Dynamical Systems on Networks: From Epidemics to Flight of Drones

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Outline of Talks

Part I: Global Stability Problems in Heterogeneous Epidemic Models

I.1: Modeling Infectious Diseases in Heterogeneous Populations

- Simple epidemic models and their dynamics
- Basic reproduction number and the threshold theorem
- Multi-group models for heterogeneous populations

I.2: Global-Stability Problem in Multi-Group Models

- Global-stability problem and Lyapunov functions
- A Lyapunov function for multi-group models
- Why is global-stability difficult to prove?

I.3: Matrix-Tree Theorem in Graph Theory

- Rooted directed trees and unicyclic graphs
- Kirchhoff's Matrix-Tree Theorem

I.4: How do all of these come together?

- Global-stability result for multi-group models.

Outline of Talks

Part II: Dynamical Systems on Networks

II.1: Dynamical systems on networks

- A network as a directed graph
- Dynamical systems on networks
- Examples

I.2: Global-stability problem

- Global-stability problem and Lyapunov functions
- A general theorem
- Applications

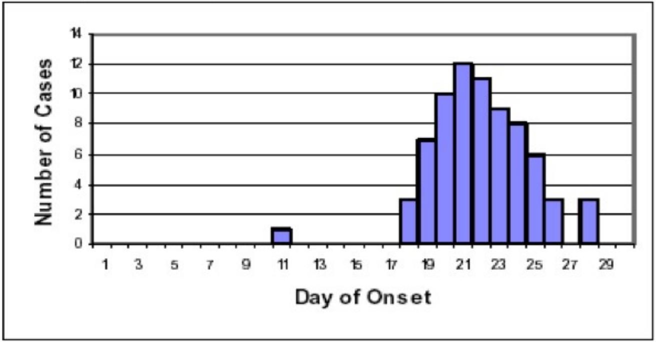
I.3: Flight formation control for drones

- Network of autonomous robotic agents
- Flight formation problems and HPC control protocol
- Simulations

I.4: Synchronization problems

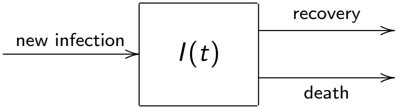
- Synchronization of metronomes, a video
- Global synchronization of coupled oscillators.

How to Model an Epidemic?

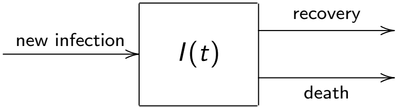


An Epidemic Curve

How to Model an Epidemic?

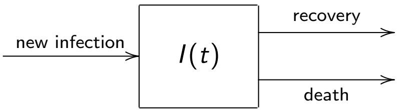


How to Model an Epidemic?



$$\begin{aligned} I'(t) &= \boxed{\text{Incidence Rate}} - \boxed{\text{Recovery Rate}} - \boxed{\text{Death Rate}} \\ &= f(I(t), S(t), N(t)) - \gamma\, I(t) - d\, I(t) \end{aligned}$$

How to Model an Epidemic?



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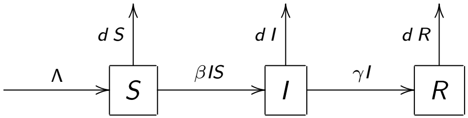
$f(I, S, N) = \beta I S$: bilinear incidence

$f(I, S, N) = \lambda \dfrac{I S}{N}$: proportionate incidence

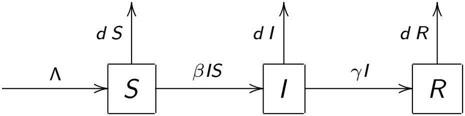
A Single-Group SIR Model

S : Susceptibles $\quad$ I : Infectious $\quad$ R : *Removed*



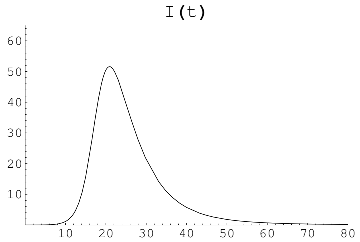
A Single-Group SIR Model

S : Susceptibles I : Infectious R : *Removed*



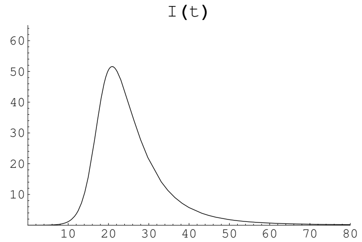
$$
\begin{aligned}
S' &= \Lambda - \beta I S - d S \\
I' &= \beta I S - (\gamma + d) I \\
R' &= \gamma I - d R
\end{aligned}
$$

A Single-Group SIR Model

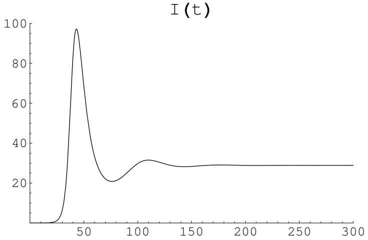


Numerical output I: epidemic case

A Single-Group SIR Model



Numerical output I: epidemic case



Numerical output II: endemic case

Threshold Theorem

The basic reproduction number is

$$R_0 = \frac{\beta \Lambda}{(\gamma + d)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d}$$

The average secondary infections produced by a single infective during its entire infectious period.

Threshold Theorem

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The average secondary infections produced by a single infective during its entire infectious period.

Theorem (Threshold Theorem)

(1) *If $R_0 \leq 1$, then the disease-free equilibrium $P_0 = (\Lambda/d, 0)$ is stable and attracts all solutions in R^2_+.*

(2) *If $R_0 > 1$, then P_0 is unstable, and a unique endemic (positive) equilibrium P^* is stable and attracts all positive solutions in R^2_+.*

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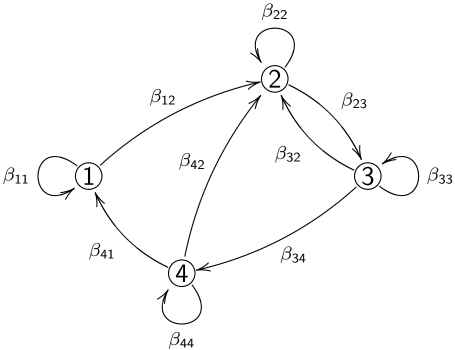
Proof uses the Poincaré-Bendixson theory for 2d systems.

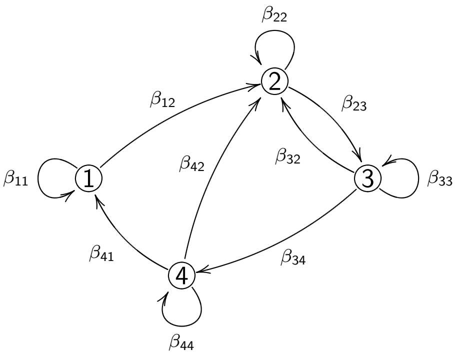
n-Group Models for Heterogeneous Populations



Each circled number represents a homogeneous group.

β_{jk} : transmission coefficient between I_j and S_k.

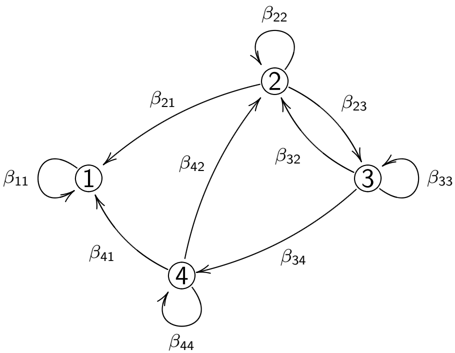
n-Group Models for Heterogeneous Populations



$$B = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 & 0 \\ 0 & \beta_{22} & \beta_{23} & 0 \\ 0 & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & 0 & \beta_{44} \end{bmatrix}$$

Transmission Matrix B is irreducible.

n-Group Models for Heterogeneous Populations



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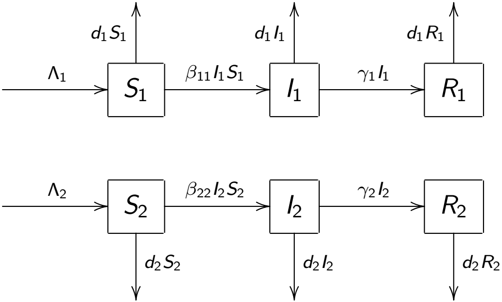
Transmission Matrix B is reducible.

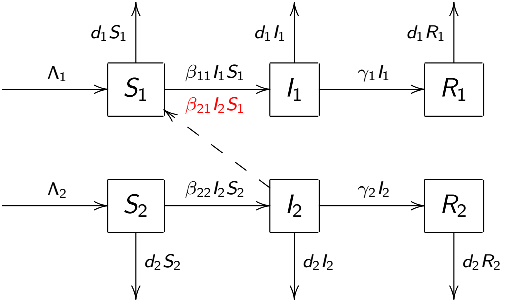
A Two-Group SIR Model



Incidence terms (bilinear):

- Group 1: $\beta_{11} I_1 S_1$
- Group 2: $\beta_{22} I_2 S_2$

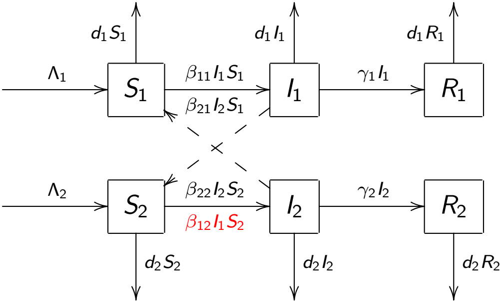
A Two-Group SIR Model



Incidence terms (bilinear):

- Group 1: $\beta_{11} I_1 S_1 + \beta_{21} I_2 S_1$
- Group 2: $\beta_{22} I_2 S_2$

A Two-Group SIR Model



Incidence terms (bilinear):

- Group 1: $\beta_{11} I_1 S_1 + \beta_{21} I_2 S_1$
- Group 2: $\beta_{22} I_2 S_2 + \beta_{12} I_1 S_2$

An n-Group SIR Model

$$\begin{cases} S_k' = \Lambda_k - d_k S_k - \sum_{j=1}^{n} \beta_{jk} I_j S_k, \\ I_k' = \sum_{j=1}^{n} \beta_{jk} I_j S_k - (d_k + \gamma_k) I_k, \end{cases} \qquad k = 1, \cdots, n.$$

Mathematical Questions:

- If $R_0 > 1$, is P^* unique?

An n-Group SIR Model

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Mathematical Questions:

- If $R_0 > 1$, is P^* unique?
- When P^* is unique, is it globally stable?

Previous Results on GAS of P^*

For Models using bilinear incidence:

- Lajmanovich and Yorke (1976)
  - n-group SIS model, by Lyapunov function
  - later extended by Nold, Hirsch
- Hethcote (1975)
  - n-group SIR model with no vital dynamics
- Thieme (1983)
  - n-group SEIRS model, small latent and immune periods
- Beretta and Capasso (1986)
  - n-group SIR model, constant group sizes
- Lin and So (1993)
  - n-group SIRS model, constant group sizes
  - β_{ij} $(i \neq j)$ small

Non-uniqueness of P^* when $R_0 > 1$

- Lin (1992)
  n-group model for HIV
- Huang, Cooke, Castillo-Chavez (1992)
  n-group model for HIV with delay

These models use proportionate incidence.

Global-Stability and Lyapunov Functions

Consider a general system of ODE

$$x' = F(x), \quad x \in D \subset \mathbb{R}^d.$$

$\bar{x}$ is an equilibrium if $F(\bar{x}) = 0$.

An equilibrium $\bar{x}$ is globally stable in D if it is locally stable and all solutions in D converge to $\bar{x}$ as $t \to \infty$.

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Global-Stability and Lyapunov Functions

Theorem (Lyapunov)

Suppose $\exists$ a Lipschitz function $V(x)$ such that

(1) $V(x) \geq V(\bar{x})$ *and* $V(x) = V(\bar{x}) \iff x = \bar{x}$.

(2) $\overset{*}{V}(x) = \nabla V(x) \cdot F(x) \leq 0, \quad x \in D$, *and*

$\overset{*}{V}(x) = 0 \iff x = \bar{x}$.

Then $\bar{x}$ is globally stable in D.

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Then $\bar{x}$ *is globally stable in* D.

$V(x(t))$ strictly decreasing along a solution $x(t)$

Constructing a Lyapunov Function for the n-Group Model

Consider a candidate

$$V = \sum_{k=1}^{n} v_k \Big[\underbrace{(S_k - S_k^* \ln S_k) + (I_k - I_k^* \ln I_k)}_{\text{A Lyapunov function for a single-group model}} \Big]$$

Constructing a Lyapunov Function for the n-Group Model

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Choose appropriate v_k so that $\overset{*}{V}(x)$ is negative definite.

Derivative of V

$$V' =$$

$$\sum_{k=1}^{n} v_k \Big[(S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{I_k} I_k') \Big]$$

Derivative of V

$$V' =$$

$$\sum_{k=1}^{n} v_k \Big[(S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{I_k} I_k') \Big]$$

$$= \sum_{k=1}^{n} v_k \Big[d_k S_k^* \Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \Big) \Big]$$

$$+ \sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big]$$

$$+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

Derivative of V

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$$= \sum_{k=1}^{n} v_k \Big[d_k S_k^* \Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \Big) \Big] \qquad \leq 0$$

$$+ \sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big]$$

$$+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

Derivative of V

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$$+ \sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \qquad \equiv 0$$

$$+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

Derivative of V

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$$H_n := \sum_{j,k=1}^{n} v_k \, \bar{\beta}_{kj} \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

Derivative of V

$$
\begin{aligned}
V' = & \\
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= & \sum_{k=1}^{n} v_k \Big[d_k S_k^* \Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \Big) \Big] \\
+ & \sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \\
+ & \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)
\end{aligned}
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$$

Choosing Constants v_k

Choose v_k so that

$$\sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \equiv 0$$

for all $I_1, \cdots, I_n > 0$.

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for all $I_1, \cdots, I_n > 0$. This is equivalent to

$$\begin{bmatrix} \beta_{11} I_1^* S_1^* & \cdots & \beta_{n1} I_n^* S_1^* \\ \vdots & \ddots & \vdots \\ \beta_{1n} I_1^* S_n^* & \cdots & \beta_{nn} I_n^* S_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} \beta_{j1} I_j^* S_1^* v_1 \\ \vdots \\ \sum_{j=1}^{n} \beta_{jn} I_j^* S_n^* v_n \end{bmatrix}$$

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since, at P^*,

$$(d_k + \gamma_k) = \sum_{j=1}^{n} \beta_{jk} I_j^* S_k^*.$$

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Set $\bar{\beta}_{jk} = \beta_{jk} I_j^* S_k^*$. Then $(v_1, \cdots, v_k)$ are determined by the linear system

$$\begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

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The solution space is 1d and a basis is given by

$$v_k = C_{kk}, \quad \text{the } k\text{-th principal minor}, \quad k = 1, \cdots, n.$$

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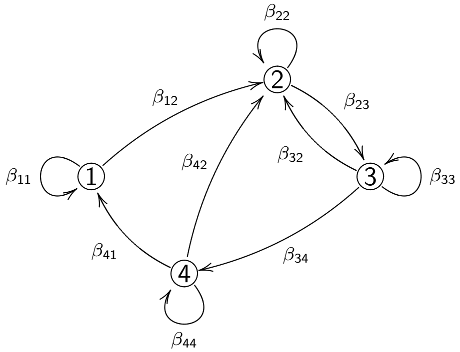
Need to show

$$V' \le H_n = \sum_{j,k=1}^{n} v_k \bar{\beta}_{kj} \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k}\Big) \le 0,$$

for all $S_1, I_1, \cdots, S_n, I_n \ge 0$.

Directed Graphs and Rooted Spanning Trees

Let G be a directed graph with vertex set $V(G) = \{1, \cdots, n\}$ and weight matrix $B = (\beta_{ij})_{n\times n}$.



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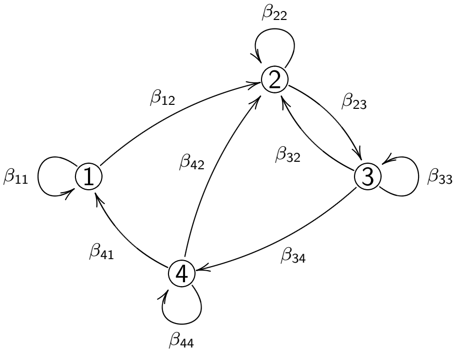

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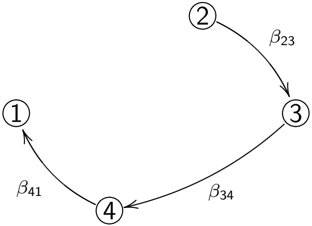

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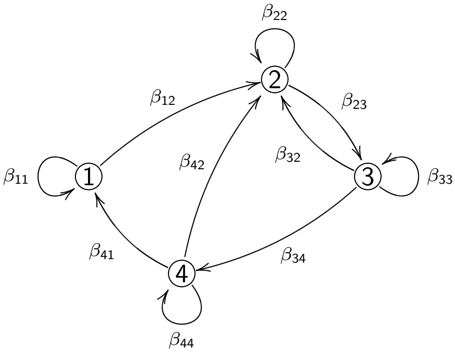

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The Matrix-Tree Theorem

Let $B = (\bar{\beta}_{ij})_{n \times n}$ be the weight matrix of graph G.

The Kirchhoff Matrix (combinatorial Laplacian) of B is

$$\overline{B} = \begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix}.$$

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Theorem (Matrix Tree Theorem, Kirchhoff 1847)

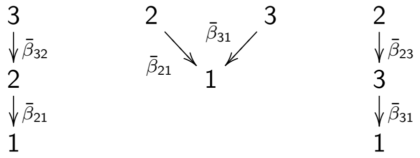
$$C_{kk} = \sum_{T\in\mathbb{T}_k} w(T).$$

$\mathbb{T}_k$: The set of spanning trees rooted at vertex k.

Solving System $\bar{B}\,v = 0$: $n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:



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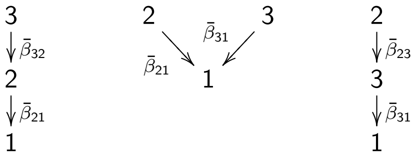

Furthermore

$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

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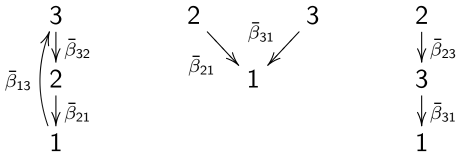

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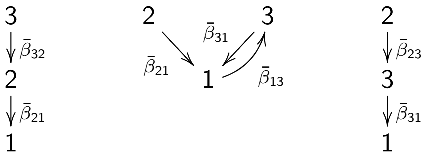

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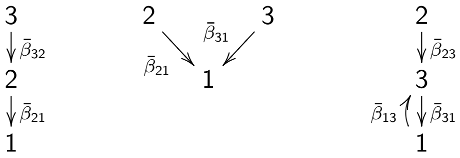

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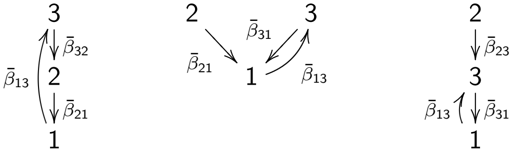

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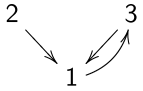
Each product is the weight of a unicyclic graph with a cycle of length $1 \le r \le 3$.

Unicyclic Graphs and Rooted Trees

How does a unicyclic graph correspond to products in $v_k \beta_{kj}$?



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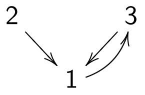

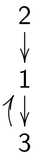

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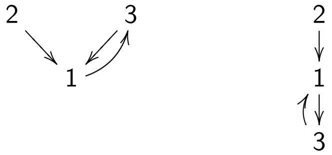

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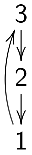
Another Unicyclic Graph



$$v_1\bar{\beta}_{13} = {\color{red}\bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13}} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

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$$v_2\bar{\beta}_{21} = \bar{\beta}_{13}\bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{12}\bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{31}\bar{\beta}_{12}\bar{\beta}_{21}$$

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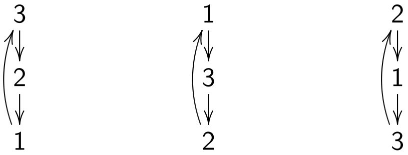

$$
\begin{aligned}
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\end{aligned}
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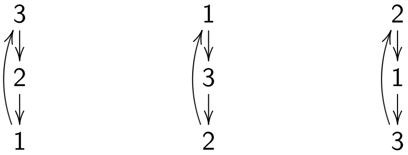

$$v_1\bar\beta_{13} = \bar\beta_{32}\bar\beta_{21}\bar\beta_{13} + \bar\beta_{21}\bar\beta_{31}\bar\beta_{13} + \bar\beta_{23}\bar\beta_{31}\bar\beta_{13}$$
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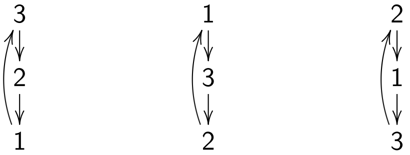

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$$V' \le H_n = \sum_{j,k=1}^{n} v_k\,\bar\beta_{kj}\Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*}\frac{I_j}{I_j^*}\frac{I_k^*}{I_k}\Big)$$

H_n is Summed over all Unicyclic Graphs

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and

$$H_Q = w(Q) \cdot \sum_{(p,q) \in E(C_Q)} \Big(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \Big)$$

$$= w(Q) \cdot \Big[2r - \sum_{(p,q) \in E(C_Q)} \Big(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \Big) \Big]$$

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Finally, because C_Q is a cycle,

$$\prod_{(p,q)\in E(C_Q)} \frac{S_p^*}{S_p}\cdot\frac{S_p}{S_p^*}\cdot\frac{I_p}{I_p^*}\cdot\frac{I_q^*}{I_q} = \prod_{(p,q)\in E(C_Q)} \frac{I_p}{I_p^*}\cdot\frac{I_q^*}{I_q} = 1.$$

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Main Result

For the *n*-group SIR model with bilinear incidence,

Theorem (Guo, Li, Shuai, 2007)

Assume that transmission matrix B is irreducible.

If $R_0 > 1$, then P^* is unique and is globally stable in R_+^{2n}.

The same graph-theoretical approach can be used to:

Build Lyapunov function V for a large-scale system

$$V = \sum_{k=1}^{n} c_k V_k$$

using the known Lyapunov function V_k for each component.