A Research Story: Compound Equations and Dynamics. Part 3

James Muldowney, University of Alberta

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Curves and Surfaces

A smooth curve $\gamma$ in $\mathbb{R}^n$ is a $C^1$ function $s \rightarrow x(s)$, $s \in I \subset \mathbb{R}$, $x(s) \in \mathbb{R}^n$.

A measure of the length of $\gamma$ is

$$l(\gamma) = \int_{\gamma} dl \overset{\text{def}}{=} \int_{I} \left\| \frac{dx}{ds}(s) \right\| ds$$

where $\| \cdot \|$ is a norm on $\mathbb{R}^n$. For example, the euclidean norm

$$\|x\| = \sqrt{(x_1)^2 + \cdots + (x_n)^2}$$

gives the usual measure of length

$$l(\gamma) = \int_{I} \sqrt{\frac{dx_1}{ds}^2 + \cdots + \frac{dx_n}{ds}^2} ds$$
A smooth $2$-surface $\sigma$ in $\mathbb{R}^n$ is a $C^1$ function $(s_1, s_2) \rightarrow x(s_1, s_2)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$, $x(s_1, s_2) \in \mathbb{R}^n$.

A measure of the area of $\sigma$ is

$$a_2(\sigma) = \int_{\sigma} da \overset{\text{def}}{=} \int_U \|x_{s_1} \wedge x_{s_2}\| \, ds_1 \, ds_2$$

where $x_{s_i} = \frac{\partial}{\partial s_i} x(s_1, s_2)$ and $\| \cdot \|$ is a norm on $\mathbb{R}^{(2)}$. If $\| \cdot \|$ is the Euclidean norm we have

$$a_2(\sigma) = \int_U \sqrt{\sum_{1 \leq i < j \leq n} \left( \frac{\partial}{\partial (s_1, s_2)} \frac{\partial}{\partial (s_1, s_2)} x_i, x_j \right)^2} \, ds_1 \, ds_2$$

where

$$\frac{\partial}{\partial (s_1, s_2)} \frac{\partial}{\partial (s_1, s_2)} (x_i, x_j) = \det \begin{bmatrix} \frac{\partial x_i}{\partial s_1} & \frac{\partial x_i}{\partial s_2} \\ \frac{\partial x_j}{\partial s_1} & \frac{\partial x_j}{\partial s_2} \end{bmatrix}. $$
A smooth $k$-surface $\sigma$ in $\mathbb{R}^n$ is a $C^1$ function $$(s_1, \ldots, s_k) \rightarrow x(s_1, \ldots, s_k), s_1, \ldots, s_k \in U \subset \mathbb{R}^k, x(s_1, \ldots, s_k) \in \mathbb{R}^n.$$ A measure of the $k$-area of $\sigma$ is

$$a_k(\sigma) = \int_{\sigma} da_k \overset{\text{def}}{=} \int_U \|x_{s_1} \land \cdots \land x_{s_k}\| \, ds_1 \cdots ds_k$$

where $x_{s_i} = \frac{\partial}{\partial s_i} x(s_1, \ldots, s_k)$ and $\| \cdot \|$ is a norm on $\mathbb{R}^k$. If $\| \cdot \|$ is the Euclidean norm we have

$$a_k(\sigma) = \int_U \sqrt{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\partial (x_{i_1}, \ldots, x_{i_k})}{\partial (s_1, \ldots, s_k)}^2} \, ds_1 ds_2$$

where

$$\frac{\partial (x_{i_1}, \ldots, x_{i_k})}{\partial (s_1, \ldots, s_k)} = \det \begin{bmatrix} \frac{\partial x_{i_1}}{\partial s_1} & \frac{\partial x_{i_1}}{\partial s_2} & \cdots & \frac{\partial x_{i_1}}{\partial s_k} \\ \frac{\partial x_{i_2}}{\partial s_1} & \frac{\partial x_{i_2}}{\partial s_2} & \cdots & \frac{\partial x_{i_2}}{\partial s_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{i_k}}{\partial s_1} & \frac{\partial x_{i_k}}{\partial s_2} & \cdots & \frac{\partial x_{i_k}}{\partial s_k} \end{bmatrix}.$$
Nonlinear Differential Equations

\[ f \in C^1 \left( \mathbb{R}^n \rightarrow \mathbb{R}^n \right) \]

\[ \dot{x} = f(x) \quad (N) \]

Solution: \[ x(t) = \phi(t) = \phi(t, x_0) \], is uniquely determined by \( x(0) = x_0 \) and, for simplicity, we will only consider equations for which solutions exist for all \( t > 0 \)

If \( \phi(t, x_0) = x_0 \) for all \( t \), then \( x_0 \) is called an equilibrium.

If \( \phi(t + \omega) = \phi(t), \omega > 0 \), the solution is periodic of period \( \omega \).

An orbit (positive semi-orbit) is a set \( \{ \phi(t) : 0 \leq t < \infty \} \).

The orbit of an equilibrium is a single point.

The orbit of a periodic solution is a simple closed curve (Jordan curve).
Linearization about a solution $\phi(t)$:

$$ \dot{y} = \frac{\partial f}{\partial x}(\phi(t)) y $$

(L)

Solution is:

$$ y(t) = \frac{\partial \phi}{\partial x_0}(t, x_0) y(0), \quad x_0 = \phi(0) $$

$$ Y(t) = \frac{\partial \phi}{\partial x_0}(t, x_0). \text{fundamental matrix, } Y(0) = I $$

"Proof": $y = \phi(t, x_0)$ solves $\dot{y} = f(y)$

$$ \Rightarrow \frac{\partial \phi}{\partial t}(t, x_0) = f(\phi(t, x_0)) $$

Differentiate with respect to $x_0$

$$ \Rightarrow \frac{\partial^2 \phi}{\partial t \partial x_0}(t, x_0) = \frac{\partial^2 \phi}{\partial x_0 \partial t}(t, x_0) = \frac{\partial f}{\partial x}(\phi(t, x_0)) \frac{\partial \phi}{\partial x_0}(t, x_0) $$

$$ \Rightarrow \dot{Y} = \frac{\partial f}{\partial x}(\phi(t)) Y $$
The $k$–th compound equation of $(L)$ is:

$$\dot{z} = \frac{\partial f^{[k]}}{\partial x} \left( \phi (t) \right) z$$  \hspace{1cm} (L_k)

Solution: $z(t) = \frac{\partial \phi^{(k)}}{\partial x_0} (t, x_0) z(0), x_0 = \phi(0)$

The case $k = n$ of $(L_k)$ is the Liouville equation:

$$\dot{z} = \text{div} f \left( \phi (t) \right) z$$ \hspace{1cm} (L_n)

Solution: $z(t) = \det \frac{\partial \phi}{\partial x_0} (t, x_0) z(0), x_0 = \phi(0)$
Suppose that $D \subset \mathbb{R}^n$ has finite $n$-dimensional measure $a_n(D)$, then the measure of $\phi(t, D)$ is

$$a_n(\phi(t, D)) = \int_{x \in \phi(t, D)} dx = \int_{x_0 \in D} \left| \det \frac{\partial \phi}{\partial x_0}(t, x_0) \right| dx_0$$

\((L_n) \Rightarrow \det \frac{\partial \phi}{\partial x_0}(t, x_0) = \exp \left[ \int_0^t \text{div} \, f(\phi(s, x_0)) \, ds \right].\) So, for example, if \(\text{div} \, f < 0\) in $\mathbb{R}^n$, then the measure of the set $\phi(t, D)$ decreases with time.

When $n = 2$ this observation implies that no simply connected region where $\text{div} \, f < 0$ can contain a non-trivial periodic orbit of $(L)$. This is known as *Bendixson’s Condition*. Most textbooks prove this as a very nice application of Green’s Theorem.
Stability of the linearized equations \((L)\) and its compounds \((L_k)\) have many implications for the dynamics of \((N)\)

If \(\gamma_0 : x = x_0(s), 0 \leq s \leq 1\) is a curve in \(\mathbb{R}^n\), then \(\gamma_t : x = \phi(t, x_0(s)), 0 \leq s \leq 1\) is also a curve in \(\mathbb{R}^n\) for each \(t \geq 0\).

\[
\begin{align*}
  l\gamma_0 &= \int_0^1 \left\| \frac{d}{ds} x_0(s) \right\| \, ds \\
  l\gamma_t &= \int_0^1 \left\| \frac{d}{ds} \phi(t, x_0(s)) \right\| \, ds = \int_0^1 \left\| \frac{\partial \phi}{\partial x_0}(t, x_0(s)) \frac{d}{ds}(x_0(s)) \right\| \, ds \\
  &\leq \int_0^1 \left\| \frac{\partial \phi}{\partial x_0}(t, x_0(s)) \right\| \left\| \frac{d}{ds}(x_0(s)) \right\| \, ds
\end{align*}
\]

We can conclude for example that, if \(\left\| \frac{\partial \phi}{\partial x_0}(t, x_0) \right\| \xrightarrow{t \to \infty} 0\) uniformly with respect to \(x_0 \in \mathbb{R}^n\), then

- there is at most one equilibrium of \((N)\) and,
- any equilibrium attracts all other orbits
If \( \sigma_0 : (s_1, s_2) \rightarrow x(s_1, s_2) \) is a 2-surface in \( \mathbb{R}^n \) then so also is \( \sigma_t : (s_1, s_2) \rightarrow \phi(t, x(s_1, s_2)) \).

We can use similar ideas to get higher dimensional Bendixson Conditions to rule out the existence of periodic orbits. These are conditions on \((L_2)\) that typically imply that some measure of surface area decreases in the dynamics. Another related type of condition would imply that \( a_2 \sigma_t \rightarrow 0 \).

The central idea is to observe that a periodic orbit \( \gamma \) is invariant in the dynamics, \( \phi(t, \gamma) = \gamma \). So, if \( \Sigma_0 \) is any surface which has \( \gamma \) as its boundary, then \( \Sigma_t = \phi(t, \Sigma_0) \) is also a surface with \( \gamma \) as boundary. But if, among all surfaces with boundary \( \gamma \), \( \Sigma_0 \) is a surface with minimum area and \((N)\) diminishes area we would contradict the minimality of \( \Sigma_0 \). So no such invariant closed curve can exist.
The following are Bendixson conditions for various measures of 2-surface area. Each reduces to the classical result when $n = 2$:

$$\lambda_1 + \lambda_2 < 0 \text{ (RA Smith)}$$

$$\max_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r, s} \left( |\frac{\partial f_r}{\partial x_q}| + |\frac{\partial f_s}{\partial x_q}| \right) \right\} < 0$$

$$\max_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r, s} \left( |\frac{\partial f_q}{\partial x_r}| + |\frac{\partial f_q}{\partial x_s}| \right) \right\} < 0$$

$$\lambda_{n-1} + \lambda_n > 0$$

$$\min_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r, s} \left( |\frac{\partial f_r}{\partial x_q}| + |\frac{\partial f_s}{\partial x_q}| \right) \right\} > 0$$

$$\min_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r, s} \left( |\frac{\partial f_q}{\partial x_r}| + |\frac{\partial f_q}{\partial x_s}| \right) \right\} > 0$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$ are the eigenvalues of $\frac{1}{2} \left( \frac{\partial f^*}{\partial x} + \frac{\partial f}{\partial x} \right)$
General Compounds


$\mathcal{X} \subset \mathcal{Y}$: General compound $A^{[k]} \in \mathcal{L} \left( \wedge^k \mathcal{X} \rightarrow \wedge^k \mathcal{Y} \right)$. $0 \leq m \leq k$

$$A^{[k,m]} \left( v^1 \wedge \cdots \wedge v^k \right) \overset{\text{def}}{=} \sum_{(\varepsilon_1, \ldots, \varepsilon_k)} A^{\varepsilon_1} v^1 \wedge A^{\varepsilon_2} v^2 \wedge \cdots \wedge A^{\varepsilon_k} v^k$$

$\varepsilon_i \in \{0, 1\}$, $\varepsilon_1 + \cdots + \varepsilon_k = m$, $A^0 = I$

$$A^{[k,0]} = I^{(k)}, \quad A^{[k,1]} = A^{[k]}, \quad A^{[k,k]} = A^{(k)}$$

$$D_h^m \left( I + hA \right)^{(k)} \bigg|_{t=0} = m! A^{[k,m]}$$
\[ D_h^m \left( I + hA \right)^{(k)} \bigg|_{t=0} = m! A^{[k,m]} \]

\[
\left( I + hA \right)^{(k)} = \sum_{m=0}^{k} h^m A^{[k,m]} \\
= hA^{[k,1]} + h^2 A^{[k,2]} + \cdots + h^k A^{[k,k]}
\]

If \( \lambda_1, \cdots, \lambda_n \) are the eigenvalues of \( A \) with eigenvectors \( v^1, \cdots, v^n \), then the eigenvalues of \( \left( I + hA \right)^{(k)} \) are

\[
h \left( \lambda_{i_1} + \cdots + \lambda_{i_k} \right) + h^2 \left( \lambda_{i_1} \lambda_{i_2} + \cdots + \lambda_{i_{k-1}} \lambda_{i_k} \right) + \cdots + h^k \left( \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \right)
\]

with eigenvectors \( v^{i_1} \wedge v^{i_2} \wedge \cdots \wedge v^{i_k} \).


