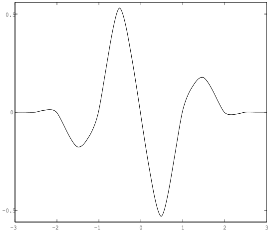
(a) ϕ



(b) ψ^1



(c) ψ^2



(d) ψ^3

Part II: Wavelet applications

The general procedure of wavelet applications in signal and image processing: For an input data v,

- Perform multi-level wavelet/framelet decomposition: $w = \mathcal{W}v$.
- Process the wavelet coefficients w to obtain new wavelet coefficients $\mathring{w}$.
- Perform multi-level wavelet/framelet reconstruction $\mathring{v} = \mathcal{V}\mathring{w}$.

Most data are supported on a bounded interval and we have to perform fast wavelet/framelet transform (FFrT) on data on bounded intervals!.

Diagram of Multi-level FFrT

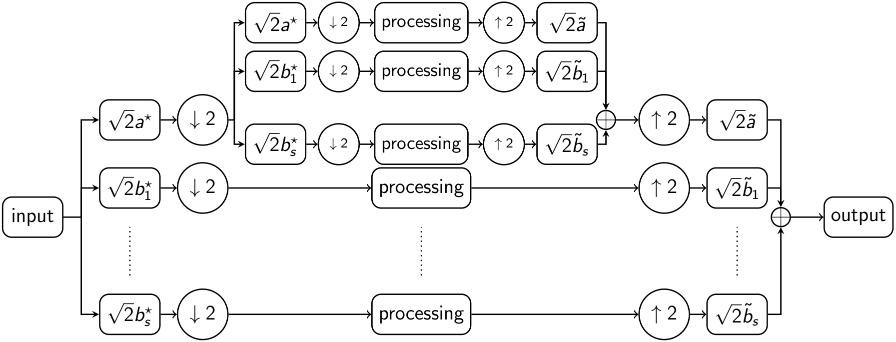
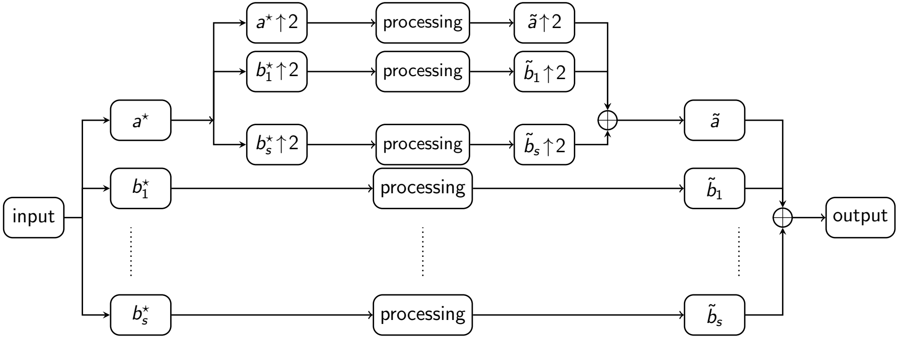


Figure: Diagram of a two-level discrete framelet transform using a dual framelet filter bank $(\{a; b_1, \dots, b_s\}, (\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\})$.

Undecimated Multi-level FFrT



Undecimated DFrT using a framelet filter bank $(\{\tilde{a};\tilde{b}_1,\ldots,\tilde{b}_s\}$, $(a;b_1,\ldots,b_s\})$, which is required to satisfy

$$\overline{\widehat{a}(\xi)}\widehat{\tilde{a}}(\xi)+\overline{\widehat{b_1}(\xi)}\widehat{\tilde{b}_1}(\xi)+\cdots+\overline{\widehat{b_s}(\xi)}\widehat{\tilde{b}_s}(\xi)=1.$$

Processing Wavelet Coefficients

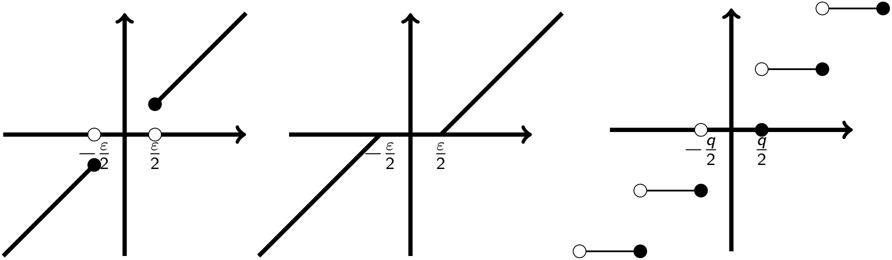


Figure: The hard thresholding, soft thresholding, and quantization.

Quantization is used for compression: convert a real number into a discrete set $\{\ldots, -2q, -q, 0, q, 2q, \ldots\}$.
The hard or soft thresholding is used for denoising and processing.

Signals on Intervals

- Signals: $v^b = \{v^b(k)\}_{k=0}^{N-1} : [0, N-1] \cap \mathbb{Z} \to \mathbb{C}$.
- Extend v^b from interval $[0, N-1]$ to $\mathbb{Z}$.
- $\mathcal{T}_{u(\cdot-2m)} v = [\mathcal{T}_u v](\cdot - m), \quad m \in \mathbb{Z}$.
- extend v^b from $[0, N-1] \cap \mathbb{Z}$ to a sequence v on $\mathbb{Z}$ by any method that the reader prefers.
- zero-padding: $v(k) = 0$ for $k \in \mathbb{Z} \backslash [0, N-1]$.
- Must recover $v^b(0), \ldots, v^b(N-1)$ exactly.
- Keep all $[\mathcal{S}_{\tilde{u}_\ell} \mathcal{T}_{u_\ell} v](n), n = 0, \ldots, N-1$.
- $\text{fsupp}(\tilde{u}) = [n_-, n_+]$ with $n_- \leqslant 0$ and $n_+ \geqslant 0$.

Periodic Extension

Proposition

Let $u \in l_1(\mathbb{Z})$ be a filter and $v^b = \{v^b(k)\}_{k=0}^{N-1}$. Extend v^b into an N-periodic sequence v on $\mathbb{Z}$ as follows:

$$v(Nn + k) := v^b(k), \qquad k = 0, \dots, N-1, \quad n \in \mathbb{Z}.$$

Then the following properties hold:

(i) *$u * v$ is an N-periodic sequence on $\mathbb{Z}$;*

(ii) *$\mathcal{S}_u v$ is a $2N$-periodic sequence on $\mathbb{Z}$;*

(iii) *If N is even, then $\mathcal{T}_u v$ is an $\frac{N}{2}$-periodic;*

(iv) *If N is odd, then $\mathcal{T}_u v$ is an N-periodic.*

Tight Framelet Filter Bank

- A tight framelet filter bank from B_2, where

$$a = \{\tfrac{1}{4}, \underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{4}\}_{[-1,1]},$$
$$b_1 = \{\tfrac{1}{4}, \underline{-\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{4}\}_{[-1,1]},$$
$$b_2 = \{-\tfrac{\sqrt{2}}{4}, \underline{\mathbf{0}}, -\tfrac{\sqrt{2}}{4}\}_{[-1,1]}.$$

- A test input data:

$$v = \{-21, -22, -23, -23, -25, 38, 36, 34\}_{[0,7]}$$

Example: Tight Framelet Filter Bank

We extend v^b to an 8-periodic sequence v on $\mathbb{Z}$, given by

$$v = \{\ldots, -25, 38, 36, 34, \underline{\mathbf{-21}, -22, -23, -23, -25, 38, 36, 34}, -21, -22, -23, -23,$$

Then all sequences $\mathcal{T}_a v, \mathcal{T}_{b_1} v, \mathcal{T}_{b_2} v$ are 4-periodic and

$$w_0 = \tfrac{\sqrt{2}}{2}\mathcal{T}_a v = \tfrac{\sqrt{2}}{2}\{\ldots, -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, \underline{\mathbf{-15}, -\tfrac{91}{2}, -\tfrac{35}{2}, 72,} -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, .$$

$$w_1 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_1} v = \tfrac{\sqrt{2}}{2}\{\ldots, -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \underline{\mathbf{-28}, -\tfrac{1}{2}, \tfrac{61}{2}, -2,} -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \ldots\},$$

$$w_2 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_2} v = \{\ldots, -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \underline{\mathbf{-27}, -\tfrac{1}{2}, -\tfrac{65}{2}, 0,} -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \ldots\}.$$

It is also easy to check that $\frac{\sqrt{2}}{2}(\mathcal{S}_{u_0} w_0 + \mathcal{S}_{u_1} w_1 + \mathcal{S}_{u_2} w_2) = v$. But

$$\|w_0\|^2 + \|w_1\|^2 + \|w_2\|^2 = \frac{15571}{2} + \frac{3437}{2} + \frac{3571}{2} = \frac{22579}{4} \approx 5644.8$$

$$\|v\|^2 = 6504.$$

Symmetric Extension

Proposition

Let $u \in l_1(\mathbb{Z})$ *such that* $u(2c-k) = \epsilon u(k)$. *Extend* $v^b = \{v^b(k)\}_{k=0}^{N-1}$, *with both endpoints non-repeated (EN), into* $(2N-2)$*-periodic* v *by* $v(k) = v^b(2N-2-k)$.

(i) *Then* $u^\star * v$ *is* $(2N-2)$*-periodic with*

$$[u^\star * v](-2c-k) = [u^\star * v](2N-2-2c-k) = \epsilon[u^\star * v](k),$$

and $[-\lfloor c \rfloor, N-1-\lceil c \rceil]$ *is its control interval.*

(ii) *If* $c \in \mathbb{Z}$, *then* $\mathcal{T}_u v$ *is* $(N-1)$*-periodic with*

$$[\mathcal{T}_u v](-c-k) = [\mathcal{T}_u v](N-1-c-k) = \epsilon[\mathcal{T}_u v](k),$$

and $[\lceil -\frac{c}{2} \rceil, \lfloor \frac{N-1-c}{2} \rfloor]$ *is a control interval of* $\mathcal{T}_u v$.

Proposition

Let $u \in l_1(\mathbb{Z})$ with $\epsilon \in \{-1, 1\}$ and $c \in \frac{1}{2}\mathbb{Z}$. Extend v^b, with both endpoints repeated (ER), into 2N-periodic v by $v(k) = v^b(2N - 1 - k)$.

(i) *Then $u^\star * v$ is 2N-periodic with*

$$[u^\star * v](-1 - 2c - k) = [u^\star * v](2N - 1 - 2c - k) = \epsilon[u^\star * v](k),$$

and $[-\lfloor \frac{1}{2} + c \rfloor, N - \lceil \frac{1}{2} + c \rceil]$ is its control interval.

(ii) *If $c - \frac{1}{2} \in \mathbb{Z}$, then $\mathcal{T}_u v$ is an N-periodic sequence:*

$$[\mathcal{T}_u v](-\tfrac{1}{2} - c - k) = [\mathcal{T}_u v](N - \tfrac{1}{2} - c - k) = \epsilon[\mathcal{T}_u v](k),$$

and $[\lceil -\frac{1}{4} - \frac{c}{2} \rceil, \lfloor \frac{N}{2} - \frac{1}{4} - \frac{c}{2} \rfloor]$ is its control interval.

The LeGall biorthogonal wavelet filter bank is given by

$$a=\{\tfrac14,\underline{\mathbf{\tfrac12}},\tfrac14\}_{[-1,1]},\qquad \tilde a=\{-\tfrac18,\tfrac14,\underline{\mathbf{\tfrac34}},\tfrac14,-\tfrac18\}_{[-2,2]}$$

$$b=\{\tfrac18,\underline{\mathbf{\tfrac14}},-\tfrac34,\tfrac14,\tfrac18\}_{[-1,3]},\qquad \tilde b=\{\underline{\mathbf{\tfrac14}},-\tfrac12,\tfrac14\}_{[0,2]},$$

Extend v^b by both endpoints non-repeated (EN):

$$v=\{\ldots,-25,-23,-23,-22,\underline{\mathbf{-21},-22,-23,-23,-25,38,36,34},36,38,-$$

Then $\mathcal{T}_{\tilde a}v$ is 7-periodic and is symmetric about 0, $7/2$:

$$w_0=\tfrac{\sqrt2}{2}\mathcal{T}_{\tilde a}v=\tfrac{\sqrt2}{2}\{\ldots,-\tfrac{133}{4},-\tfrac{91}{2},\underline{\mathbf{-42},-\tfrac{91}{2},-\tfrac{133}{4},\tfrac{349}{4}},\tfrac{349}{4},-\tfrac{133}{4},\ldots\},$$

and $\mathcal{T}_{\tilde b}v$ is 7-periodic and is symmetric about $-\frac12$, 3:

$$w_1=\tfrac{\sqrt2}{2}\mathcal{T}_{\tilde b}v=\tfrac{\sqrt2}{2}\{\ldots,-2,\tfrac{65}{2},0,\underline{\mathbf{0},1,\tfrac{65}{2},-2},\tfrac{65}{2},1,0,\ldots\}.$$

Example: Tight Framelet Filter Bank from B_2

A tight framelet filter bank from B_2, where

$$a = \{\tfrac{1}{4}, \underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \quad b_1 = \{\tfrac{1}{4}, \underline{-\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \quad b_2 = \{-\tfrac{\sqrt{2}}{4}, \underline{\mathbf{0}}, -\tfrac{\sqrt{2}}{4}\}_{[-1,1]}.$$

Extend v^b with both endpoints non-repeated (EN). Then all $\mathcal{T}_a v, \mathcal{T}_{b_1} v, \mathcal{T}_{b_2} v$ are 7-periodic and symmetric about 0 and $7/2$:

$$w_0 = \tfrac{\sqrt{2}}{2}\mathcal{T}_a v = \tfrac{\sqrt{2}}{2}\{\ldots, 72, -\tfrac{35}{2}, -\tfrac{91}{2}, \underline{-\mathbf{43}, -\tfrac{91}{2}, -\tfrac{35}{2}, 72}, 72, -\tfrac{35}{2}, -\tfrac{91}{2}, \ldots\},$$

$$w_1 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_1} v = \tfrac{\sqrt{2}}{2}\{\ldots, 2, -\tfrac{61}{2}, \tfrac{1}{2}, \underline{\mathbf{0}, -\tfrac{1}{2}, \tfrac{61}{2}, -2}, 2, -\tfrac{61}{2}, \tfrac{1}{2}, \ldots\}.$$

$$w_2 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_2} v = \{\ldots, 0, -\tfrac{65}{2}, -\tfrac{1}{2}, \underline{\mathbf{1}, -\tfrac{1}{2}, -\tfrac{65}{2}, 0}, 0, -\tfrac{65}{2}, -\tfrac{1}{2}, 1, \ldots\}.$$

Compare with framelet coefficients through periodic extension:

$$w_0 = \tfrac{\sqrt{2}}{2}\mathcal{T}_a v = \tfrac{\sqrt{2}}{2}\{\ldots, -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, \underline{-\mathbf{15}, -\tfrac{91}{2}, -\tfrac{35}{2}, 72}, -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, .$$

$$w_1 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_1} v = \tfrac{\sqrt{2}}{2}\{\ldots, -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \underline{-\mathbf{28}, -\tfrac{1}{2}, \tfrac{61}{2}, -2}, -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \ldots\}.$$

$$w_2 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_2} v = \{\ldots, -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \underline{-\mathbf{27}, -\tfrac{1}{2}, -\tfrac{65}{2}, 0}, -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \ldots\}.$$

Tensor Product (Separable) Tight Framelet

- Let $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ be a 1D dual framelet filter bank.
- If $s = 1$, it is called a biorthogonal wavelet filter bank.
- Tensor product filters: $[u_1 \otimes \cdots \otimes u_d](k_1, \ldots, k_d) = u_1(k_1) \cdots u_d(k_d)$.
- Tensor product two-dimensional dual framelet filter bank:

$$\Big(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\} \otimes \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\} \otimes \{a; b_1, \ldots, b_s\}\Big).$$

That is,

$$\{a; b_1, \ldots, b_s\} \otimes \{a; b_1, \ldots, b_s\} = \{a \otimes a; b_1 \otimes a, \ldots, b_s \otimes a, \\ b_1 \otimes b_1, \ldots, b_s \otimes b_1, \ldots, b_s \otimes b_1, \ldots, b_s \otimes\}$$

consists of one low-pass tensor product filter $a \otimes a$ and total $(s+1)^2 - 1 = s^2 + 2s$ high-pass tensor product filters.
- Advantages: fast and simple algorithm.

Wavelets for Image/Signal Compression

- The most popular wavelets used for the compression purpose are biorthogonal wavelet filter banks with symmetry and $4 \sim 6$ vanishing moments, in particular, LeGall and Daubechies 7/9 biorthogonal wavelet filter banks.
- For filter banks having symmetry, signals on intervals and images on rectangles are often extended by symmetry extension.
- Dauchies orthogonal wavelet filter banks are also used. Due to lack of symmetry, periodic extension of data on intervals and rectangles is often used.
- Grey scale images I : each entry of I takes discrete values $[0, 1, \ldots, 255]$ ($2^8 = 256$).
- Color images I has three channel: Red (R), Green (G), Blue (B). With each entry in each channel takes values discrete values $[0, 1, \ldots, 255]$.

Most Popular Wavelets for Compression

- LeGall biorthogonal wavelet filter bank $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$:

$$a = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \qquad \tilde{a} = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}$$

$$b = \{\tfrac{1}{8}, \underline{\mathbf{\tfrac{1}{4}}}, -\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{8}\}_{[-1,3]}, \qquad \tilde{b} = \{\underline{\mathbf{\tfrac{1}{4}}}, -\tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]},$$

where $b(k) = (-1)^{1-k}\overline{\tilde{a}(1-k)}$ and $\tilde{b}(k) = (-1)^{1-k}\overline{a(1-k)}$. Both a and $\tilde{a}$ have order 2 sum rules, while both b and $\tilde{b}$ have 2 vanishing moments. Use $\{a; b\}$ for reconstruction.

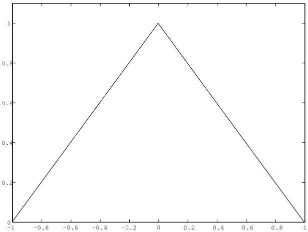
- Dauchies 7/9 biorthogonal wavelet filter bank $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$:

$$a = \{-\tfrac{t}{64}, \tfrac{2-t}{32}, \tfrac{16+t}{64}, \underline{\mathbf{\tfrac{6+t}{16}}}, \tfrac{16+t}{64}, \tfrac{2-t}{32}, -\tfrac{t}{64}\}_{[-3,3]},$$

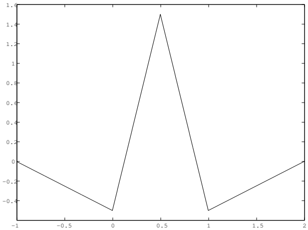
$$\tilde{a} = \{\tfrac{t^2-4t+10}{256}, \tfrac{t-4}{64}, \tfrac{-t^2+6t-14}{64}, \tfrac{20-t}{64}, \underline{\mathbf{\tfrac{3t^2-20t+110}{128}}}, \tfrac{20-t}{64}, \tfrac{-t^2+6t-14}{64}, \tfrac{t-4}{64}, \tfrac{t^2-4t+10}{256}\}_{[-4,4]},$$

where $t \approx 2.92069$ such that both a and $\tilde{a}$ have order 4 sum rules, while both b and $\tilde{b}$ have 4 vanishing moments.

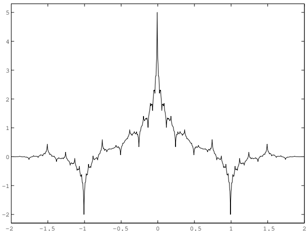
LeGall Biorthogonal Wavelets



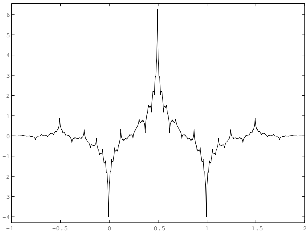
(a) ϕ



(b) ψ

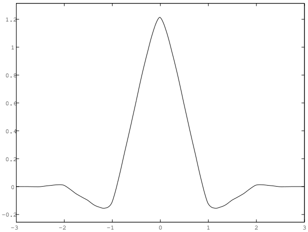


(c) $\tilde{\phi}$

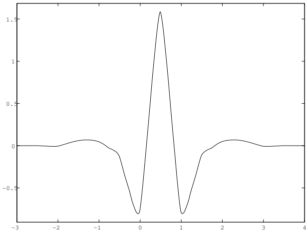


(d) $\tilde{\psi}$

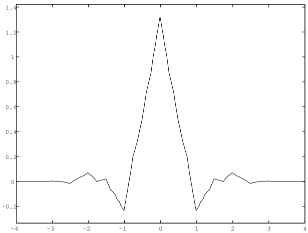
Daubechies 7/9 Biorthogonal Wavelets



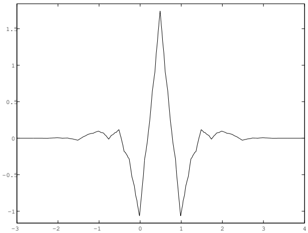
(e) ϕ



(f) ψ



(g) $\tilde{\phi}$



(h) $\tilde{\psi}$

Quantize Wavelet Coefficients for Compression

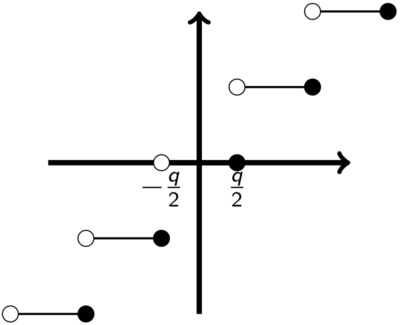
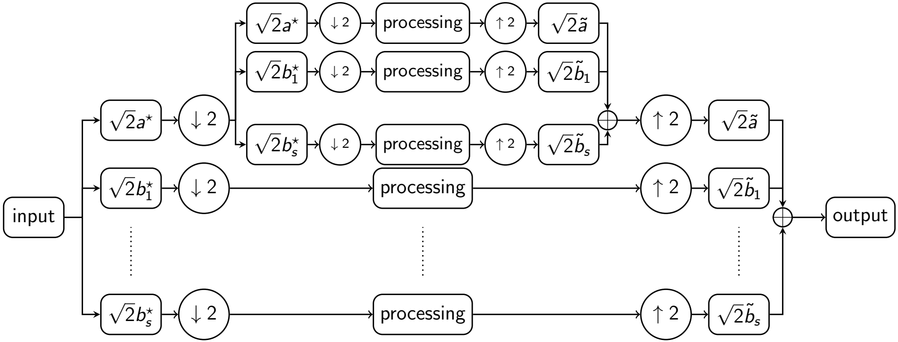


Figure: Quantization is used to convert a real number into a discrete set $\{\ldots, -2q, -q, 0, q, 2q, \ldots\}$ so that it becomes bit stream. Coefficients with large magnitude are record and small coefficients are dropped and replaced by 0. Positions of large coefficients have to be recorded as well. For efficient coding, most popular methods for selecting and recording wavelet coefficients are EZW (embedded Zerotrees) and SPIHT (Set Partitioning Hierarchical Trees).

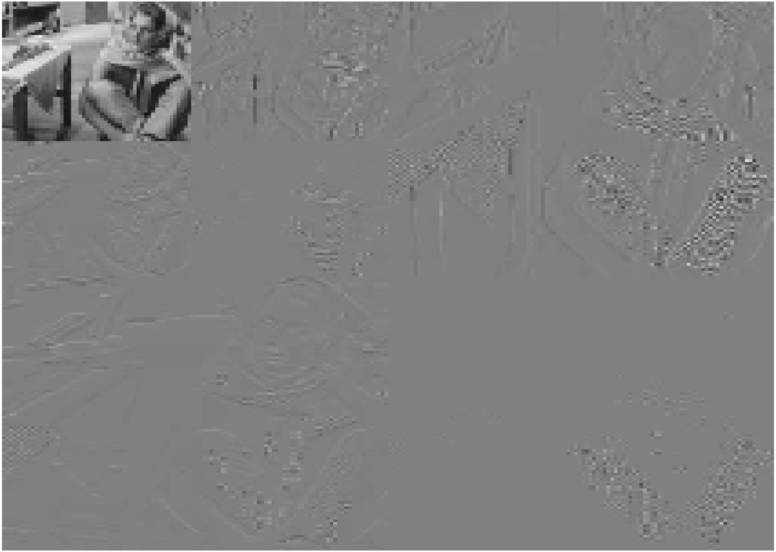
Use Maximum Multi-level FFrT



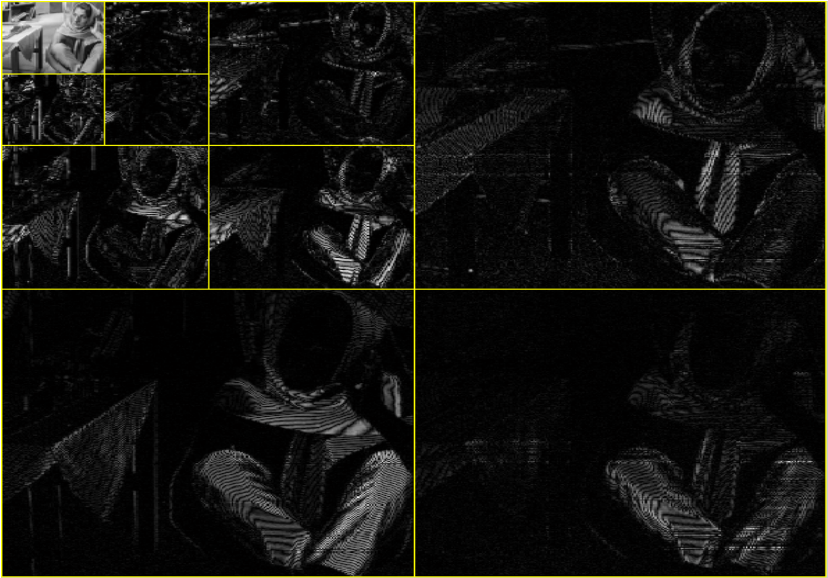
Explanation: Because $\widehat{a}(0) = \sum_{k\in\mathbb{Z}} a(k) = 1$, $v * (\sqrt{2}a^\star)$ is just averaging the signal v by the low-pass a and then amplify it by $\sqrt{2}$. If we perform the decomposition for L level, then the low-pass wavelet coefficients are roughly averaged v amplified by $2^{L/2}$ times.

Test Image: Barbara

Tree Structure of Wavelet coefficients



Tree Structure of Wavelet coefficients



Tree structure of wavelet coefficients

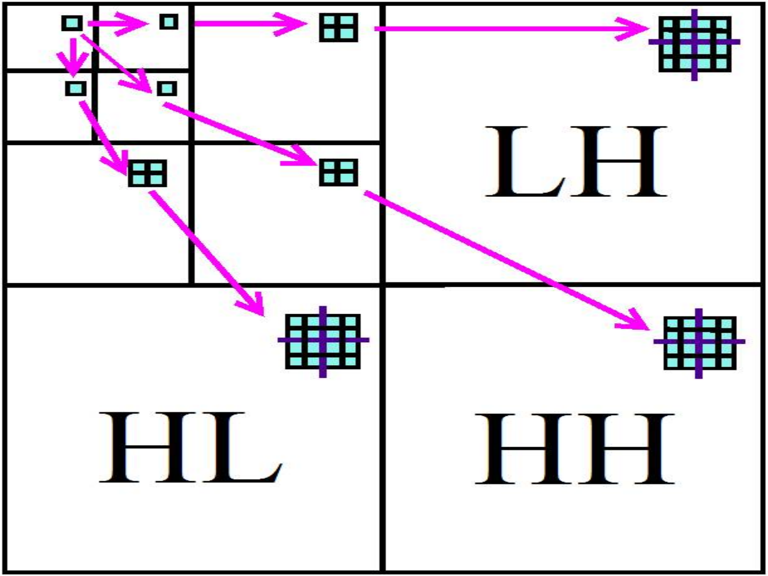


Image compression by wavelets

The original image of Lena

Image compression by wavelets

Compressed Lena image with compression ratio 32

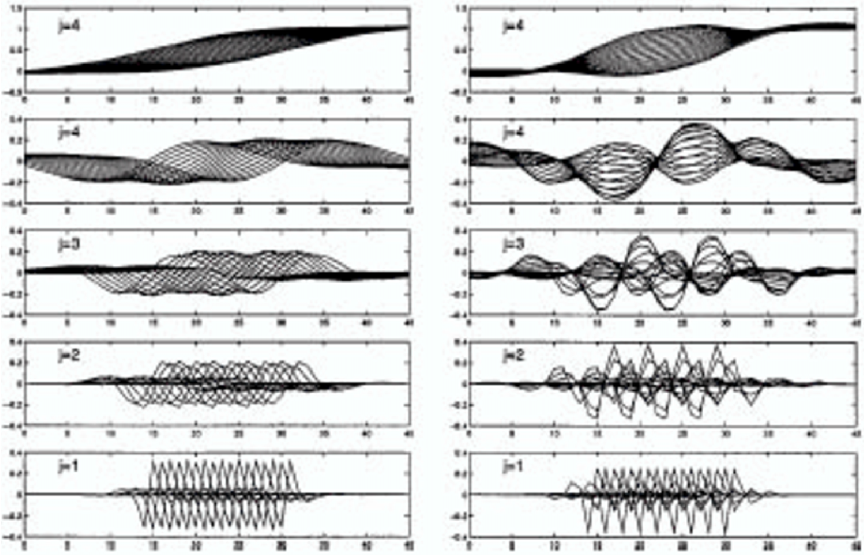
Image compression by wavelets

Compressed Lena image with compression ratio 128

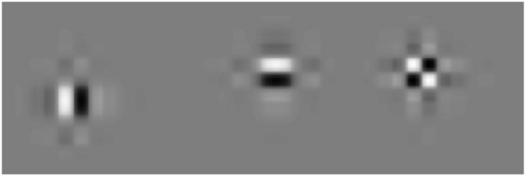
For Other Signal and Image Processing

- Beyond compression purpose, framelet filter banks are often used instead.
- Orthogonal wavelets and biorthogonal wavelets suffer a few key desired properties: translation invariant (see next page from Selesnick's paper) and directionality.
- For image denoising, people often use undecimated wavelet transforms, which are just special cases of framelets.
- Processing wavelet or framelet coefficients through thresholding is a key issue. Statistics and probability theory are often involved.

Translation Invariance: Framelets vs Wavelets

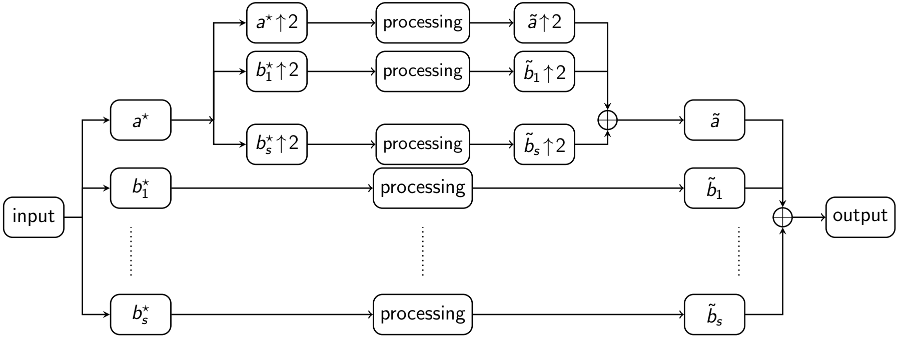


Shortcomings of Separable Real-valued Wavelets



- Two Shortcomings: lack directionality and translation invariance.
- Tensor product real-valued wavelets have only two directions: vertical $\phi(x)\psi(y)$, horizontal $\psi(x)\phi(y)$. But not $\psi(x)\psi(y)$.
- To improve translation invariance: Undecimated wavelet transform, undecimated DCT, real-valued tight framelets, complex wavelets, dual tree complex wavelets,...
- To improve directionality, curvelets, shearlets, contourlets, dual tree complex wavelets, steerable filter banks,bandlets,brushlets,

Undecimated Wavelet/Framelet Transform



Undecimated DFrT using a framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, (a; b_1, \ldots, b_s\})$, which is required to satisfy

$$\overline{\widehat{a}(\xi)}\widehat{\tilde{a}}(\xi) + \overline{\widehat{b_1}(\xi)}\widehat{\tilde{b}_1}(\xi) + \cdots + \overline{\widehat{b_s}(\xi)}\widehat{\tilde{b}_s}(\xi) = 1.$$

Tensor Product Complex Tight Framelets

- $\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}$ which is 2π-periodic.
- $\{a; b_1, \dots, b_s\}$ is a tight framelet filter bank if

$$|\widehat{a}(\xi)|^2 + |\widehat{b_1}(\xi)|^2 + \cdots + |\widehat{b_s}(\xi)|^2 = 1,$$

$$\widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} + \widehat{b_1}(\xi)\overline{\widehat{b_1}(\xi+\pi)} + \cdots + \widehat{b_s}(\xi)\overline{\widehat{b_s}(\xi+\pi)} = 0.$$

- Tensor product (separable) tight framelet filter bank in 2D:

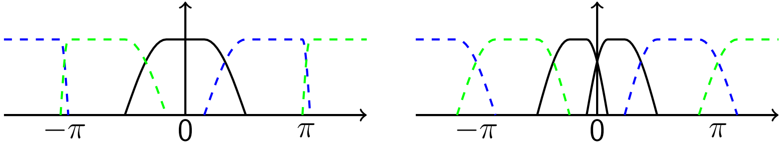
$$\{a; b_1, \dots, b_s\} \otimes \{a; b_1, \dots, b_s\}$$

where $a \otimes b(m,n) := a(m)b(n)$, $m, n \in \mathbb{Z}$.

- For $c_L < c_R$ and $\varepsilon_L, \varepsilon_R > 0$, the bump function is

$$\chi_{[c_L,c_R];\varepsilon_L,\varepsilon_R}(\xi) := \begin{cases} 0, & \xi \leqslant c_L - \varepsilon_L \text{ or } \xi \geqslant c_R + \varepsilon_R, \\ \cos\left(\frac{\pi(c_L+\varepsilon_L-\xi)}{4\varepsilon_L}\right), & c_L - \varepsilon_L < \xi < c_L + \varepsilon_L, \\ 1, & c_L + \varepsilon_L \leqslant \xi \leqslant c_R - \varepsilon_R, \\ \cos\left(\frac{\pi(\xi-c_R+\varepsilon_R)}{4\varepsilon_R}\right), & c_R - \varepsilon_R < \xi < c_R + \varepsilon_R. \end{cases}$$

Directional Complex Tight Framelets TP-$\mathbb{C}$TF$_m$



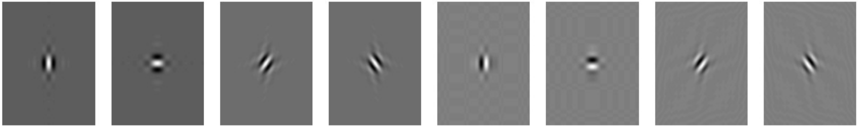
- Tight framelet filter banks $\mathbb{C}\mathrm{TF}_3 := \{a; b^+, b^-\}$ and $\mathbb{C}\mathrm{TF}_4 := \{a^+, a^-; b^+, b^-\}$: $\widehat{a}$ and $\widehat{a^+}$ (black); $\widehat{b^+}$ (blue):

$$\widehat{a^+} := \chi_{[0,c];\varepsilon,\varepsilon}, \quad \widehat{b^+} := \chi_{[c,\pi];\varepsilon,\varepsilon}, \quad a^- := \overline{a^+}, \quad b^- := \overline{b^+},$$

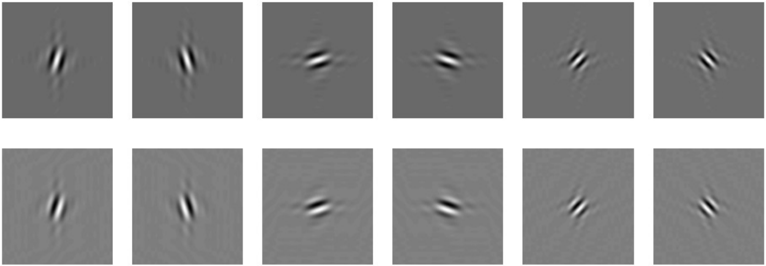
- The tensor product tight framelet TP-$\mathbb{C}\mathrm{TF}_m := \otimes^d \mathbb{C}\mathrm{TF}_m$.
- TP-$\mathbb{C}\mathrm{TF}_m (m \geqslant 3)$: Han, Math. Model. Nat. Phenom. **8** (2013) 18–47; Han/Zhao, SIAM J. Imag. Sci. **7** (2014) 997–1034.

Discrete Affine Systems TP-$\mathbb{C}$TF$_3$ and TP-$\mathbb{C}$TF$_4$

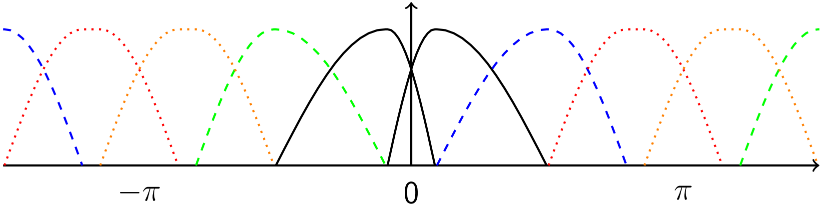
Two-dimensional TP-$\mathbb{C}$TF$_3$ has 4 directions:



Two-dimensional TP-$\mathbb{C}$TF$_4$ has 6 directions:



Directional Complex Tight Framelets TP-$\mathbb{C}$TF$_6$

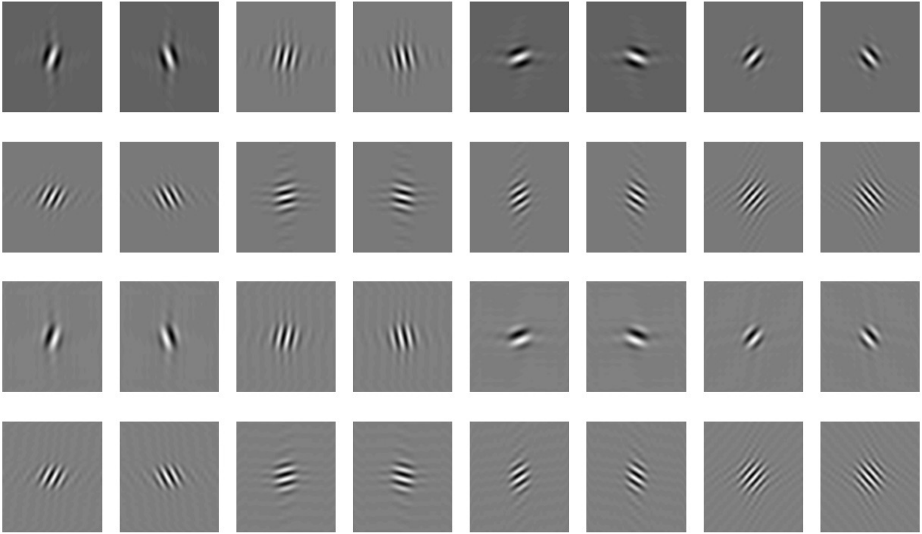


- Tight framelet filter bank $\mathbb{C}\mathrm{TF}_6 := \{a^+, a^-; b_1^+, b_2^+, b_1^-, b_2^-\}$: black lines for $\widehat{a^+}$ and $\widehat{a^-}$; dashed lines for $\widehat{b_1^+}$ and $\widehat{b_1^-}$; dotted lines for $\widehat{b_2^+}$ and $\widehat{b_2^-}$: $a^- := \overline{a^+}$, $b_1^- := \overline{b_1^+}$, $b_2^- := \overline{b_2^+}$, and

$$\widehat{a^+} := \chi_{[0,c];\varepsilon,\varepsilon}, \quad \widehat{b_1^+} := \chi_{[c_1,c_2];\varepsilon,\varepsilon}, \quad \widehat{b_2^+} := \chi_{[c_2,\pi];\varepsilon,\varepsilon}.$$

- The tensor product tight framelet TP-$\mathbb{C}$TF$_6 := \otimes^d \mathbb{C}\mathrm{TF}_6$.
- Take advantages of wavelets and Discrete Cosine Transform.

Discrete Affine Systems TP-$\mathbb{C}$TF$_6$ (14 directions)



Signal/Image/Video Denoising and Inpainting

- Let $\mathrm{g} = (\mathrm{g}_1, \ldots, \mathrm{g}_d)^\mathsf{T}$ be an observed corrupted data:

$$\mathrm{g}_j = \begin{cases} \mathrm{f}_j + \mathrm{n}_j, & \text{if } j \in \Omega, \\ \mathrm{m}_j, & \text{if } j \in \Omega^c := \{1, \ldots, d\} \backslash \Omega, \end{cases}$$

- f=true data (signal/image.video).
- $\Omega \subseteq \{1, \ldots, d\}$ is a (known or unknown) observable region.
- n= i.i.d. Gaussian noise with zero mean and variance σ^2.
- m= either unknown missing pixel or impulse noise.
- Goal: Recover the true unknown data f from the corrupted g by suppressing noise n or inpainting unknown m.
- Denoising problem if $\Omega = \{1, \ldots, d\}$.
- Inpainting problem if Ω is known.
- Removing mixed noise problem if Ω is unknown.

Image Denoising

- Image denoising model is

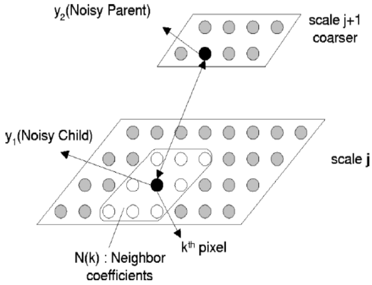
$$\mathbf{z} = \mathbf{x} + \mathbf{n}.$$

where

- $\mathbf{z}$=noisy image,
- $\mathbf{x}$=true image,
- $\mathbf{n}$=i.i.d. Gaussian noise of zero mean and variance σ^2.
- Coefficients $\mathbf{y} = D^\mathsf{T}\mathbf{z}$ after transform with a tight frame D.
- Perform thresholding on $\mathbf{y}$ to get $\tilde{\mathbf{y}}$.
- Take inverse transform to get a reconstructed image $\tilde{\mathbf{x}} = D\tilde{\mathbf{y}}$.

Bivariate Shrinkage

- Image model $\mathbf{z} = \mathbf{x} + \mathbf{n}$. where $\mathbf{z}$=noisy image, $\mathbf{x}$=true image, and $\mathbf{n}$=i.i.d. Gaussian noise of zero mean and variance σ^2.
- Coefficients $\mathbf{y} = D\mathbf{z}$ after transform with transform/dictionary D
- Bivariate shrinkage by Selesnick: $T = \frac{\sqrt{3}\sigma^2}{\sigma_x\sqrt{y_1^2+y_2^2}}$.



This figure is from Selesnick's paper.

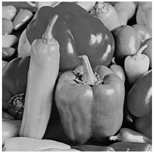
Test Images, Videos, and Inpainting Masks

(a) Barbara

(b) Lena

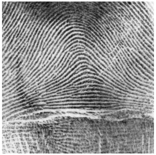
(c) C.man

(d) House



(e) Peppers

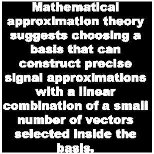
(f) Boat



(g) F.print

(h) Mobile

(i) C.guard



(j) Mask

Denoising Results: Barbara

	Barbara					
σ	UDWT	DT	$\mathbb{C}$TF4	$\mathbb{C}$TF6(Gain)	$\mathbb{C}$TF4	$\mathbb{C}$TF6(Gain)
5	36.90	37.36	37.41	**37.82**(0.46)	37.75	**38.10**(0.74)
10	32.66	33.52	33.62	**34.14**(0.48)	34.10	**34.47**(0.95)
15	30.31	31.38	31.47	**32.02**(0.64)	31.97	**32.32**(0.94)
20	28.71	29.87	29.91	**30.49**(0.62)	30.43	**30.77**(0.90)
25	27.52	28.70	28.71	**29.31**(0.61)	29.26	**29.57**(0.87)
30	26.58	27.77	27.74	**28.34**(0.57)	28.32	**28.61**(0.84)
50	24.27	25.26	25.21	**25.71**(0.45)	25.69	**26.02**(0.76)

The larger PSNR ($= 10\log_{10}\frac{255^2}{MSE}$) the better performance, where $MSE(u,v) := \frac{1}{|S|}\sum_{k\in S}|u(k)-v(k)|^2$ is the mean squared error.

Denoising Results: Barbara

The original image is on the left, the noisy image with $\sigma = 30$ is in the middle, and the denoised image is on the right.

Denoising Results: Lena

	Lena					
σ	UDWT	DT	$\mathbb{C}$TF4	$\mathbb{C}$TF6(Gain)	$\mathbb{C}$TF4	$\mathbb{C}$TF6(Gain)
5	38.05	38.25	38.10	**38.35** (0.10)	38.43	**38.53** (0.28)
10	34.84	35.19	35.14	**35.45** (0.26)	35.59	**35.70** (0.51)
15	33.09	33.47	33.50	**33.77** (0.30)	33.88	**34.01** (0.54)
20	31.85	32.23	32.31	**32.55** (0.22)	32.63	**32.77** (0.54)
25	30.89	31.26	31.38	**31.58** (0.32)	31.62	**31.78** (0.56)
30	30.10	30.47	30.61	**30.78** (0.31)	30.79	**30.96** (0.49)
50	27.83	28.21	28.41	**28.50** (0.29)	28.49	**28.64** (0.43)

UDWT=Undecimated Discrete Wavelet Transform.
DT=Dual Tree Complex Wavelet Transform.

Denoising Comparison for Barbara Image

	UDWT	DTCWT	TV	Shearlet	TP-$\mathbb{C}$TF$_6$
Redundancy $\rightarrow$	13	4	N/A	49	10.7
$\sigma = 10$	32.64	33.52	31.57	33.69	**34.14**
$\sigma = 15$	30.30	31.38	28.99	31.61	**32.02**
$\sigma = 20$	28.70	29.87	27.28	30.10	**30.49**
$\sigma = 25$	27.50	28.70	26.06	28.93	**29.31**
$\sigma = 30$	26.56	27.77	25.17	27.97	**28.34**

DTCWT=Dual Tree Complex Wavelet Transform.
TP-$\mathbb{C}$TF$_6$=Han and Zhao, SIAM J. Imag. Sci. 7 (2014), 997–1034.
UDWT=Undecimated Discrete Wavelet Transform.
TV=Rudin-Osher-Fatemi (ROF) model using higher-order scheme.
Shearlet=shearlet frames in W. Lim, IEEE T. Image Process., 2013.

Measure of performance: $\text{PSNR} = 10 \log_{10} \frac{255^2}{MSE}$.
The larger PSNR value the better performance.

Denoising Using Different Shrinkages & TP-$\mathbb{C}TF_6$

Lena	bishrink	GSM	UDGSM	MPGSM
$\sigma = 10$	35.45	35.50	35.70	**35.72**
$\sigma = 15$	33.77	33.73	34.01	**34.03**
$\sigma = 20$	32.55	32.48	32.77	**32.80**
$\sigma = 25$	31.58	31.50	31.78	**31.83**
$\sigma = 30$	30.78	30.70	30.96	**31.01**

Table: Lena PSNR values of advanced statistical models.

UDGSM=Undecimated TP-$\mathbb{C}TF_6$ using GSM (Gaussian Scale Mixture).

Denoising Using Different Shrinkages & TP-$\mathbb{C}TF_6$

Barbara	bishrink	GSM	UDGSM	MPGSM
$\sigma = 10$	34.14	34.21	34.47	**34.81**
$\sigma = 15$	32.02	32.08	32.32	**32.82**
$\sigma = 20$	30.49	30.59	30.77	**31.39**
$\sigma = 25$	29.31	29.45	29.57	**30.26**
$\sigma = 30$	28.34	28.52	28.61	**29.33**

Table: Barbara PSNR values of advanced statistical models.

UDGSM=Undecimated TP-$\mathbb{C}TF_6$ using GSM (Gaussian Scale Mixture).

Image Inpainting Model: $\mathbf{y} = \chi_\Omega \mathbf{x} + \mathbf{n}$

- Let $\Omega \subseteq \{1, \ldots, d\}$ be an observable region.

$$y_j = \begin{cases} x_j + n_j, & j \in \Omega, \\ \text{arbitray (unknown)}, & j \notin \Omega. \end{cases}$$

- $\mathbf{y} = (y_1, \ldots, y_d)^\mathsf{T}$ is the given observed image on Ω.
- $\mathbf{n} = (n_1, \ldots, n_d)^\mathsf{T}$ is i.i.d. Gaussian noise with variance σ^2.
- The inpainting mask Ω^c is known in advance.
- Goal: recover the unknown true image $\mathbf{x}$ by restoring missing pixels of $\mathbf{x}$ outside Ω and suppress its noise on Ω.
- Solve the inpainting problem $\mathbf{y} = \chi_\Omega \mathbf{x} + \mathbf{n}$ through the minimization scheme:

$$\min_{\mathbf{c} \in \mathbb{R}^n} \frac{1}{2} \|\chi_\Omega \mathcal{D}\mathbf{c} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{c}\|_1 + \kappa \|(I - \mathcal{D}^T \mathcal{D})\mathbf{c}\|_2^2,$$

where $\mathcal{D} \in \mathbb{R}^{n \times d}$ is a tight frame satisfying $\mathcal{D}\mathcal{D}^T = I_d$ and a reconstructed image is given by $\mathbf{x} = \mathcal{D}\mathbf{c}$.

Random Missing Pixels or Corrupted by Texts

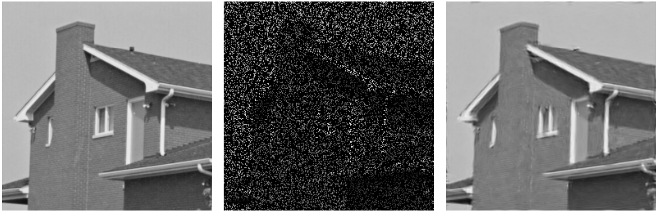


Figure: 80% missing pixels. Recovered by our algorithm: PSNR=31.67.

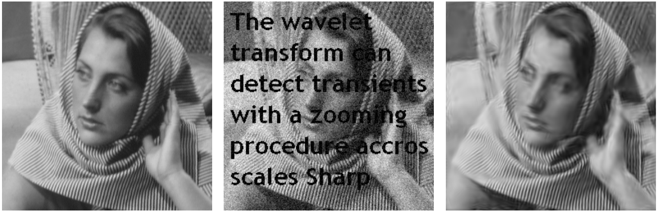


Figure: Corrupted by text with $\sigma = 20$. Recovered with PSNR$= 28.93$.

Image Inpainting Algorithm

Initialization: $\boldsymbol{x}_0 = 0$, $\lambda = \lambda_0$, $\ell = 0$.
while not convergent **do**
  $\boldsymbol{c}_{\ell+1} = \text{Thresholding}_\lambda(\mathcal{V}^\mathsf{T}(\chi_\Omega \boldsymbol{y} + (I - \chi_\Omega)\boldsymbol{x}_\ell))$.
  $\boldsymbol{x}_{\ell+1} = \mathcal{V}\boldsymbol{c}_{\ell+1}$.
  error $= \|(I - \chi_\Omega) - -(\boldsymbol{x}_\ell - \boldsymbol{x}_{\ell+1})\|_2 / \|\boldsymbol{y}\|_2$.
  if error $<$ tolerence **then**
    Update the thresholding value λ.
  end if
  $\ell = \ell + 1$.
end while
return $\boldsymbol{x}_{\ell+1}$.

Some Image Inpainting Algorithms

1. [CCS08]: Cai-Chan-Shen, Appl Comput Harmon Anal (2008). using Ron-Shen spline tight framelet $\{a; b_1, \dots, b_4\}$
2. [COS09] Cai-Osher-Shen, Multiscale Modeling Simul. (2009).
3. [CCS10]: Cai-Chan-Shen, Inverse Probl Imaging (2010). using Ron-Shen piecewise linear spline tight framelet $\{a; b_1, b_2\}$ with DCT.
4. [ESD05]: Elad-Stark-Donoho, Appl Comput Harmon Anal (2005), using curvelets and DCT.
5. [LSS13] Li-Shen-Suter, IEEE Trans Image Process. (2013), using undecimated DCT-Haar wavelets.
6. [L13] Lim, IEEE Trans Image Process. (2013), using undecimated compactly supported shearlet frames.
7. [SHB14] Shen-Han-Braverman, Image inpainting using directional tensor product complex tight framelets TP-$\mathbb{C}$TF$_6$.

Comparison of Results: Inpainting Without Noise

	Barbara	C.man	Lena	House	Peppers
CCS08 (25)	32.82	30.28	32.11	37.08	31.41
COS09 (11)	34.98	30.48	31.91	37.54	31.10
CCS10 (11)	35.19	30.56	31.83	37.36	31.07
ESD05 (4.9)	35.24	31.02	33.59	35.41	31.32
LSS13 (49)	36.60	31.99	33.82	35.76	32.27
L13 (49)	36.07	32.35	34.82	**39.66**	32.49
SHB14 (10.7)	**38.38**	**32.74**	**35.03**	39.60	**33.21**

CCS08 (25)	27.77	26.75	28.32	31.01	27.10
COS09 (11)	29.01	26.70	28.05	31.18	26.76
CCS10 (11)	29.35	26.88	28.14	31.32	26.79
ESD05 (4.9)	29.50	27.23	29.16	33.06	27.52
LSS13 (49)	30.46	28.58	29.50	32.45	28.05
L13 (49)	30.56	28.52	30.26	33.65	29.17
SHB14 (10.7)	**32.02**	**28.77**	**30.70**	**34.65**	**29.19**

Image Inpainting With Noise

σ	Barbara	C.man	Lena	House	Peppers
5	34.88	31.49	33.08	36.08	31.42
10	32.17	29.87	31.21	33.64	29.69
20	28.93	27.56	28.64	30.88	27.45
30	26.95	26.07	26.89	29.02	25.94
50	24.62	24.13	24.74	26.61	23.94
5	30.69	28.24	29.85	33.15	28.41
10	29.11	27.28	28.70	31.56	27.47
20	26.89	25.78	26.93	29.22	25.86
30	25.44	24.67	25.67	27.63	24.64
50	23.62	22.97	23.96	25.57	22.86

Other inpainting algorithms have much lower PSNR values, typically 4 to 12db less than SHB14 using TP-$\mathbb{C}$TF$_6$.

Remove Mixed Gaussian and Impulse Noises

Gaussian and Pepper–and–Salt impulse noise. Cameraman: $\sigma = 0$, $p = 0.3$, PSNR $= 32.50$. Lena: $\sigma = 15$, $p = 0.5$, PSNR $= 30.95$.

Gaussian and Random-valued impulse noises: Barbara: $\sigma = 30$, $p = 0.2$, PSNR $= 25.93$. Peppers: $\sigma = 20$, $p = 0.1$, PSNR $= 27.31$.

Remove Gaussian & Pepper-and-Salt Noise

	AOP	TP-$\mathbb{C}$TF$_6$	AOP	TP-$\mathbb{C}$TF$_6$	AOP	TP-$\mathbb{C}$TF$_6$
σ p	256×256 Cameraman		256×256 House		256×256 Peppers	
5 0.1	31.09	**32.97**(1.88)	36.35	**38.16**(1.81)	31.29	**32.11**(0.82)
5 0.3	29.02	**31.12**(2.10)	34.38	**36.23**(1.85)	28.79	**29.55**(0.76)
15 0.1	27.44	**29.24**(1.80)	29.32	**32.83**(3.51)	27.42	**28.85**(1.43)
15 0.3	26.45	**27.75**(1.31)	29.22	**31.90**(2.68)	26.46	**27.37**(0.91)
σ p	256×256 Cameraman		256×256 House		256×256 Peppers	
5 0.1	36.40	**37.65**(1.25)	29.39	**34.52**(5.13)	33.80	**35.53**(1.73)
5 0.3	34.74	**36.33**(1.59)	27.43	**33.68**(6.25)	31.66	**33.68**(2.02)
15 0.1	29.39	**33.12**(3.73)	26.14	**30.63**(4.49)	28.48	**30.73**(2.25)
15 0.3	29.16	**31.89**(2.74)	25.25	**29.48**(4.23)	27.96	**29.68**(1.72)

AOP, TV-based, SIAM J. Imaging, 5 (2013),1227–1245.

TP-$\mathbb{C}$TF$_6$, Shen/Han/Braverman, J. Math. Imaging Vis., 54 (2016), 64–77.

Remove Gaussian & Random-valued Impulse Noise

σ	p	PARIGI	TP-$\mathbb{C}$TF$_6$	PARIGI	TP-$\mathbb{C}$TF$_6$	PARIGI	TP-$\mathbb{C}$TF$_6$
σ	p	512 × 512 Baboon		512 × 512 Barbara		512 × 512 Bridge	
5	0.1	24.81	**28.30**(3.49)	31.55	**35.20**(3.65)	26.96	**29.32**(2.36)
5	0.3	23.05	**24.08**(1.03)	29.28	**29.56**(0.28)	25.45	**25.51**(0.06)
15	0.1	23.63	**25.71**(2.08)	28.88	**30.50**(1.62)	25.34	**26.48**(1.14)
15	0.3	21.81	**23.28**(1.47)	27.33	**27.93**(0.60)	23.38	**24.59**(1.21)
σ	p	512 × 512 Boat		512 × 512 Goldhill		512 × 512 Lena	
5	0.1	31.41	**33.66**(2.25)	32.60	**34.23**(1.63)	34.72	**36.57**(1.85)
5	0.3	**28.81**	28.34(-0.47)	**30.64**	29.99(-0.65)	**32.57**	31.91(-0.66)
15	0.1	28.21	**30.08**(1.87)	29.08	**30.13**(1.05)	30.31	**32.80**(2.49)
15	0.3	26.57	**27.68**(1.11)	27.99	**28.66**(0.67)	29.22	**31.03**(1.81)

Observable region Ω is unknown, σ^2 is the variance of noise, and $p = \#\Omega^c/d$ is the noise ratio.

PARIGI, patch-based, SIAM J. Imaging, 5 (2012),1140–1174.

TP-$\mathbb{C}$TF$_6$, Shen/Han/Braverman, J. Math. Imaging Vis., 54 (2016), 64–77.

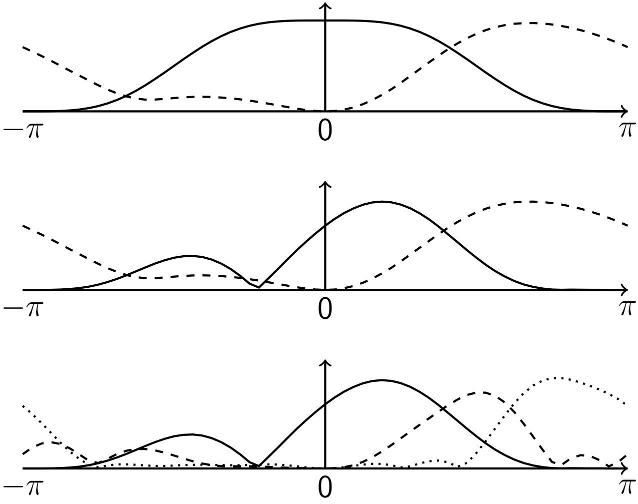
Remove Gaussian & Random-valued Impulse Noise

	K-ALS	TP-$\mathbb{C}$TF$_6$	K-ALS	TP-$\mathbb{C}$TF$_6$	K-ALS	TP-$\mathbb{C}$TF$_6$
σ p	256 × 256 C.man		256 × 256 House		256 × 256 Peppers	
5 0.1	28.25	**29.71**(1.46)	35.71	**36.83**(1.11)	28.47	**30.61**(2.14)
5 0.3	24.38	**25.21**(0.83)	**30.52**	30.39(-0.13)	25.08	**25.37**(0.29)
15 0.1	**27.45**	27.32(-0.13)	32.59	**32.60**(0.02)	27.33	**27.78**(0.45)
15 0.3	23.69	**24.73**(1.03)	28.06	**29.44**(1.37)	23.96	**24.65**(0.69)
σ p	512 × 512 Lena		512 × 512 Barbara		512 × 512 Boat	
5 0.1	35.07	**36.65**(1.58)	32.12	**35.18**(3.06)	31.86	**33.43**(1.57)
5 0.3	29.96	**31.22**(1.27)	26.42	**26.89**(0.47)	**27.63**	27.53(-0.1)
15 0.1	32.10	**32.71**(0.61)	29.70	**30.60**(0.9)	29.68	**29.98**(0.3)
15 0.3	27.66	**29.63**(1.97)	25.14	**25.52**(0.38)	25.97	**26.34**(0.37)

K-ALS, patch-based, SIAM J. Imaging, 6 (2013),526–562.

TP-$\mathbb{C}$TF$_6$, Shen/Han/Braverman, J. Math. Imaging Vis., 54 (2016), 64–77.

Frequency Separation Property and Directionality



Frequency Separation Property and Directionality

- $\widehat{b_\ell^+}(\xi) \approx 0$ for $\xi \in [-\pi, 0]$ while $\widehat{b_\ell^-}(\xi) \approx 0$ for $\xi \in [0, \pi]$.
- By $\widehat{\psi_\ell}(\xi) := \widehat{b_\ell}(\xi/2)\widehat{\phi}(\xi/2)$, frequency separation property:

$$\widehat{\psi_1}(\xi) \approx \chi_{[\zeta_1 - c_1, \zeta_1 + c_1], \varepsilon_1, \varepsilon_1}, \quad \widehat{\psi_2}(\xi) \approx \chi_{[\zeta_2 - c_2, \zeta_2 + c_2], \varepsilon_2, \varepsilon_2}.$$

- Define a real-valued isotropic function f by $\widehat{f} = g$ with

$$g(\xi_1, \xi_2) = \chi_{[-c_1, c_1], \varepsilon_1, \varepsilon_1}(\xi_1)\chi_{[-c_2, c_2], \varepsilon_2, \varepsilon_2}(\xi_2).$$

- Tensor product framelet $\psi := \psi_1 \otimes \psi_2$ satisfies

$$\psi^{[r]}(x) = f(x)\cos((\zeta_1, \zeta_2) \cdot x), \quad \psi^{[i]}(x) = f(x)\sin((\zeta_1, \zeta_2) \cdot x).$$

- Directionality is induced by $\cos(\zeta \cdot x)$ and $\sin(\zeta \cdot x)$ waves.
- Take advantages of wavelets and Discrete Cosine Transform.

Summary

- Four types: Orthogonal or biorthogonal wavelet filter banks; tight or dual framelet filter banks.
- Advantages of wavelets: sparse multiscale representation, fast wavelet/framelet transform, good spatial and frequency localization, singularity detection, etc.
- A subdivision scheme is an iterative local averaging scheme for generating smooth curves and only uses low-pass filters.
- Signal/image compression often uses orthogonal wavelets or symmetric biorthogonal wavelets.
- Signal/image denoising often uses tight or dual framelets.
- Three steps of wavelet applications: wavelet decomposition, coefficient thresholding, wavelet reconstruction.