Matching under transferable utility

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Plan of the lectures

Today: introductory material.

- What is optimal transport?
- What is known? What sort of mathematics is involved?
- Why should I care? What can I do with it? Applications?

Monday: a deeper look at one selected topic. At the end of today's talk, we can vote to decide on the topic. The choices include:

- Matching theory (economics): what sort of patterns emerge when agents match together (for instance, workers and firms on the labour market, or husbands and wives on the marriage market).
- Density functional theory (physics/chemistry): how does a system of electrons organize itself to minimize interaction energy.
- Curvature and entropy (geometry): How does curvature relate to the behavior of densities along interpolations?

Both talks will focus on **ideas** and we will try to avoid getting bogged down in too many details.

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- $Y \subset \mathbb{R}^m$ represents the set of worker types, differentiated by m characteristics, such as age, home location, experience,....
- In discrete models, there are $x^1, x^2, ..., x^k \in X$ types of firms and $y^1, ..., y^l \in Y$ types of workers. There are $f_i := f(x^i)$ firms of type i and $g_j := g(y^j)$ workers of type j.

More on the basic model: matchings

- Assume that each each firm hires **exactly** one worker, and each worker takes **exactly** one job (these assumptions can be relaxed, but we'll keep it simple here). In this case, we'd better have $\sum_{i=1}^{k} f(x^i) = \sum_{j=1}^{l} g(x^j)$.

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- If a firm of type x hires a worker of type y, they generate a surplus of $s(x, y)$. We can think of this as the profit firm x would earn if they had worker y working for them. By varying the worker's wages, this surplus can be divided any they want.

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- A matching is a $k \times l$ matrix γ with nonnegative entries, $\gamma_{ij} \geq 0$, such that $\sum_{i=1}^{k} \gamma_{ij} = g(y^j)$, and $\sum_{j=1}^{l} \gamma_{ij} = f(x^i)$. γ_{ij} is the number of workers of type j hired by firms of type i.

More on the basic model: stability

- Functions $u(x)$ and $v(y)$ are called payoff functions for γ if $u(x^i) + v(y^j) = s(x^i, y^j)$ whenever $\gamma_{ij} \neq 0$. They represent a *division of the surplus*; $v(y^j)$ is the salary payed to worker y^j, $u(x^i)$ is the profit kept by the firm.

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- *How does this capture stability?*

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- We look for a matching, $\gamma(x,y) \geq 0$, with $\int_X \gamma(x,y)dx = g(y)$ and $\int_Y \gamma(x,y)dy = f(x)$, and payoff functions with $u(x) + v(y) = s(x,y)$ whenever $\gamma(x,y) \neq 0$.

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- The matching is stable if we can find payoff functions with $u(x) + v(y) \geq s(x, y)$ for **all** x, y.
- The continuous limit is useful, as we can exploit calculus and geometry/topology to understand the solution.

Connection with optimal transport

- Let $\Gamma(f, g)$ be the set of all matchings.

Theorem (Shapley-Shubik 1971, Gretsky-Ostroy-Zame 1992)

A matching is stable if and only if it maximizes $\int_{X\times Y} s(x,y)\gamma(x,y)dxdy$ almong $\gamma \in \Gamma(f,g)$.

- This is *exactly* the Monge-Kantorovich problem (we could rewrite it to minimize $\int_{X\times Y} c(x,y)\gamma(x,y)dxdy$ for $c(x,y) = -s(x,y)$).
- Shapley-Shubik dealt with the discrete case (in which case you get a discrete optimal transport, or assignment, problem).

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- For any matching $\bar{\gamma}(x, y)$ we have

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- The last line **doesn't** depend on $\bar{\gamma}$.
- If $\bar{\gamma}=\gamma$, the inequality $u(x)+v(y)\geq s(x,y)$ is an equality on the points where $\gamma(x,y)\neq 0$, so we get

$$\int_{X\times Y} s(x,y)\gamma(x,y)dxdy = \int_X u(x)f(x)dx + \int_Y v(y)g(y)dy$$

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- Let $u(x)$ and $v(y)$ solve the dual problem. Then $u(x) + v(y) \geq s(x, y)$ for all x, y and

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- This is only possible if $u(x) + v(y) = s(x, y)$ whenever $\gamma(x, y) > 0$.

An application

Corollary

There exists at least one stable matching.

- The proof is by continuity-compactness in the right topology.
- This is not just mathematical tomfoolery. In matching with **non-transferable utility**, there **might not** be any stable matching!
- Other information, such as uniqueness and structure of the solution, can be deduced under certain conditions.

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- Economic interpretation: $y \mapsto \frac{\partial s}{\partial x}$ (marginal suplus) is **increasing**. So $y \mapsto s(x^1, y) - s(x^0, y)$ is increasing if $x^1 > x^0$. Having a higher end worker (more experienced, perhaps) makes a bigger difference for a higher end (larger, maybe) firm.

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- This falls apart when $n \neq m$ (P 12). When $m = 1$, but $n > 1$, explicit solutions and regularity can be recovered under some conditions (Chiappori-McCann-P 15).

Low dimensional solutions

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- This holds for the discrete case, too.

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- This might mean you don't have enough (or the correct) characteristics (your model should be multi-dimensional).

Extensions/related problems

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- There are many other economic problems that relate to optimal transport (even those that aren't transferable-utility matching problems). See Galichon's book.

Some references

- A. Galichon *Optimal transport methods in economics.* Princeton University Press, 2015.
- I. Ekeland *Notes on optimal transportation.* Econ. Theory, 42, p.437 -459, 2010.
- G. Carlier *Optimal transporation and economic applications.* Lecture note for the IMA short course, New mathematical models in economics and finance, 2010.