Basics on Wavelet Theory and Its Applications

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Summer Enrichment Program at UofA

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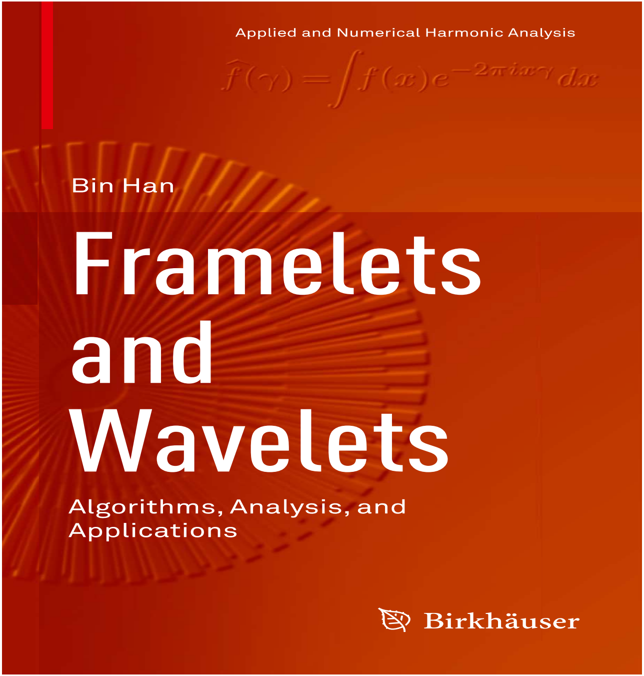
Outline of Mini-Course Talks

- What is the discrete wavelet transform and why is it useful?
- General discrete wavelet transforms and types of wavelets.
- Wavelets and framelets in the discrete and function settings.
- Wavelet applications to signal and image processing.
- Wavelet-based methods for numerical solutions to partial differential equations (PDEs).

Declaration: Some figures and graphs in this talk are from the book [Bin Han, Framelets and Wavelets: Algorithms, Analysis and Applications, Birkhäuser/Springer, 2017] and various other sources from Internet, or from published papers, or produced by `matlab`, `maple`, or `C` programming. [Details and sources of all graphs can be provided upon request of the audience.]


Applied and Numerical Harmonic Analysis
$\widehat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$
Bin Han
Framelets and Wavelets
Algorithms, Analysis, and Applications
Birkhäuser

How to represent data economically?

- In today's world, most data and signals are in digital format: digital TV, movies, songs, signals, images, videos,...
- How to represent data effectively (as few numbers as possible)?
- How to detect the sharp changes in data?
- Key advantages of wavelets:
    - sparse representation.
    - multiscale tree structure.
    - fast computational algorithms.

Record information effectively

Given a particular signal to you:
$[-21, -22, -23, -23, -25, 38, 36, 34]$.
If you are allowed to send out only one number about this signal,

which number shall you choose?

Your answer(s):

Record information effectively

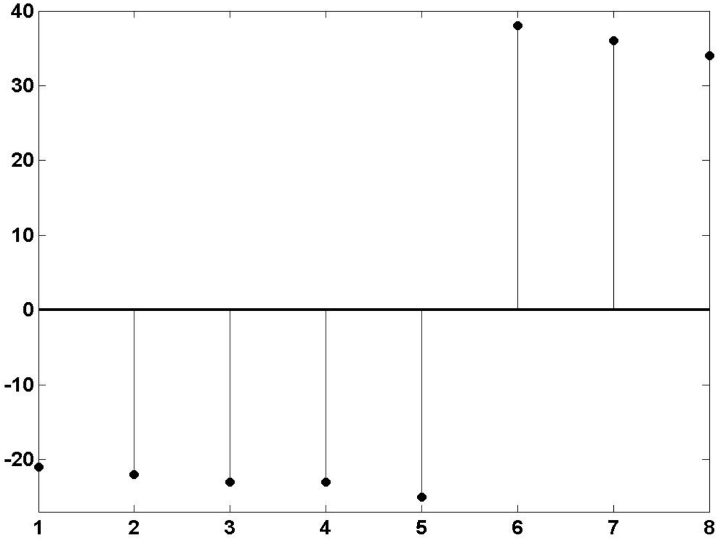
Given a particular signal to you:
$[-21, -22, -23, -23, -25, 38, 36, 34]$.
If you are allowed to send out only one number about this signal,

which number shall you choose?

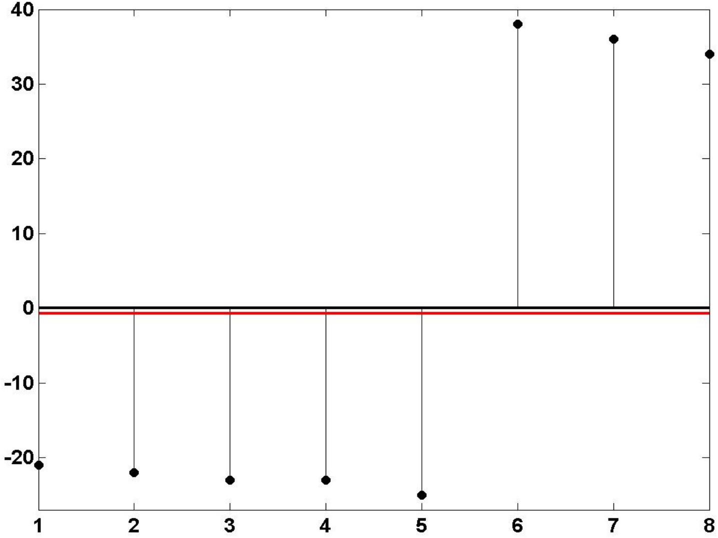
Your answer(s): Average

$$\frac{-21-22-23-23-25+38+36+34}{8} = -0.75.$$

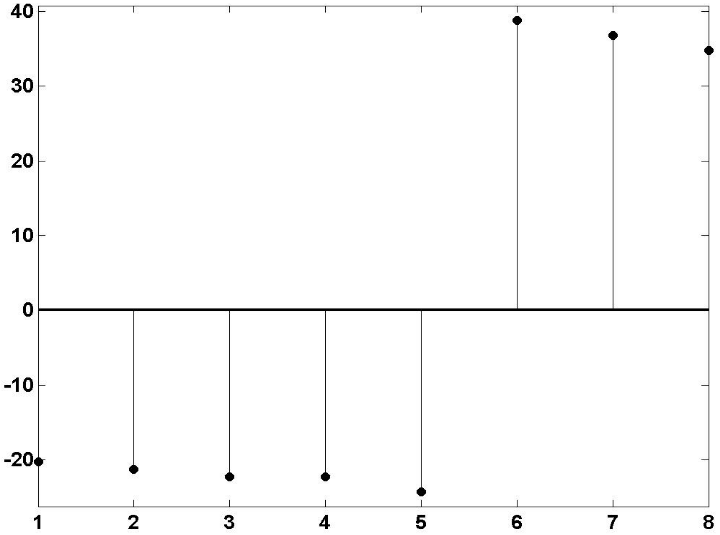
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



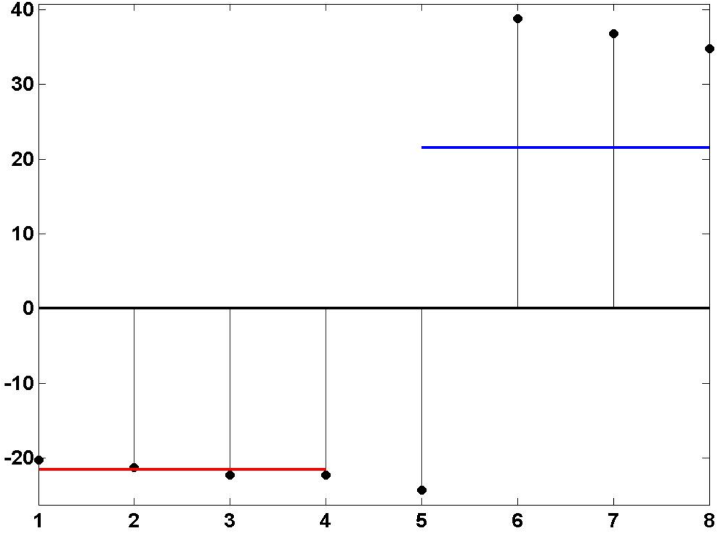
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



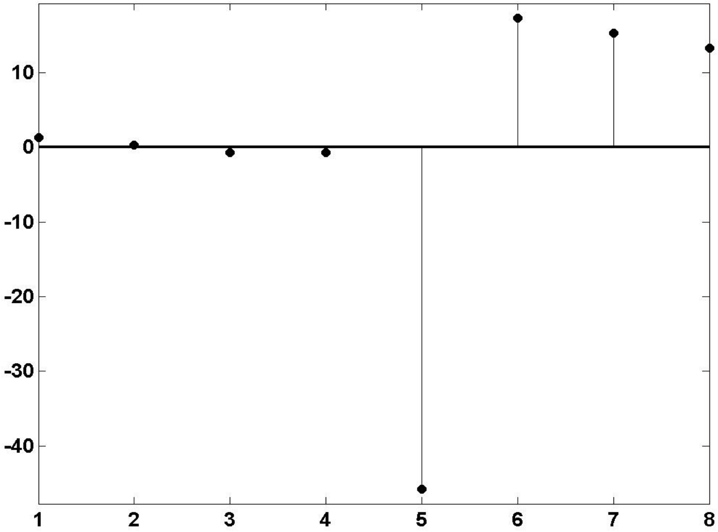
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



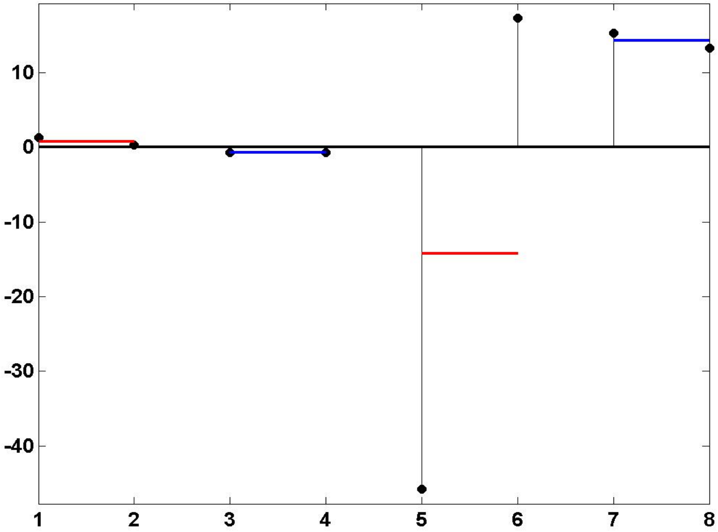
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



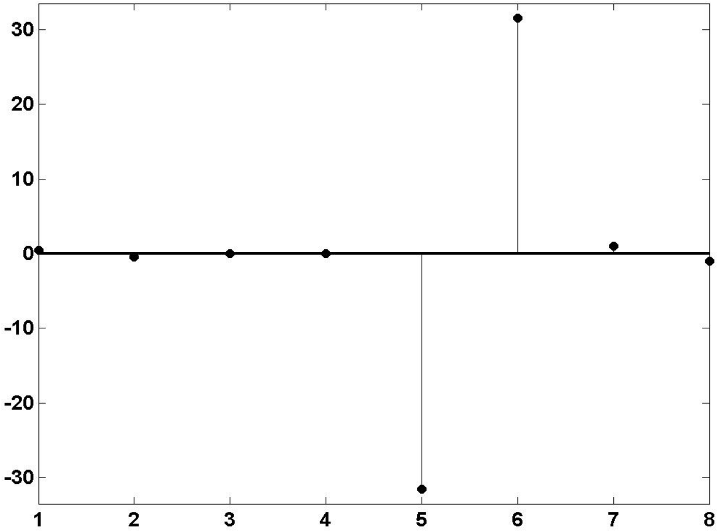
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



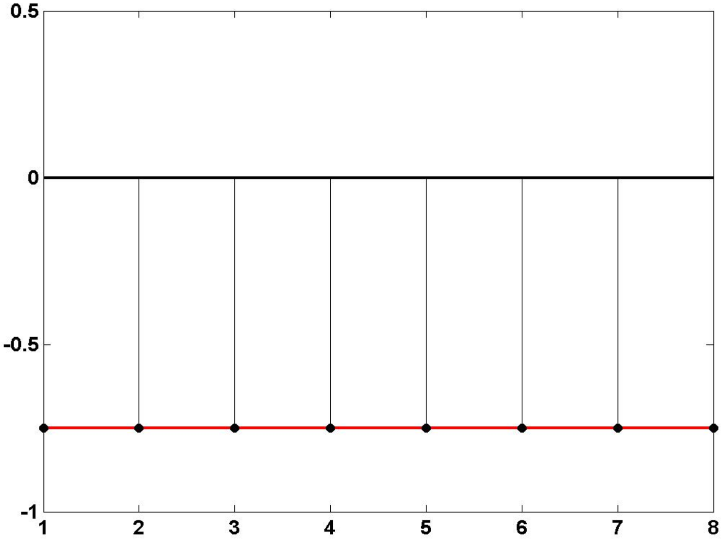
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



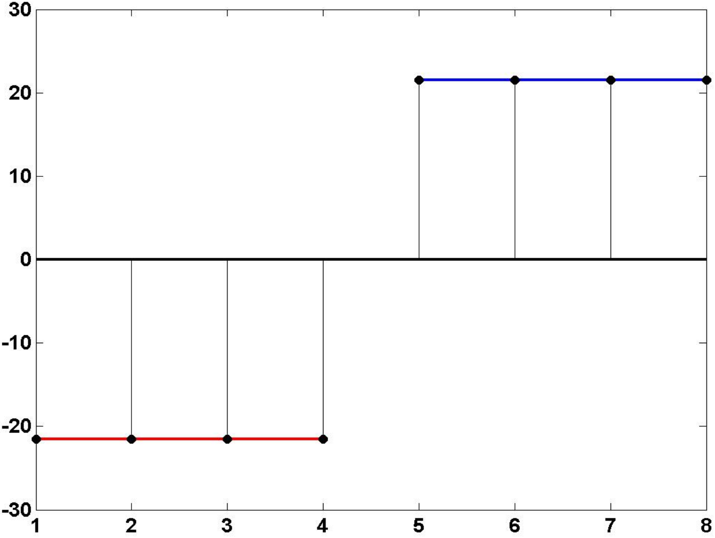
The idea of wavelets using numbers

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- Averages at level 1 (A1): -0.75,
- Average at level 2 (A2): -21.5, 21.5
- Averages at level 3 (A3): 0.75, -0.75, -14.25, 14.25.
- Averages at level 4 (A4): 0.5, -0.5, 0, 0, -31.5, 31.5, 1, -1.

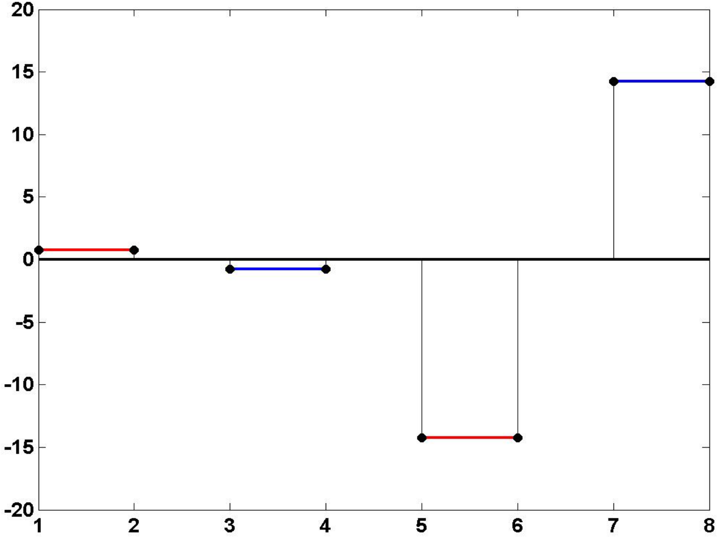
Graph of wavelet coefficients $A1$



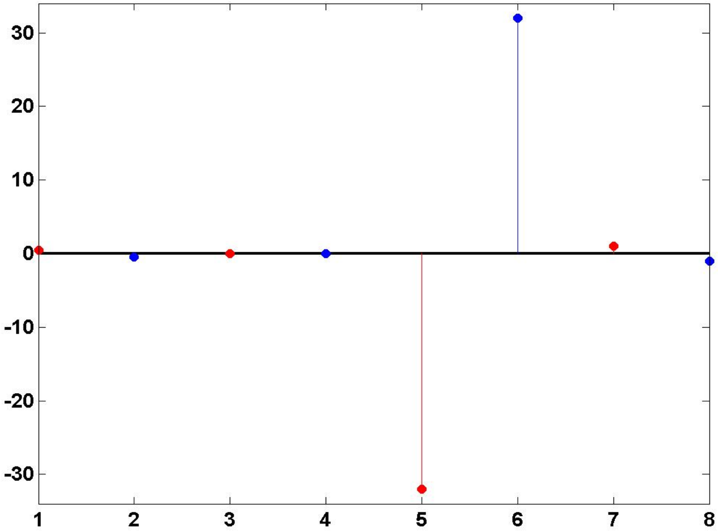
Graph of wavelet coefficients $A2$



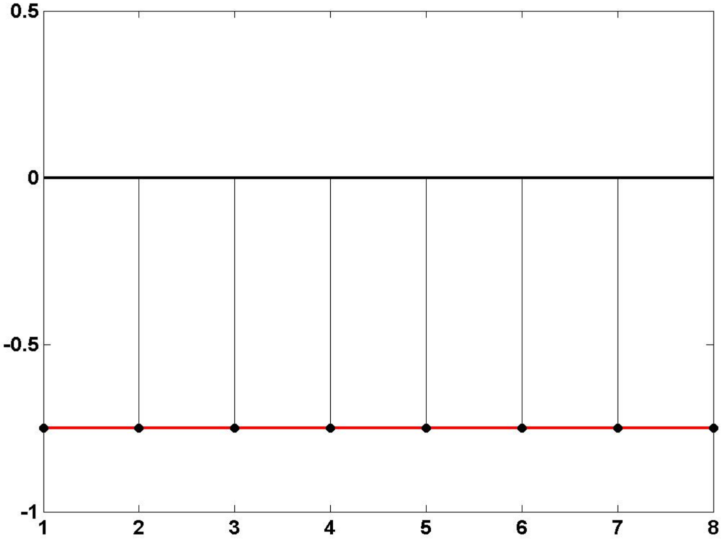
Graph of wavelet coefficients $A3$



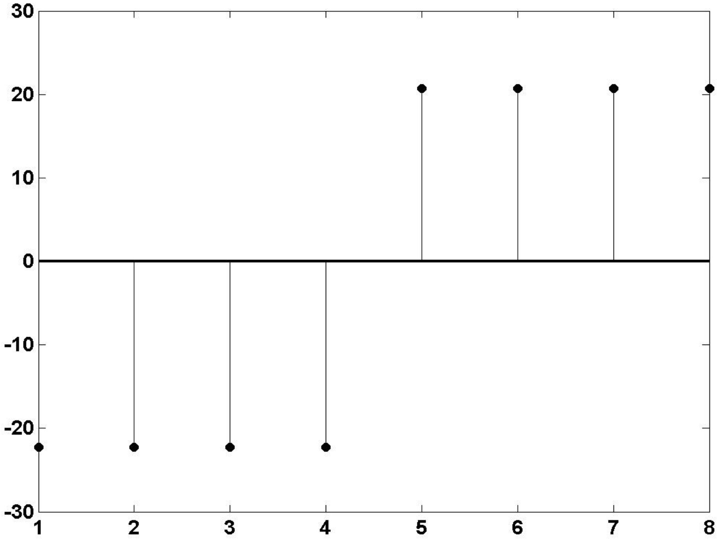
Graph of wavelet coefficients $A4$



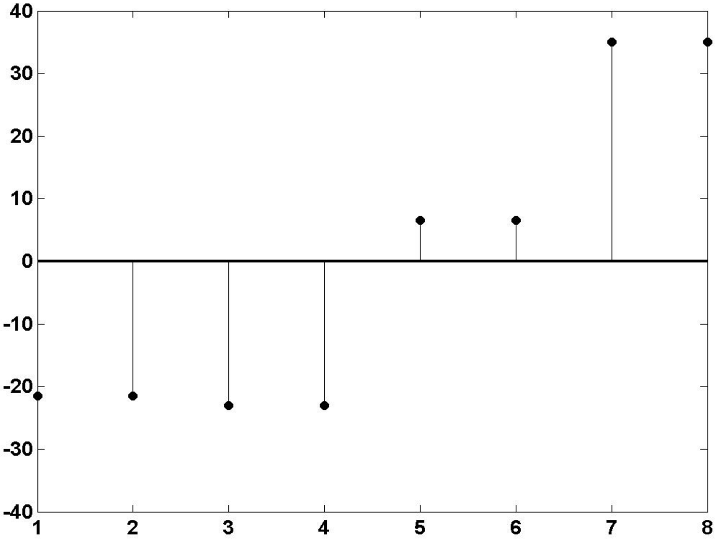
Reconstruction: $A1$ (1 number)



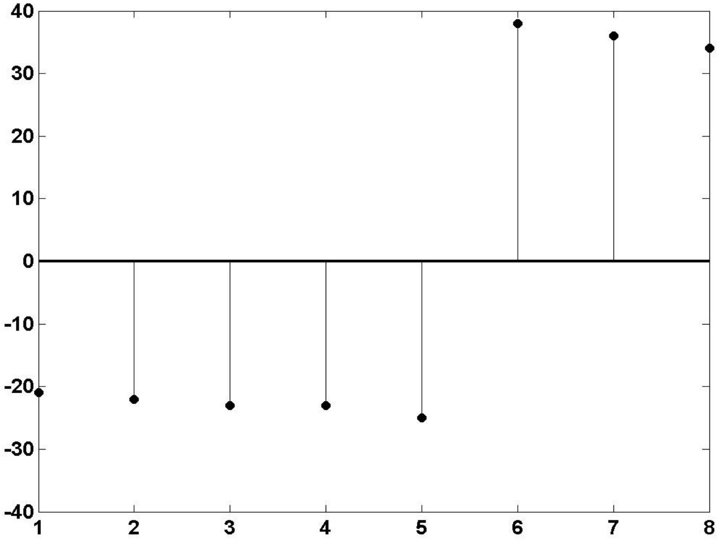
Reconstruction: $A1 + A2$ (2 numbers)



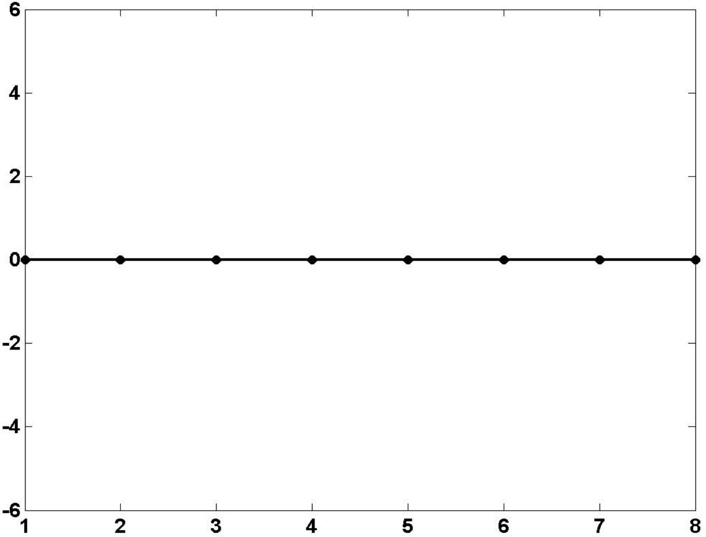
Reconstruction: $A1 + A2 + A3$ (4 numbers)



Reconstruction: $A1 + A2 + A3 + A4$ (8 numbers)



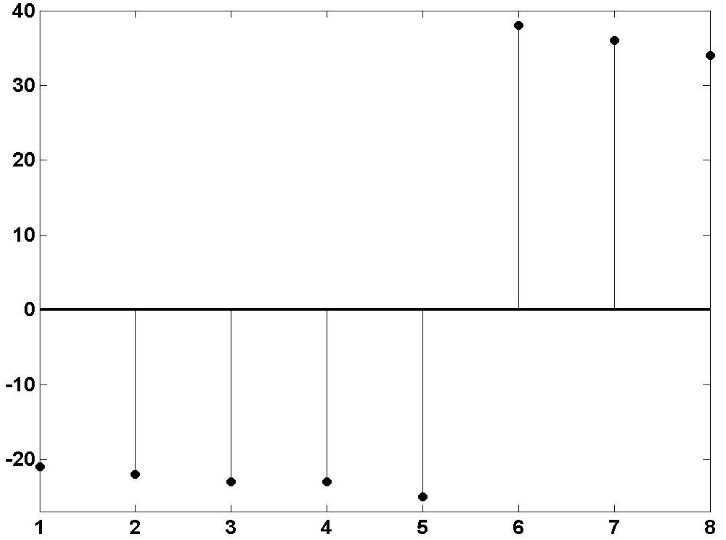
Comparison: Original−Reconstructed



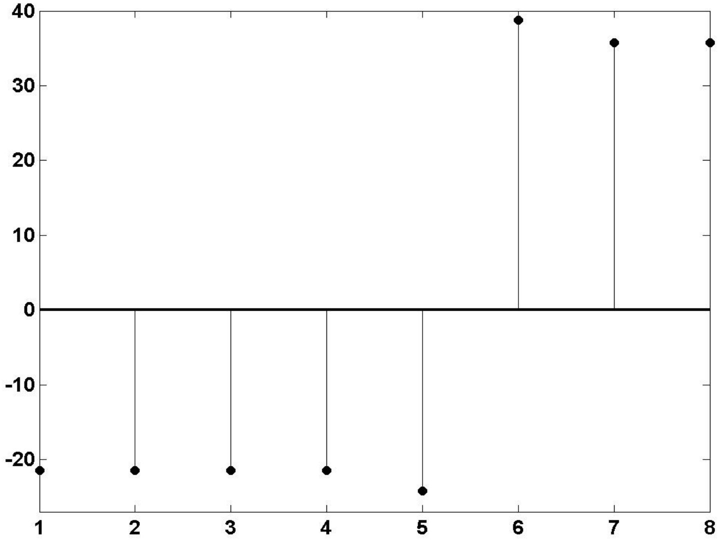
Why wavelets?

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- Averages at level 1 (A1): $\boxed{\text{-0.75}}$,
- Average at level 2 (A2): -21.5, 21.5
- Averages at level 3 (A3): $\boxed{0.75}$, -0.75, -14.25, 14.25.
- Averages at level 4 (A4): $\boxed{0.5}$, -0.5, $\boxed{0}$, 0, -31.5, 31.5, $\boxed{1}$, -1

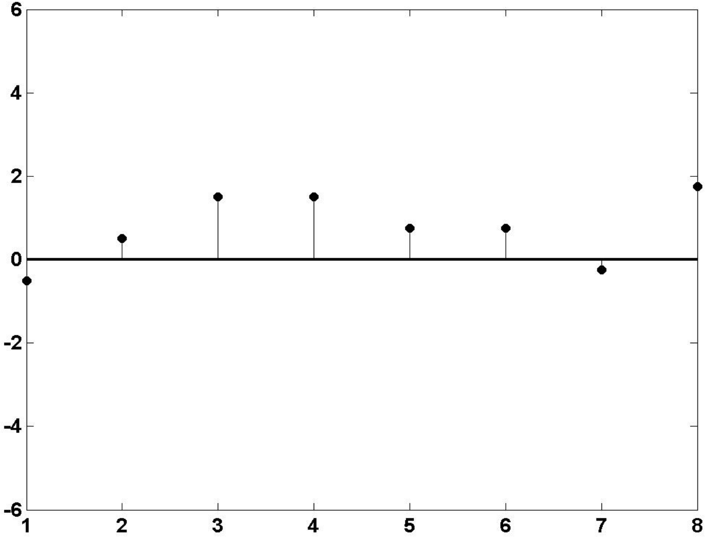
Comparison: Original



Reconstructed with 3 numbers by thresholding



Comparison: Original—Reconstructed



How to compute wavelet coefficients fast?

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- Averages at level 1 (A1): -0.75,
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Are we missing something for wavelets? or can we expect more from wavelets?

How to compute wavelet coefficients fast?

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Are we missing something for wavelets? or can we expect more from wavelets?

For applications,
a fast computational algorithm
is highly demanded!

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- Averages: $[-22.25, 20.75]$. Differences: $[0.75, -14.25]$.

Fast Wavelet Transform (FWT): Decomposition

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- Compare:

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Fast Wavelet Transform (FWT): Reconstruction

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- Averages: $[-22.25, 20.75]$. Differences: $[0.75, -14.25]$.
- Averages: $[-0.75]$. Differences: $[-21.5]$.
- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.

Fast Wavelet Transform (FWT): Reconstruction

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
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- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$

Fast Wavelet Transform (FWT): Reconstruction

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
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- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.

Fast Wavelet Transform (FWT): Reconstruction

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
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- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.
- Add detail $[0.75, -0.75 - 14.25, 14.25] \Rightarrow [-21.5, -23, 6.5, 35]$.

Fast Wavelet Transform (FWT): Reconstruction

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- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
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- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.
- Add detail $[0.75, -0.75 - 14.25, 14.25] \Rightarrow [-21.5, -23, 6.5, 35]$.
- Subdivide$[-21.5, -23, 6.5, 35] \to$ $[-21.5, -21.5, -23, -23, 6.5, 6.5, 35, 35]$.

Fast Wavelet Transform (FWT): Reconstruction

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
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- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.
- Add detail $[0.75, -0.75 - 14.25, 14.25] \Rightarrow [-21.5, -23, 6.5, 35]$.
- Subdivide$[-21.5, -23, 6.5, 35] \to$ $[-21.5, -21.5, -23, -23, 6.5, 6.5, 35, 35]$.
- Add detail $[0.5, -0.5, 0, 0, -31.5, 31.5, 1, -1]$ to get $[-21, -22, -23, -23, -25, 38, 36, 34]$.

Discrete Fourier Transform (DFT)

- Discrete Fourier transform (DFT): $[x(0), \dots, x(N-1)] \to [\widehat{x}(0), \dots, \widehat{x}(N-1)]$:

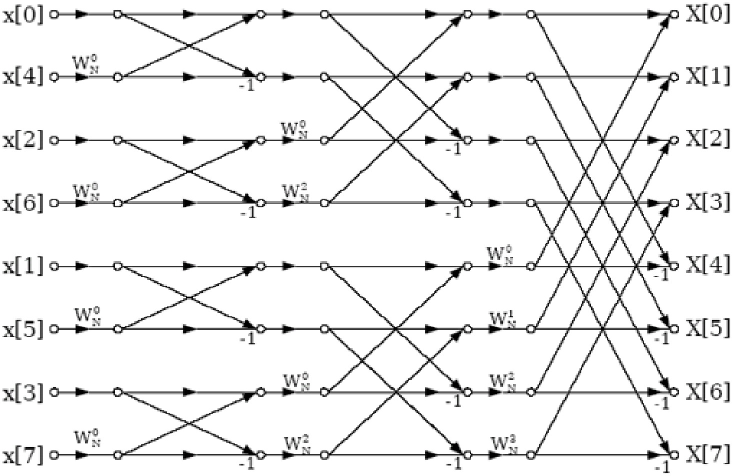
$$\widehat{x}(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}, \qquad k = 0, \dots, N-1.$$

- Inverse discrete Fourier transform (iDFT): $[\widehat{x}(0), \dots, \widehat{x}(N-1)] \to [x(0), \dots, x(N-1)]$:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \widehat{x}(k) e^{i2\pi kn/N}, \qquad n = 0, \dots, N-1.$$

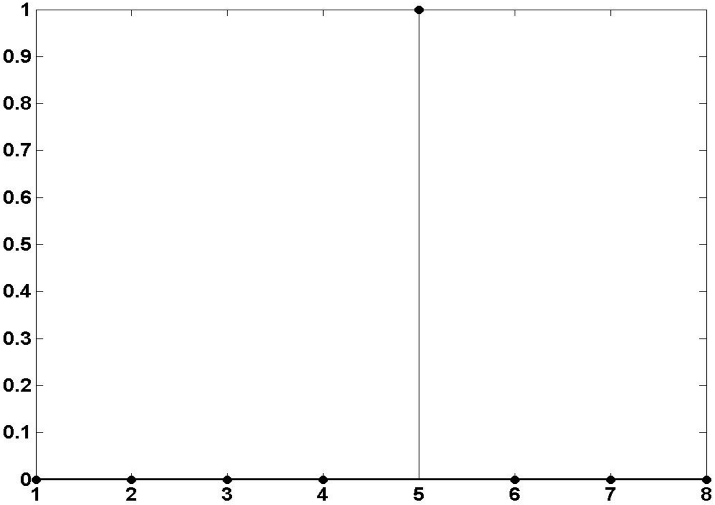
Fast Fourier Transform: Complexity $\mathscr{O}(N \log N)$

$\widehat{x}$ can be computed efficiently by Fast Fourier Transform (FFT), for example, through the Butterfly Scheme:



Fourier transform Lacks Time Localization

$x = [0, 0, 0, 0, 1, 0, 0, 0] \longleftrightarrow \widehat{x} = [1, -1, 1, -1, 1, -1, 1, -1].$



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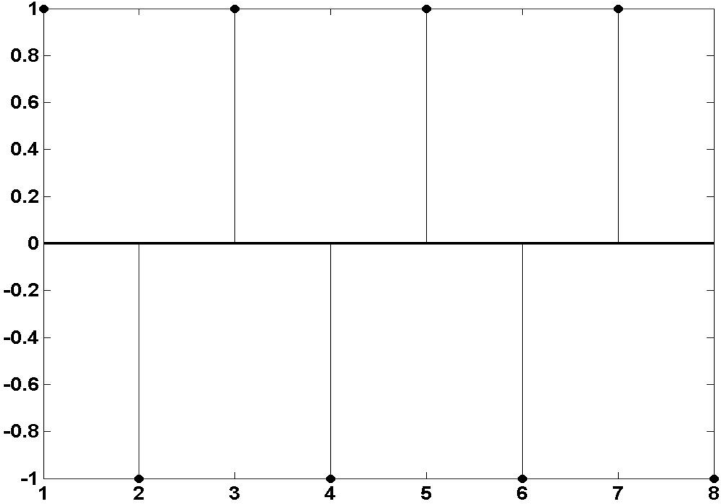
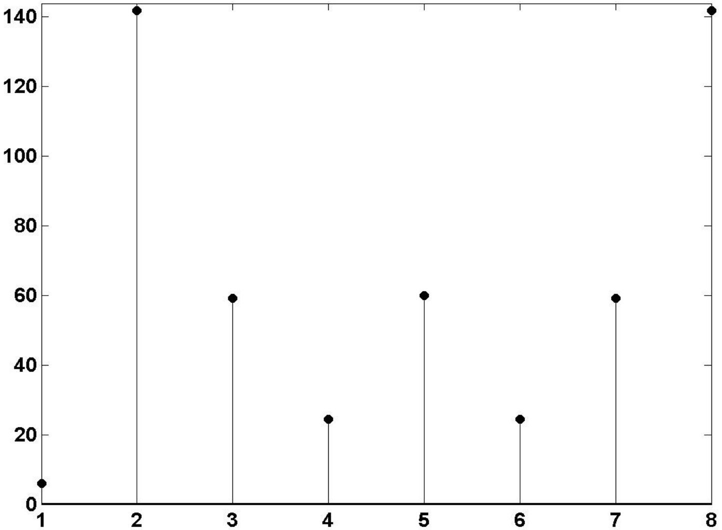


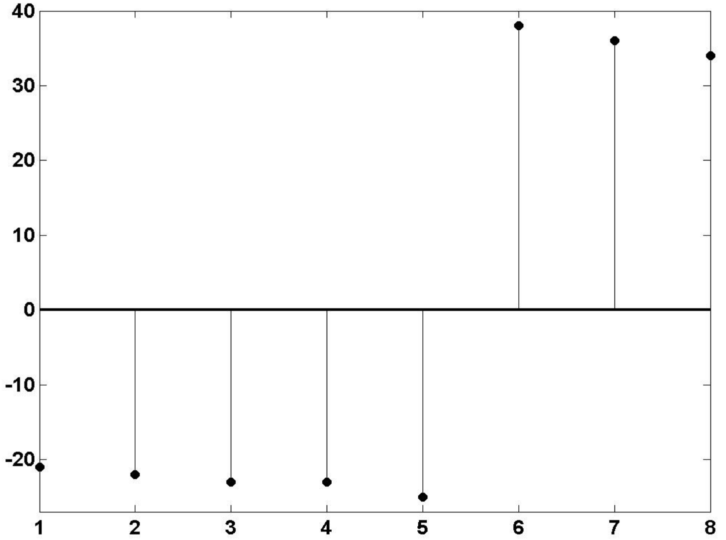
How does Fourier Transform Perform?

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- $|\widehat{x}| = [6, 141.74, 59.21, 24.51, 60, 24.51, 59.21, 141.74]$.
- $\widehat{x} = [-6, 1.87 + 141.73i, -59 - 5i, 6.12 + 23.74i, -60, 6.12 - 23.74i, -59 + 5i, 1.87 - 141.73i]$.
- Reconstructed using 3 largest coefficients:
- $\tilde{x} = [-7.03, -17.22, -42.93, -17.89, -7.97, 32.22, 27.93, 32.89]$.

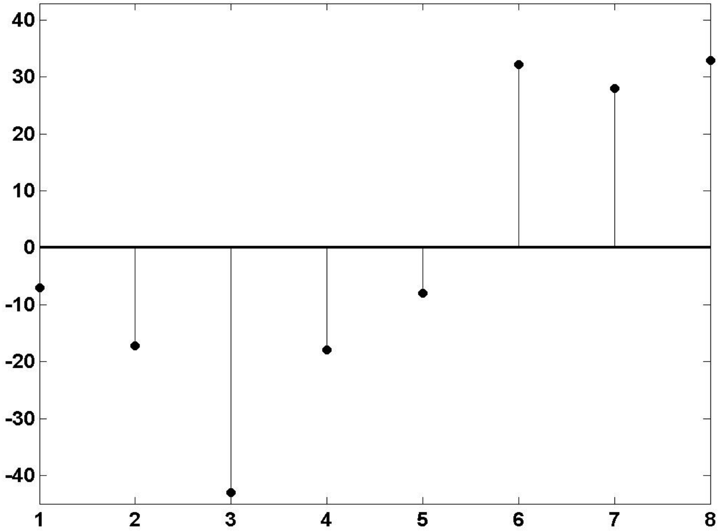
Magnitude of Fourier Coefficients



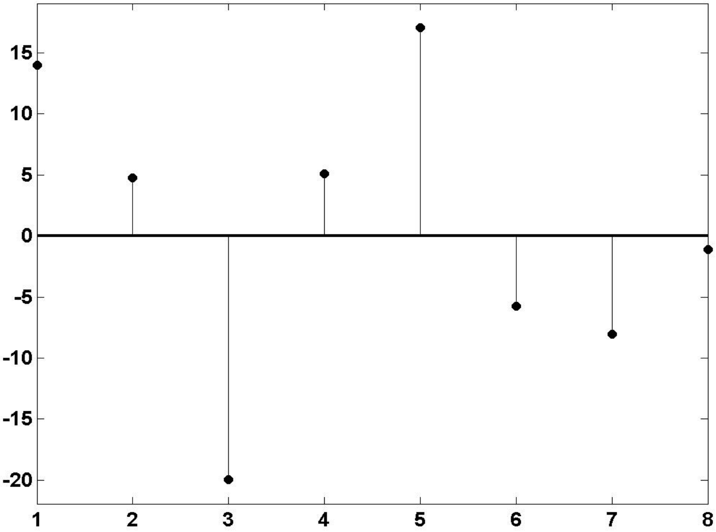
Comparison: Original



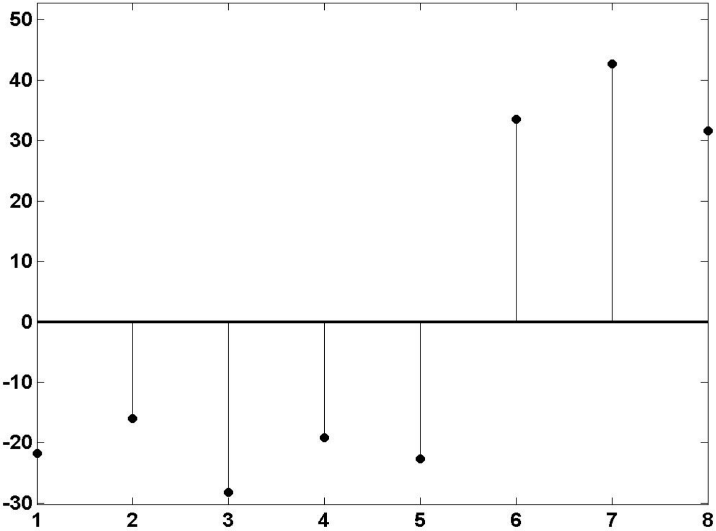
Reconstructed with 3 Numbers by Thresholding



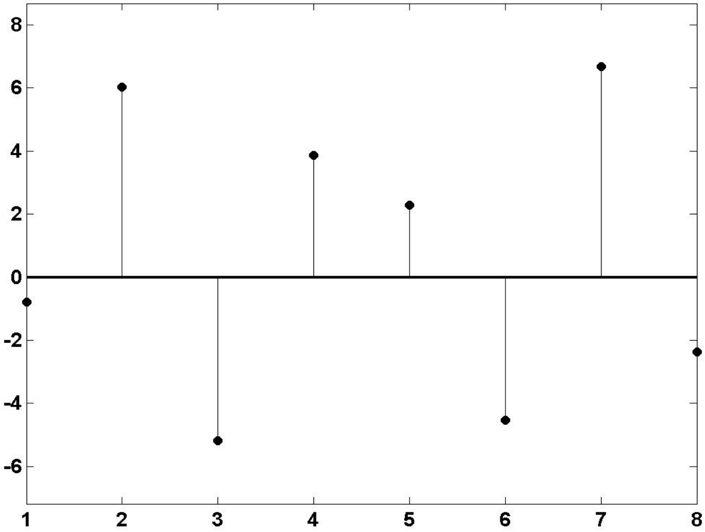
Comparison: Original—Reconstructed



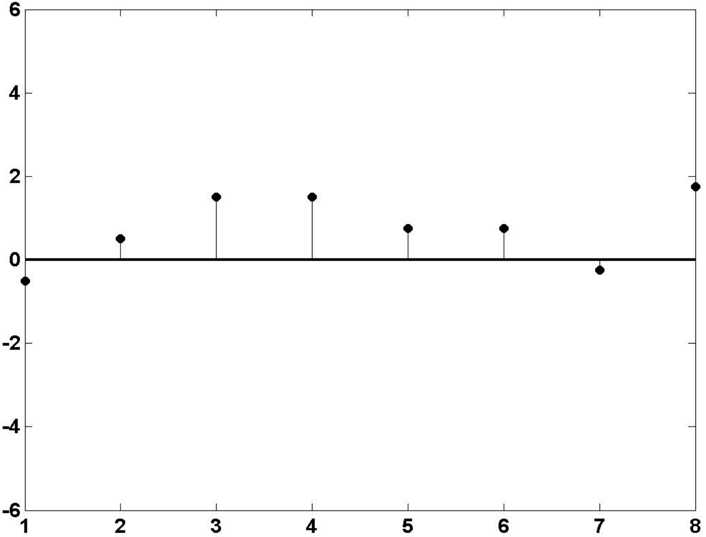
Reconstructed with 5 Numbers by Thresholding



Comparison: Original—Reconstructed



Comparison with Wavelets: Original-Reconstructed



Revisit: Multiscale Wavelet Transform

- Decomposition:
- $[-21, -22, -23, -23, -25, 38, 36, 34]$.
- $[-21.5, -23, 6.5, 35, 0.5, 0, -31.5, 1]$.
- $[-22.25, 20.75, 0.75, -14.25, 0.5, 0, -31.5, 1]$.
- $[-0.75, -21.5, 0.75, -14.25, 0.5, 0, -31.5, 1]$.
- Processing: Drop small wavelet coefficients.
- Reconstruction:
- $[-0.75, -21.5, 0.75, -14.25, 0.5, 0, -31.5, 1]$.
- Apply subdivision scheme: $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- $[-22.25, 20.75, 0.75, -14.25, 0.5, 0, -31.5, 1]$.
- $[-21.5, -23, 6.5, 35, 0.5, 0, -31.5, 1]$.
- $[-21, -22, -23, -23, -25, 38, 36, 34]$.

Convolutions, Downsampling and Upsampling

- Model signals by sequences in $\ell(\mathbb{Z})$: $v = \{v(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$.
- Model filters by finitely supported sequences $\ell_0(\mathbb{Z})$: $u = \{u(k)\}_{k\in\mathbb{Z}}$ such that $u(k) \neq 0$ for finitely many $k \in \mathbb{Z}$.
- Convolutions: For $j \in \mathbb{Z}$,

$$[u * v](j) := \sum_{k\in\mathbb{Z}} u(j-k)v(k)$$
$$= \cdots + u(1)v(j-1) + u(0)v(j) + u(-1)v(j+1) + \cdots$$

- Downsampling: For $v = \{v(k)\}_{k\in\mathbb{Z}}$, $[v\!\downarrow\!2](k) := v(2k)$ for $k \in \mathbb{Z}$. That is, only keep $v(k)$ on even integer positions $k \in 2\mathbb{Z}$.
- Upsampling: Padding zero in between,

$$[v\!\uparrow\!2](k) = v(k/2) \text{ if } k \text{ is even} \quad \text{and} \quad [v\!\uparrow\!2](k) = 0 \text{ if } k \text{ is odd.}$$

- Flipping filters: $u^\star(k) := \overline{u(-k)}$ for $k \in \mathbb{Z}$.

z-transform and Fourier series

- For $u \in \ell(\mathbb{Z})$, its z-transform is given by

$$\mathrm{u}(z) := \sum_{k\in\mathbb{Z}} u(k) z^k, \qquad z \in \mathbb{C}\backslash\{0\}.$$

- Its Fourier transform $\widehat{u}$ is 2π-periodic and is given by

$$\widehat{u}(\xi) := \mathrm{u}(e^{-i\xi}) = \sum_{k\in\mathbb{Z}} u(k) e^{-ik\xi}, \qquad \xi \in \mathbb{R}.$$

- The z transform of $u * v$ is $\mathrm{u}(z)\mathrm{v}(z)$ and $\widehat{u * v}(\xi) = \widehat{u}(\xi)\widehat{v}(\xi)$.
- The z-transform of $v\!\downarrow\!2$ is $[v(z^{1/2}) + v(-z^{1/2})]/2$ and $\widehat{v\!\downarrow\!2}(\xi) = [\widehat{v}(\xi/2) + \widehat{v}(\xi/2+\pi)]/2$.
- The z-transform of $v\!\uparrow\!2$ is $\mathrm{v}(z^2)$ and $\widehat{v\!\uparrow\!2}(\xi) = \widehat{v}(2\xi)$.
- $\mathrm{u}^\star(z) = \overline{\mathrm{u}(1/\overline{z})}$ and $\widehat{u^\star}(\xi) = \overline{\widehat{u}(\xi)}$.

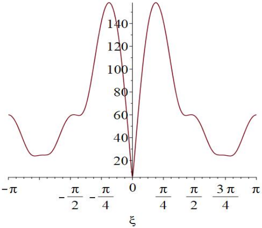
Haar Wavelet Decomposition

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]_{[0,7]}$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- $v = [\ldots, 0, 0, -21, -22, -23, -23, -25, 38, 36, 34, 0, 0, \ldots]$.
- Define the Haar low-pass filter a and high-pass filter b:

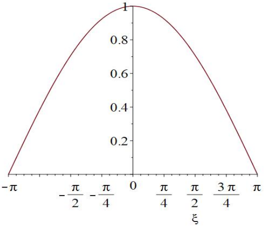
$$a = \{\tfrac{1}{2}, \tfrac{1}{2}\}_{[0,1]}, \qquad b = \{\tfrac{1}{2}, -\tfrac{1}{2}\}_{[0,1]}.$$

- $\widehat{v * a^\star}(\xi) = \widehat{v}(\xi)\widehat{a^\star}(\xi) = \widehat{v}(\xi)\overline{\widehat{a}(\xi)}$, that is, $v * a^\star$ is $[\ldots, \underline{0}, -10.5, \underline{21.5}, -22.5, \underline{-23}, -24, \underline{6.5}, 37, \underline{35}, 17, \underline{0}, \ldots]$.
- $[v * a^\star]\downarrow 2 = [\ldots, \underline{0}, \underline{21.5}, \underline{-23}, \underline{6.5}, \underline{35}, \underline{0}, \ldots]$.
- $v * b^\star = [\ldots, \underline{0}, 10.5, \underline{0.5}, 0.5, \underline{0}, 1, \underline{-31.5}, 1, \underline{1}, 17, \underline{0}, \ldots]$.
- $[v * b^\star]\downarrow 2 = [\ldots, \underline{0}, \underline{0.5}, \underline{0}, \underline{-31.5}, \underline{1}, \underline{0}, \ldots]$.

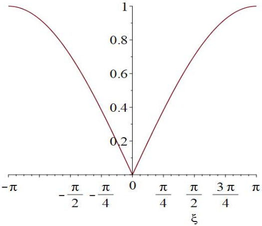
Undecimated Frequency Viewpoint



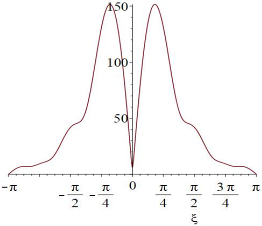
(a) $|\widehat{v}(\xi)|$



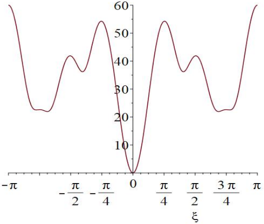
(b) $|\widehat{a}(\xi)|$



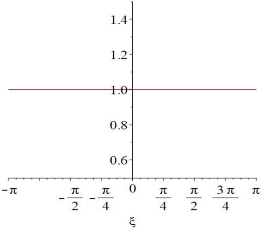
(c) $|\widehat{b}(\xi)|$



(d) $|\widehat{v}(\xi)\overline{\widehat{a}(\xi)}|$



(e) $|\widehat{v}(\xi)\overline{\widehat{b}(\xi)}|$



(f) $|\widehat{a}|^2 + |\widehat{b}|^2$

Undecimated Haar Wavelet Transform

- $a = \{\frac{1}{2}, \frac{1}{2}\}_{[0,1]}$ and $b = \{\frac{1}{2}, -\frac{1}{2}\}_{[0,1]}$.
- $|\widehat{a}(\xi)|^2 = \cos^2(\xi/2)$ and $|\widehat{b}(\xi)|^2 = \sin^2(\xi/2)$.
- $|\widehat{a}(\xi)|^2 + |\widehat{b}(\xi)|^2 = 1$.
- Perfect reconstruction:

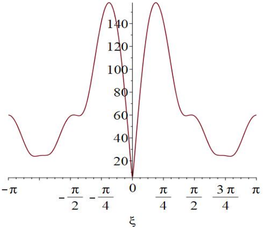
$$\widehat{v}(\xi) = \widehat{v}(\xi)[\overline{\widehat{a}(\xi)}\widehat{a}(\xi)+\overline{\widehat{b}(\xi)}\widehat{b}(\xi)] = [\widehat{v}(\xi)\overline{\widehat{a}(\xi)}]\widehat{a}(\xi)+[\widehat{v}(\xi)\overline{\widehat{b}(\xi)}]\widehat{b}(\xi).$$

- The undecimated Haar wavelet transform is

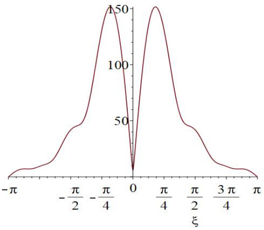
$$v = (v * a^\star) * a + (v * b^\star) * b.$$

- Energy preservation: $\|v\|^2_{\ell_2(\mathbb{Z})} = \|v * a^\star\|^2_{\ell_2(\mathbb{Z})} + \|v * b^\star\|^2_{\ell_2(\mathbb{Z})}$, where $\|v\|^2_{\ell_2(\mathbb{Z})} := \sum_{k\in\mathbb{Z}} |v(k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{v}(\xi)|^2 d\xi$.
- Lengths of $v * a^\star$ and $v * b^\star$ are the same as that of v!.

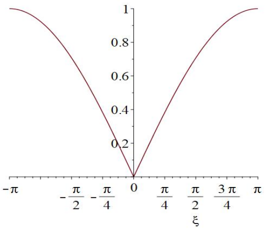
Decimated Frequency Viewpoint



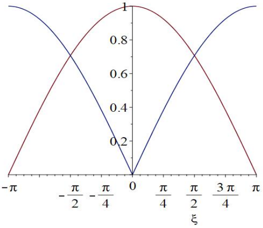
(a) $|\widehat{v}(\xi)|$



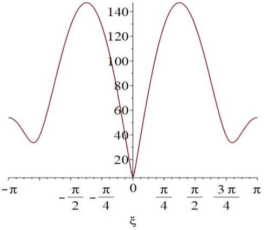
(b) $|\widehat{v}(\xi)\overline{\widehat{a}(\xi)}|$



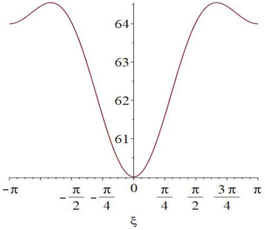
(c) $|\widehat{v}(\xi)\overline{\widehat{b}(\xi)}|$



(d) $|\widehat{a}|$ and $|\widehat{b}|$



(e) $|\widehat{[v * a^\star]\downarrow 2}|$



(f) $|\widehat{[v * b^\star]\downarrow 2}|$

(Decimated) Haar Wavelet Transform

- $a = \{\frac{1}{2}, \frac{1}{2}\}_{[0,1]}$ and $b = \{\frac{1}{2}, -\frac{1}{2}\}_{[0,1]}$.
- $|\widehat{a}(\xi)|^2 = \cos^2(\xi/2)$ and $|\widehat{b}(\xi)|^2 = \sin^2(\xi/2)$.
- $\widehat{[v * a^\star]\downarrow 2}(\xi) = \widehat{v}(\xi/2)\overline{\widehat{a}(\xi)} + \widehat{v}(\xi/2+\pi)\overline{\widehat{a}(\xi/2+\pi)}$.
- $\widehat{[v * b^\star]\downarrow 2}(\xi) = \widehat{v}(\xi/2)\overline{\widehat{b}(\xi)} + \widehat{v}(\xi/2+\pi)\overline{\widehat{b}(\xi/2+\pi)}$.
- $v = 2^{-1/2}[(\sqrt{2}[v * a^\star]\downarrow 2)\uparrow 2] * a + 2^{-1/2}[(\sqrt{2}[v * b^\star]\downarrow 2)\uparrow 2] * b$,

$$\begin{aligned}\widehat{v}(\xi) =& [\widehat{v}(\xi)\overline{\widehat{a}(\xi)} + \widehat{v}(\xi+\pi)\overline{\widehat{a}(\xi+\pi)}]\widehat{a}(\xi)\\ &+ [\widehat{v}(\xi)\overline{\widehat{b}(\xi)} + \widehat{v}(\xi+\pi)\overline{\widehat{b}(\xi+\pi)}]\widehat{b}(\xi)\end{aligned}$$

- The above perfect reconstruction holds if and only if

$$\overline{\widehat{a}(\xi)}\widehat{a}(\xi) + \overline{\widehat{b}(\xi)}\widehat{b}(\xi) = 1, \quad \overline{\widehat{a}(\xi+\pi)}\widehat{a}(\xi) + \overline{\widehat{b}(\xi+\pi)}\widehat{b}(\xi) = 0.$$

- Such $\{a; b\}$ is called an orthogonal wavelet filter bank.
- Energy preservation:
$\|v\|^2_{\ell_2(\mathbb{Z})} = \|\sqrt{2}(v * a^\star)\downarrow 2\|^2_{\ell_2(\mathbb{Z})} + \|\sqrt{2}(v * b^\star)\downarrow 2\|^2_{\ell_2(\mathbb{Z})}$.

Revisit Haar Wavelet Decomposition

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]_{[0,7]}$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- $v = [\ldots, 0, 0, -21, -22, -23, -23, -25, 38, 36, 34, 0, 0, \ldots]$.
- $a = \{\frac{1}{2}, \frac{1}{2}\}_{[0,1]}$ and $b = \{\frac{1}{2}, -\frac{1}{2}\}_{[0,1]}$.
- $\widehat{v * a^\star}(\xi) = \widehat{v}(\xi)\widehat{a^\star}(\xi) = \widehat{v}(\xi)\overline{\widehat{a}(\xi)}$, that is, $v * a^\star$ is $[\ldots, \underline{0}, -10.5, \underline{21.5}, -22.5, \underline{-23}, -24, \underline{6.5}, 37, \underline{35}, 17, \underline{0}, \ldots]$.
- $[v * a^\star]\downarrow 2 = [\ldots, \underline{0}, \underline{21.5}, \underline{-23}, \underline{6.5}, \underline{35}, \underline{0}, \ldots]$.
- $v * b^\star = [\ldots, \underline{0}, 10.5, \underline{0.5}, 0.5, \underline{0}, 1, \underline{-31.5}, 1, \underline{1}, 17, \underline{0}, \ldots]$.
- $[v * b^\star]\downarrow 2 = [\ldots, \underline{0}, \underline{0.5}, \underline{0}, \underline{-31.5}, \underline{1}, \underline{0}, \ldots]$.
- $\|v\|^2_{\ell_2} = 6504 = 5109 + 1395 = \|v * a^\star\|^2_{\ell_2} + \|v * b^\star\|^2_{\ell_2}$.
- $\|v\|^2_{\ell_2} = 6504 = 4517 + 1987 = \|\sqrt{2} v * a^\star \downarrow 2\|^2_{\ell_2} + \|\sqrt{2} v * b^\star \downarrow 2\|^2_{\ell_2}$.

Differences: Tight Frames and Orthonormal Bases

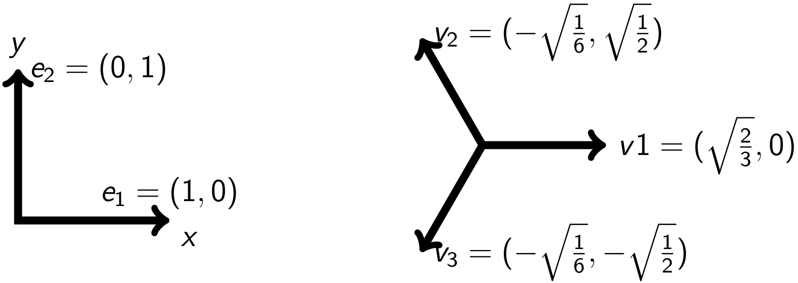


Figure: Left: Orthonormal basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$ in $\mathbb{R}^2$. Right: Tight frame $\{v1 = (\sqrt{\frac{2}{3}}, 0), v_2 = (-\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}), v_3 = (-\sqrt{\frac{1}{6}}, -\sqrt{\frac{1}{2}})\}$ in $\mathbb{R}^2$.

$$v = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2$$
$$v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \langle v, u_3 \rangle u_3.$$

A tight frame generalizes an orthonormal basis by having redundancy and more elements.

Summary

- Four types of wavelets and framelet:
  - Wavelets: orthogonal wavelets and biorthogonal wavelets
  - Framelets: tight framelets and dual framelets.
- For compression purpose such as signal/image compression and wavelet applications to numerical PDEs, we often use (bi)orthogonal wavelet filter banks $\{a; b\}$: (1) More restrictive conditions on filter banks; (2) no redundance.
- For data sciences applications such as signal and image processing, we often use dual (or tight) framelet filter banks $\{a; b_1, \ldots, b_s\}$: (1) less restrictive conditions on filter banks (flexibility); (2) enjoy redundancy.
- The undecimated Haar wavelet filter bank can be written as the tight framelet filter bank $\{a; a(\cdot - 1), b, b(\cdot - 1)\}$.
- Haar orthogonal wavelet filter bank is simple, but more wavelet filter banks with some additional desired properties are needed.