Compound Matrices

$m \times n$ matrix:

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \; 1 \leq i \leq m, \; 1 \leq j \leq n$$

$p \times p$ minor:

$$a_{r_1 \ldots r_p}^{s_1 \ldots s_p} = \det \begin{bmatrix} a_{r_i}^{s_j} \end{bmatrix}, \; 1 \leq i, j \leq p,$$

*minor* of $A$ determined by the rows $r_1, \ldots, r_p$ and the columns $s_1, \ldots, s_p$

examples:

$$a_{11}^{12} = \begin{vmatrix} a_1^1 & a_1^2 \\ a_1^1 & a_1^2 \end{vmatrix} = 0$$

$$a_{12}^{12} = \begin{vmatrix} a_1^1 & a_2^1 \\ a_2^1 & a_2^2 \end{vmatrix}, \; a_{12}^{13} = \begin{vmatrix} a_1^1 & a_2^1 \\ a_3^1 & a_3^2 \end{vmatrix}$$
\[ a^{123}_{122} = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_2^1 & a_2^2 & a_2^3 \end{vmatrix} = 0 \]

\[ a^{123}_{123} = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{vmatrix}, \quad a^{124}_{123} = \begin{vmatrix} a_1^1 & a_1^2 & a_1^4 \\ a_2^1 & a_2^2 & a_2^4 \\ a_3^1 & a_3^2 & a_3^4 \end{vmatrix} \]
$m = n > p$, cofactor matrix:

$$A_{r_1 \ldots r_p}^{s_1 \ldots s_p}$$

is the cofactor of $a_{r_1 \ldots r_p}^{s_1 \ldots s_p}$, i.e. it is the signed minor determined by the rows complementary to rows $r_1, \ldots, r_p$ and by the columns complementary to columns $s_1, \ldots, s_p$ multiplied by $(-1)^{r_1+s_1+\ldots+r_p+s_p}$.

If $p = n$, define

$$A_{12\ldots n}^{12\ldots n} = 1.$$
3 × 3 matrix $A = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix}$:

$A_1^1 = a_{23}^2 = \begin{vmatrix} a_2^2 & a_3^3 \\ a_3^2 & a_3^3 \end{vmatrix}$, $A_2^3 = -a_{13}^{12} = -\begin{vmatrix} a_1^1 & a_1^2 \\ a_3^3 & a_3^3 \end{vmatrix}$

$A_{12}^{12} = a_3^2, A_{13}^{12} = -a_2^3$ and $A_{123}^{123} = 1.$

Note that in this case we have

$$\det A = a_i^1 A_i^1 + a_i^2 A_i^2 + a_i^3 A_i^3, \quad i = 1, 2, 3$$

and

$$0 = a_i^1 A_r^1 + a_i^2 A_r^2 + a_i^3 A_r^3, \quad \text{if} \ i \neq r.$$

$n \times n$ matrix $A$: Laplace expansion

By rows $\det A = \sum_{j=1}^{n} a_i^j A_i^j, \quad i = 1, 2, \ldots, n$

By columns $\det A = \sum_{i=1}^{n} a_i^j A_i^j, \quad j = 1, 2, \ldots, n$
If $A$ is a $n \times n$ matrix and $1 \leq k \leq n$, then the Laplace expansions by minors are:

$$\det A = \sum_{1 \leq s_1 < \ldots < s_k \leq n} a_{r_1 \ldots r_k}^{s_1 \ldots s_k} A_{r_1 \ldots r_k}^{s_1 \ldots s_k}, \text{ if } 1 \leq r_1 < \ldots < r_k \leq n$$

$$\det A = \sum_{1 \leq r_1 < \ldots < r_k \leq n} a_{r_1 \ldots r_k}^{s_1 \ldots s_k} A_{r_1 \ldots r_k}^{s_1 \ldots s_k}, \text{ if } 1 \leq s_1 < \ldots < s_k \leq n$$

Note: $0 = \sum_{(s)} a_{r_1 \ldots r_k}^{s_1 \ldots s_k} A_{t_1 \ldots t_k}^{s_1 \ldots s_k}, \text{ if } (r) \neq (t)$, and $0 = \sum_{(r)} a_{r_1 \ldots r_k}^{s_1 \ldots s_k} A_{r_1 \ldots r_k}^{t_1 \ldots t_k}, \text{ if } (s) \neq (t)$. 
$n \times n$ matrix $A$: Cofactor matrix of $A$:

$$\text{cof} A = \begin{bmatrix} A^i_j \end{bmatrix}, \ i, j = 1, \ldots, n$$

Adjugate (or classical adjoint) matrix of $A$:

$$\text{adj} A = (\text{cof} A)^T.$$

Properties:

$$A (\text{adj} A) = (\text{adj} A) A = (\det A) I$$

$$A^{-1} = \frac{1}{\det A} \text{adj} A$$

$$\det (\text{cof} A) = \det (\text{adj} A) = (\det A)^{n-1}$$
Multiplicative Compounds

An $n \times m$ matrix $A$, $1 \leq k \leq \min\{n, m\}$

$k$–th multiplicative compound is the $\binom{n}{k} \times \binom{m}{k}$ matrix

$$A^{(k)} = \begin{bmatrix} a_{r_1 \ldots r_k}^{s_1 \ldots s_k} \\ a_{(r)}^{(s)} \end{bmatrix}$$

The entry in the $r$–th row and the $s$–th column of $A^{(k)}$ is $a_{r_1 \ldots r_k}^{s_1 \ldots s_k} = a_{(r)}^{(s)}$, where $(r) = (r_1, ..., r_k)$ is the $r$–th member of the lexicographic ordering of the integers $1 \leq r_1 < r_2 < ... < r_k \leq m$ and $(s) = (s_1, ..., s_k)$ is the $s$–th member in the lexicographic (dictionary) ordering of all $k$-tuples of the integers $1 \leq s_1 < s_2 < ... < s_k \leq n$:

$$1 \leq r_1 < r_2 < r_3 \leq 5$$

$$(1) = (123), (2) = (124), (3) = (125), (4) = (134), (5) = (135),$$

$$(6) = (145), (7) = (234), (8) = (235), (9) = (245), (10) = (345).$$
Example:

\[
A = \begin{bmatrix}
  a^1_1 & a^2_1 \\
  a^1_2 & a^2_2 \\
  \vdots & \vdots \\
  a^1_m & a^2_m \\
\end{bmatrix}_{m \times 2}, \quad A^{(2)} = \begin{bmatrix}
  a^{12}_1 \\
  a^{12}_2 \\
  \vdots \\
  a^{12}_{m-1,m} \\
\end{bmatrix}_{(m) \times 1}
\]

Binet-Cauchy Theorem:

\[
AB = C \Rightarrow A^{(k)}B^{(k)} = C^{(k)}
\]

Sylvester’s Theorem:

\[
\det A^{(k)} = (\det A)^{\binom{n-1}{k-1}}
\]
Linear Differential Equations

\[ \dot{x} = A(t)x \quad (L) \]

\( t \in [0, \infty), \quad x \in \mathbb{R}^n, \quad t \to A(t)_{n \times n} \) continuous.

A solution \( x(t) \) of \((L)\) is uniquely determined by its value \( x(t_0) \) at any point \( t_0 \in [0, \infty) \).

\( X(t)_{n \times m} \) is a solution matrix of \((L)\) if \( \dot{X}(t) = A(t)X(t) \)

\( X(t) \) is a fundamental matrix of \((L)\) if it is \( n \times n \), non-singular and

\[ \dot{X}(t) = A(t)X(t) \]
The columns of a fundamental matrix span the solution space of \((L)\): 
\(x(t)\) is a solution of \((L)\) \iff there exists \(c \in \mathbb{R}^n\) such that 
\[ x(t) = X(t) c. \]
Equivalently, the columns of \(X(t)\) are solutions of \((L)\) which span the solution space of \((L)\).

In particular, each column of \(X(t)\) is a solution of \((L)\).

Suppose that \(X(t)\) is a fundamental matrix of \((L)\), then a \(n \times n\) matrix \(Y(t)\) is a fundamental matrix of \((L)\) if and only if there is a constant non-singular matrix \(C\) such that \(Y(t) = X(t) C.\)
Any continuously differentiable $n \times n$ matrix $X(t)$ is a fundamental matrix for some linear differential equation $(L) \iff X(t)$ is non-singular:

\[
\begin{align*}
A(t) &= \dot{X}(t)X^{-1}(t) \\
\dot{X}(t) &= A(t)X(t)
\end{align*}
\]
Compound Differential Equations

Recall, from Sylvester's Theorem, \( \det X(t)^{(k)} = (\det X(t))^{{(n-1)}_{k-1}} \) so that \( \det X(t) \neq 0 \Rightarrow \det X^{(k)}(t) \neq 0 \). So \( Y(t) = X^{(k)}(t) = [x_{r_1 \ldots r_k}^{{s_1 \ldots s_k}}(t)] \) is a fundamental matrix for a \( \binom{n}{k} \)-dimensional equation. The coefficient matrix in this equation is denoted \( A^{[k]} \)

\[
\dot{y} = A^{[k]}(t)y \quad (k)
\]

the \( k \)-th compound equation of \((L)\). Note that \( A^{[1]} = A, \ A^{[n]} = \text{tr} A \)

\[
\dot{y} = A(t)y \quad (1)
\]

\[
\dot{y} = \text{tr} A(t)y \quad (n)
\]

In the case \( k = n \), \( X^{(n)}(t) = \det X(t) \), and \((n)\) is the famous Abel-Jacobi scalar equation which gives

\[
\det X(t) = \det X(t_0) \exp \left( \int_{t_0}^{t} \text{tr} A(s) \, ds \right)
\]
If $X(t)$ is a $n \times m$ solution matrix of $(L)$, then $Y(t) = X^{(k)}(t)$ is a ${n \choose k} \times {m \choose k}$ solution matrix of $(k)$

Example:

$$X = \begin{bmatrix} x^1_1 & x^2_1 \\ x^1_2 & x^2_2 \\ \vdots & \vdots \\ x^1_m & x^2_m \end{bmatrix}_{m \times 2}, \quad x^{(2)} = \begin{bmatrix} x^{12}_{12} \\ x^{12}_{13} \\ \vdots \\ x^{12}_{m-1,m} \end{bmatrix}_{(m \choose 2) \times 1}$$
Additive Compounds

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix}, \ 1 \leq i, j \leq m = n \]

\[ C = A^{[k]}, 1 \leq k \leq m = n \] is called the \( k \)-th additive compound \( A \)

\[ c_r^s = \begin{cases} a_{r_1}^{r_1} + \cdots + a_{r_k}^{r_k}, & \text{if } (r) = (s) \\ (-1)^{i+j} a_{r_i}^{s_j}, & \text{if exactly one entry } r_i \text{ in } (r) \\ 0, & \text{if } (r) \text{ differs from } (s) \text{ in two or more entries} \end{cases} \]

Additivity:

\[ (A + B)^{[k]} = A^{[k]} + B^{[k]} \]
Examples:

\( n = 2 : \)

\[
A^{[1]} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A
\]

\[
A^{[2]} = a_{11} + a_{22} = \text{tr}A
\]

\( n = 3 : \)

\[
A^{[1]} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A
\]

\[
A^{[2]} = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}
\]

\[
A^{[3]} = a_{11} + a_{22} + a_{33} = \text{tr}A
\]
\[ n = 4 : \]

\[ A^{[2]} = \begin{bmatrix}
  a_{11} + a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\
  a_{32} & a_{11} + a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\
  a_{42} & a_{43} & a_{11} + a_{44} & 0 & a_{12} & a_{13} \\
 -a_{31} & a_{21} & 0 & a_{22} + a_{33} & a_{34} & -a_{24} \\
 -a_{41} & 0 & a_{21} & a_{43} & a_{22} + a_{44} & a_{23} \\
 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33} + a_{44}
\end{bmatrix} \]

\[ A^{[3]} = \begin{bmatrix}
  a_{11} + a_{22} + a_{33} & a_{34} & -a_{24} & a_{14} \\
  a_{43} & a_{11} + a_{22} + a_{44} & a_{23} & -a_{13} \\
 -a_{42} & a_{32} & a_{11} + a_{33} + a_{44} & a_{12} \\
 a_{41} & -a_{31} & a_{21} & a_{22} + a_{33} + a_{44}
\end{bmatrix} \]

\[ A^{[4]} = a_{11} + a_{22} + a_{33} + a_{44} = trA \]
Geometrical Interpretation

Solutions $x^1(t), x^2(t)$ of $(L)$ with $n = 3$ may be interpreted as oriented line segments in $\mathbb{R}^3$ whose projections on a basis $e^1, e^2, e^3$ are

$$
\begin{bmatrix}
  x^1_1(t) \\
  x^1_2(t) \\
  x^1_3(t)
\end{bmatrix}
,$$

and whose evolution in time is governed by $(L)$. If

$$
X(t) = \begin{bmatrix}
  x^1_1(t) & x^2_1(t) \\
  x^1_2(t) & x^2_2(t) \\
  x^1_3(t) & x^2_3(t)
\end{bmatrix},
$$

then $X^{(2)}(t) = \begin{bmatrix}
  x^{12}_{12} \\
  x^{12}_{13} \\
  x^{12}_{23}
\end{bmatrix}$ satisfies $(2)$ and may be considered as an oriented 2-dimensional parallelogram in $\mathbb{R}^3$ whose projection onto the $(e^i, e^j)$ coordinate plane, $i < j$, is a parallelogram with area $x^{12}_{ij}$.
If $x^1(t), \cdots, x^k(t)$ are considered as an ordered set of oriented line segments in $\mathbb{R}^n$ changing with time, then $y(t) = x^{r_1 r_2 \cdots r_k}(t)$ may be interpreted as the projection of the corresponding $k$-dimensional oriented parallelopiped in $\mathbb{R}^n$ onto the $k$-dimensional coordinate subspace spanned by $e_{r_1}, \cdots, e_{r_k}$. 
\[(\exp A)^{(k)} = \exp \left( A^{[k]} \right) \]

\[\frac{d}{dt} \left( I + tA \right)^{(k)} \bigg|_{t=0} = A^{[k]} \]

The last expression is sometimes taken as the definition of \( A^{[k]} \)