A Research Story: Compound Equations and Dynamics. Part 2

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Exterior Products

Vector Space: $u \in \mathbb{X}$

Dual Space: $v \in \mathbb{X}^* = \mathcal{L}(\mathbb{X} \to \mathbb{R})$, the space of real linear functionals $v(u)$ on $\mathbb{X}$

Bilinear Map $\langle \cdot, \cdot \rangle : \mathbb{X}^* \times \mathbb{X} \to \mathbb{R}$

$$\langle v, u \rangle = v(u)$$

Multilinear Maps: $u^1, \cdots, u^k \in \mathbb{X}$. Define $u^1 \wedge u^2 \wedge \cdots \wedge u^k : (\mathbb{X}^*)^k \to \mathbb{R}$ by

$$u^1 \wedge u^2 \wedge \cdots \wedge u^k (v_1, v_2, \cdots, v_k) \overset{def}{=} \det \left[v_i \left(u^j \right) \right] = \det \left[\left\langle v_i, u^j \right\rangle \right]$$

if $v_i \in \mathbb{X}^*$ $i = 1, 2, \cdots, k$.

$\bigwedge^k \mathbb{X}$ is the set of all finite linear combinations of the maps $u^1 \wedge u^2 \wedge \cdots \wedge u^k$.

Similarly $v_1 \wedge v_2 \wedge \cdots \wedge v_k : \mathbb{X}^k \longrightarrow \mathbb{R}$ is defined by

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \left(u^1, u^2, \cdots, u_k\right) \stackrel{def}{=} \det \left[\left\langle v_i, u^j \right\rangle\right].$$

and $\bigwedge^k \mathbb{X}^*$ is the set of all finite linear combinations of these.

The expression

$$\left\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k, u^1 \wedge u^2 \wedge \cdots \wedge u^k \right\rangle \stackrel{def}{=} \det \left[\left\langle v_i, u^j \right\rangle\right]$$

extended by linearity gives a bilinear map $\langle \cdot, \cdot \rangle : \bigwedge^k \mathbb{X}^{\,*} \times \bigwedge^k \mathbb{X} \rightarrow \mathbb{R}$.

Let $\left\{e^j\right\}$ be a basis for $\mathbb{X}$. Define real functionals e_i on the basis by $\left\langle e_i, e^j \right\rangle = \delta_i^j$. Extend e_i to $\mathbb{X}$ by linearity. if $u \in \mathbb{X}$, then

$$v = \sum_j c_j e^j, \quad c_i = \langle e_i, v \rangle$$

Moreover $\left\{e^{j_1} \wedge \cdots \wedge e^{j_k}\right\}$ is a basis for $\bigwedge^k \mathbb{X}$
Note that

$$\left\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e^{j_1} \wedge \cdots \wedge e^{j_k} \right\rangle = \delta_{(i)}^{(j)}$$

If $v^1, \cdots, v^k \in \mathbb{X}$, then $v^s = \sum_j c_j^s e^j, \quad c_i^s = \langle e_i, v^s \rangle$ and

$$\begin{aligned} v^1 \wedge \cdots \wedge v^k &= \Sigma_{(j)} c_{(j)} e^{j_1} \wedge \cdots \wedge e^{j_k}, \\ c_{(i)} &= \left\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, v^1 \wedge \cdots \wedge v^k \right\rangle \\ &= \det\left[\langle e_{i_r}, v^s \rangle\right], \quad r, s = 1, \cdots, k \\ &= \det\left[c_{i_r}^s\right] \stackrel{def}{=} c_{i_1 i_2 \cdots i_k}^{12 \ldots k} \end{aligned} \tag{1}$$

Example:

$$v^1 = \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \\ v_4^1 \end{bmatrix}, v^2 = \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \\ v_4^2 \end{bmatrix} \in \mathbb{R}^4$$

$$v^1 \wedge v^2 = \begin{bmatrix} v_{12}^{12} \\ v_{13}^{12} \\ v_{14}^{12} \\ v_{23}^{12} \\ v_{24}^{12} \\ v_{34}^{12} \end{bmatrix} \in \bigwedge{}^2 \mathbb{R}^4 \cong \mathbb{R}^6$$

Compound Operators

$\mathbb{X}$,$\mathbb{Y}$ vector spaces, $A \in \mathcal{L}(\mathbb{X} \to \mathbb{Y})$
Multiplicative compound $A^{(k)} \in \mathcal{L}\left(\bigwedge^k \mathbb{X} \to \bigwedge^k \mathbb{Y}\right)$

$$A^{(k)}\left(v^1 \wedge \cdots \wedge v^k\right) \stackrel{def}{=} Av^1 \wedge \cdots \wedge Av^k, \qquad v^j \in \mathbb{X}$$

extended by linearity to $\bigwedge^k \mathbb{X}$.
$\mathbb{X} \subset \mathbb{Y}$: *Additive compound* $A^{[k]} \in \mathcal{L}\left(\wedge^k \mathbb{X} \to \wedge^k \mathbb{Y}\right)$.

$$A^{[k]}\left(v^1 \wedge \cdots \wedge v^k\right) \stackrel{def}{=} \sum_{j=1}^{k} v^1 \wedge \cdots \wedge Av^j \wedge \cdots \wedge v^k$$

extended by linearity to $\bigwedge^k \mathbb{X}$.

Matrices

$C = A : \mathbb{R}^n \to \mathbb{R}^m$

$$\left[c_r^s\right] = \langle e_r, Ae^s\rangle = \left[a_r^s\right]$$

$C = A^{(k)} : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^m$

$$\begin{aligned} c_{(r)}^{(s)} &= \left\langle e_{r_1} \wedge \cdots \wedge e_{r_k}, A^{(k)}\left(e^{s_1} \wedge \cdots \wedge e^{s_k}\right)\right\rangle \\ &= \left\langle e_{r_1} \wedge \cdots \wedge e_{r_k}, Ae^{s_1} \wedge \cdots \wedge Ae^{s_k}\right\rangle \\ &= \det\left\langle e_{r_i}, Ae^{s_j}\right\rangle = \det\left[a_{r_i}^{s_j}\right] \;\; i = 1, \cdots, k, \;\; j = 1, \cdots, k \\ &= a_{r_1 \cdots r_k}^{s_1 \cdots s_k} \end{aligned}$$

$C = A^{[k]} : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$. Replace $A^{(k)}$ by $A^{[k]}$ above

$$c_{(r)}^{(s)} = \left\langle e_{r_1} \wedge \cdots \wedge e_{r_k}, \sum_{j=1}^{k} e^{s_1} \wedge \cdots \wedge Ae^{s_j} \wedge \cdots \wedge e^{s_k}\right\rangle$$

so that $c_{(r)}^{(s)} = a_{r_1}^{r_1} + \cdots + a_{r_k}^{r_k}$, if $(r) = (s)$. $c_{(r)}^{(s)} = (-1)^{i+j} a_{r_i}^{s_j}$,if exactly one entry r_i in (r) does not occur in (s) and s_j does not occur in (r), $c_{(r)}^{(s)} = 0$ if (r) differs from (s) in two or more entries.

Properties

1. The Binet-Cauchy identity:

$$(AB)^{(k)} = A^{(k)} B^{(k)}$$

$$(AB)^{(k)} v^1 \wedge \cdots \wedge v^k = (AB) v^1 \wedge \cdots \wedge (AB) v^k$$
$$= A \left(Bv^1\right) \wedge \cdots \wedge A (B) v^k = A^{(k)} \left[\left(Bv^1\right) \wedge \cdots \wedge (B) v^k\right]$$
$$= A^{(k)} \left[B^{(k)} v^1 \wedge \cdots \wedge v^k\right] = A^{(k)} B^{(k)} v^1 \wedge \cdots \wedge v^k$$

2. Additive property:

$$(A+B)^{[k]} = A^{[k]} + B^{[k]}$$

$$(A+B)^{[k]} \left(v^1 \wedge \cdots \wedge v^k\right) = \sum_{j=1}^{k} v^1 \wedge \cdots \wedge (A+B) v^j \wedge \cdots \wedge v^k$$
$$= \sum_{j=1}^{k} v^1 \wedge \cdots \wedge Av^j \wedge \cdots \wedge v^k + \sum_{j=1}^{k} v^1 \wedge \cdots \wedge Bv^j \wedge \cdots \wedge v^k$$
$$= A^{[k]} v^1 \wedge \cdots \wedge v^k + B^{[k]} v^1 \wedge \cdots \wedge v^k = \left(A^{[k]} + B^{[k]}\right) v^1 \wedge \cdots \wedge v^k$$

3. Multiplicative-additive relationship

$$\frac{d}{dh}\left(I + hA\right)^{(k)}\Big|_{h=0} = A^{[k]}$$

Proof:

$$\left(I + hA\right)^{(k)} v^1 \wedge \cdots \wedge v^k = \left(I + hA\right) v^1 \wedge \cdots \wedge \left(I + hA\right) v^k$$

Differentiate using the product formula and set $h = 0$

$\frac{d}{dh}\left(I + hA\right)^{(k)}\Big|_{h=0} v^1 \wedge \cdots \wedge v^k = \sum_{j=1}^{k} v^1 \wedge \cdots \wedge Av^j \wedge \cdots \wedge v^k = A^{[k]} v^1 \wedge \cdots \wedge v^k$

4. Differential Equations: L: $\dot{x} = A(t)\,x$, L_k: $\dot{z} = A^{[k]}(t)\,z$

(i) If $x^1(t), \cdots, x^k(t)$ are solutions of L, then
$z(t) = x^1(t) \wedge \cdots \wedge x^k(t)$ is a solution of L_k since
$\dot{z}(t) = \sum_{j=1}^{k} x^1(t) \wedge \cdots \wedge \dot{x}^j(t) \wedge \cdots \wedge x^k(t) =$
$\sum_{j=1}^{k} x^1(t) \wedge \cdots \wedge A(t)\,x^j(t) \wedge \cdots \wedge x^k(t) = A^{[k]}(t)\,x^1(t) \wedge \cdots \wedge x^k(t)$

(ii) If $X(t)$ is a fundamental matrix for L, then $X^{(k)}(t)$ is a fundamental matrix of L_k since from **(i)**, if $x^i(t) = X(t)\,c^i$ are solutions of L,
$X^{(k)}(t)\,c^1 \wedge \cdots \wedge c^k = X(t)\,c^1 \wedge \cdots \wedge X(t)\,c^k$ is a solution of L_k

(iii) For any square matrix A
$A^{(1)} = A^{[1]}$, $A^{(n)} = \det A$, $A^{[n]} = \operatorname{tr} A$,
$(\exp A)^{(k)} = \exp\left(A^{[k]}\right)$, $\log(\exp A)^{(k)} = A^{[k]}$
$\dot{x} = Ax$. A fundamental matrix $X(t) = \exp(At)$ (A is constant) and $X(0) = I$. The fundamental matrix $Z(t)$ of $\dot{z} = A^{[k]}z$, $Z(0) = I$, is $Z(t) = \exp\left(A^{[k]}t\right)$ and also $Z(t) = X^{(k)}(t) = \exp(At)^{(k)}$ so that $\exp(At)^{(k)} = \exp\left(A^{[k]}t\right)$; let $t = 1$.

5. Eigenvalues $n \times n$ matrix A

(i) If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of $A^{(k)}$ and $A^{[k]}$ are $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$ and $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, respectively, $1 \le i_1 < \cdots < i_k \le n$.

(ii) The corresponding eigenvectors of both $A^{(k)}$ and $A^{[k]}$ are $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ if $v_{i_1}, v_{i_2}, \cdots, v_{i_k}$ are independent eigenvectors of A corresponding to $\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}$, respectively

(iii) Geršgorin disks for A :

$$\left|z - a_i^i\right| \leq \sum_{j \neq i} \left|a_i^j\right|$$

contain the eigenvalues λ_i of A

(iv) Geršgorin disks for $A^{(k)}$:

$$\left|z - a_{i_1 \cdots i_k}^{i_1 \cdots i_k}\right| \leq \sum_{(j) \neq (i)} \left|a_{i_1 \cdots i_k}^{j_1 \cdots j_k}\right|$$

contain the products $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$

(v) Geršgorin disks for $A^{[k]}$:

$$\left|z - \left(a_{i_1}^{i_1} + \cdots + a_{i_k}^{i_k}\right)\right| \leq \sum_{j \notin (i)} \left(\left|a_{i_1}^j\right| + \cdots + \left|a_{i_k}^j\right|\right)$$

contain the sums $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$

Courant-Fischer

(vi) A a real positive definite symmetric matrix, eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$\lambda_1 = \max \frac{x^* A x}{x^* x}, \ x \in \mathbb{R}^n, \ x \neq 0$$

$$\lambda_1 \lambda_2 \cdots \lambda_k = \max \frac{y^* A^{(k)} y}{y^* y}, \ y = x^1 \wedge x^2 \wedge \cdots \wedge x^k, \ y \neq 0$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = \max \frac{y^* A^{[k]} y}{y^* y}, \ y = x^1 \wedge x^2 \wedge \cdots \wedge x^k, \ y \neq 0$$

C any positive definite symmetric, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n(C)$, are the eigenvalues of C. Courant-Fischer implies

$$\lambda_1(AB) \leq \lambda_1(A)\lambda_1(B)$$

$$\lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B)$$

It follows from the compound matrices that

$$\lambda_1\lambda_2\cdots\lambda_k(AB) \leq \lambda_1\lambda_2\cdots\lambda_k(A)\,\lambda_1\lambda_2\cdots\lambda_k(B)$$

$$\begin{aligned} & [\lambda_1+\lambda_2+\cdots+\lambda_k](A+B) \\ \leq \; & [\lambda_1+\lambda_2+\cdots+\lambda_k](A) + [\lambda_1+\lambda_2+\cdots+\lambda_k](B) \end{aligned}$$

Dimension Problems

If the differential equation

$$\dot{x} = A(t)\,x$$

satisfies *certain technical conditions**

$$\dim\left\{x(t) : \lim_{t\to\infty} x(t) = 0\right\} \geq n - k + 1$$

if and only if *all solutions* of $\dot{z} = A^{[k]}(t)\,z$ satisfy

$$\lim_{t\to\infty} z(t) = 0.$$

So for example, with $k = n$ and the technical conditions*, there exists at least one non-zero solution $x(t)$ of $\dot{x} = A(t)\,x$ such that $\lim_{t\to\infty} x(t) = 0$ if and only if $\int_0^\infty \mathrm{tr}\,A(s)\,ds = -\infty$. In this case

$$A^{[n]}(t) = \mathrm{tr}\,A(t)$$

and the solutions of the scalar equation $\dot{z} = \mathrm{tr}\,A(t)\,z$ are $z(t) = z(0)\exp\left(\int_0^t \mathrm{tr}\,A(s)\,ds\right)$.