1 Project 1: The Willmore conjecture (theorem)

Recall the *Willmore conjecture* [3]. This conjecture, now a theorem [2], states that for a 2-torus $\mathbb{T}$ in $\mathbb{R}^3$, the *Willmore energy* obeys

$$W := \int H^2 dA \geq 2\pi \ ,$$

and $W = 2\pi$ if and only if $\mathbb{T}$ is the *Willmore torus* given by $x(\theta,\phi) = (a + b\cos\theta)\cos\phi$, $y(\theta,\phi) = (a + b\cos\theta)\sin\phi$, $z(\theta,\phi) = b\sin\theta$. The integral here is a 2-dimensional integral over the torus, and dA is the surface area element (often called the 2-dimensional volume element of the torus or the 2-volume element). The *mean curvature* H is the average of the two *principal curvatures*

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \ ,$$

where r_1 is the radius of the smallest "osculating circle" tangent to the torus at a point and r_2 is the radius of the largest such circle at the same point. (These two circles always lie in orthogonal planes in $\mathbb{R}^3$).

The purpose of this project is to test the conjecture on so-called *tube tori.* In doing so, we will discover the *minimizer* of the Willmore energy. This is the torus for which the above inequality becomes an equality $W := \int H^2 dA = 2\pi$. In other words, the minimizer is a certain tube torus.

To construct a tube torus, first draw a simple closed curve in $\mathbb{R}^3$. Then take all points which lie a fixed distance $b > 0$ from this curve. That's a *tube torus* (for a general torus, b could be a function that varies along the curve). A tube torus can self-intersect. If it does so, we say the torus is *immersed in* $\mathbb{R}^3$. If it doesn't self-intersect, we say the torus is *embedded in* $\mathbb{R}^3$.

Let's start with a planar circle, say the circle $r^2 := x^2 + y^2 = a^2$ in the $z = 0$ plane of $\mathbb{R}^3$. A tube torus about this circle can be written as the parametrized surface

$$\begin{aligned} x &= (a + b\cos\theta)\cos\phi \\ y &= (a + b\cos\theta)\sin\phi \\ z &= b\sin\theta \ , \end{aligned}$$

Here $\theta, \phi \in [0, 2\pi]$. Sketch this torus to get the idea.

In class we will compute the two principal curvatures at each point by finding the radii of the corresponding osculating circles. You should get $\kappa_1 = \frac{1}{b}$, $\kappa_2 = \frac{\cos\theta}{a+b\cos\theta}$ (this depends on how you define θ; those who define their θ to

be my θ minus π will get $\kappa_2 = \frac{\cos\theta}{b\cos\theta - a}$). Then we can compute the Willmore energy and find the minimizing torus.
The project will be to see if we can prove Willmore's conjecture for more general tori. This may take the form of analytical proofs, or it may be a computer program that computes W for different tori and checks whether $W > 2\pi^2$. Two possible classes for which we might be able to give proofs are the *tube tori* about a general plane curve or the *tori of rotation.* To construct a tube torus, draw a general smooth, non-self-intersecting closed curve in the plane (these are called Jordan curves). About one point, consider the disk of redius b about that point and perpendicular to the curve. Now drag this disk along the curve, keeping it orthogonal, to make a tube torus. To construct a torus of rotation, first draw a simple closed curve in the (x, z)-plane of $\mathbb{R}^3$. The curve should lie entirely to the right of the z-axis in this plane. Now rotate this curve about the z-axis. The resulting surface is a *torus of rotation.* It can be rather hard to figure out the mean curvature for these tori, so start with as simple an example as possible (perhaps try it if your closed curve is an ellipse). For some of these special cases, perhaps you can prove the conjecture; for the more complicated cases, perhaps you can write a computer program that will compute W and compare it to $2\pi^2$ to test the conjecture.
For a review of the Willmore conjecture, see [1]. For biographies of TJ Willmore, who posed the conjecture, and Fernando Codá Marques and André Neves, who proved it, see their Wikipedia pages.

References

[1] FC Marques, *The Willmore conjecture*, [https://arxiv.org/abs/1409.7664].

[2] FC Marques and A Neves, *Min-max theory and the Willmore conjecture*, Ann Math 179 (2014) 683–782.

[3] TJ Willmore, *Note on embedded surfaces*, An Stiint Univ "Al I Cuza" Iasi Ia Mat (1965) 493-496.

[4] TJ Willmore, *Mean curvature of Riemannian immersions*