Notes to IUSEP Lectures in Mathematical Finance:

A Tour from the Binomial Model to the Black-Scholes Formula

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1. Single-period binomial model

A single-period model for a financial market:

- Consider the following very simplistic model with one stock and a bank account over one period:

prices	initial value	terminal value
stock	S_0	$\begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{with probability } 1-p \end{cases}$
bank account	1	1 + r

  for numbers $S_0 > 0$, $0 \leq d < u$, $p \in (0,1)$ and r.

- Assumptions: agent can invest in or short sell (= negative investment) the stock, and they can invest in and borrow from the bank account at the same interest rate r.

- We also assume

$$d < 1 + r < u \quad \text{(no-arbitrage condition).}$$

This assumption is reasonable. Example: if we had $d = 2$, $u = 3$ and $r = 0$, the agent could borrow a positive dollar amount x from the bank account and invest in stock to make a risk-free profit:

	initial value	terminal value
bank account	$-x$	$-x(1+r) = -x$
stock	x	$\begin{cases} ux = 3x \text{ with probability } p \\ dx = 2x \text{ with probability } 1-p \end{cases}$
total	0	$\begin{cases} 3x - x = 2x \text{ with probab. } p \\ 2x - x = x \text{ with probab. } 1-p \end{cases}$

Pricing financial derivatives:

- Consider now additionally a financial derivative with given payoff as follows:

initial value	terminal value
$x = ?$	$\begin{cases} f_u & \text{if stock price} = uS_0 \text{ (probab. } p) \\ f_d & \text{if stock price} = dS_0 \text{ (probab. } 1-p) \end{cases}$

for fixed numbers f_u and f_d.

- Example: (European) call option with strike K. *A call option gives the buyer the right, but not the obligation to buy the stock at maturity for the strike price K.* In our model the option payoff is

$$\begin{cases} \max\{uS_0 - K, 0\} & \text{with probability } p \\ \max\{dS_0 - K, 0\} & \text{with probability } 1-p \end{cases}$$

  because
  · two possibilities uS_0 or dS_0 for the stock price,
  · the buyer will use (= exercise) the option only if the stock price is higher than K.

- How can we find an initial value x for the financial derivative?

  We can use a replication argument because we will see that we can obtain here the same payoff by investing in the stock and bank account.

  Consider a portfolio consisting of Δ units of the stock and Ψ units of the bank account.

initial value	terminal value
$\Delta S_0 + \Psi$	$\begin{cases} \Delta u S_0 + \Psi(1+r) & \text{if stock} = uS_0 \\ \Delta d S_0 + \Psi(1+r) & \text{if stock} = dS_0 \end{cases}$

- We find Δ and Ψ of a replicating portfolio by setting its terminal value equal to that of the financial derivative, which implies

$$f_u = \Delta u S_0 + \Psi(1+r),$$
$$f_d = \Delta d S_0 + \Psi(1+r).$$

Solving this system of two linear equations for the unknowns gives

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}, \quad \Psi = \frac{u f_d - d f_u}{(1+r)(u-d)}$$

so that the initial value of the portfolio equals

$$\begin{aligned} \Delta S_0 + \Psi &= \frac{f_u - f_d}{u-d} + \frac{u f_d - d f_u}{(1+r)(u-d)} \\ &= \frac{1}{1+r}\left(\frac{1+r-d}{u-d} f_u + \frac{u-1-r}{u-d} f_d\right). \end{aligned}$$

To avoid arbitrage (= risk-free gains), this quantity must be equal to the initial value x of the financial derivative:

$$x = \frac{1}{1+r}\left(\frac{1+r-d}{u-d} f_u + \frac{u-1-r}{u-d} f_d\right).$$

Risk-neutral probabilities:

- Define

$$q_u = \frac{1+r-d}{u-d}, \qquad q_d = \frac{u-1-r}{u-d}$$

so that we can write

$$x = \frac{1}{1+r}(q_u f_u + q_d f_d).$$

- Note that

  · $q_u + q_d = 1$,

  · $d < 1+r < u \Longrightarrow q_u > 0, q_d > 0$.

Therefore, we can consider q_u, q_d as the probabilities of a probability measure Q and we have

$$x = \frac{1}{1+r}E^Q[f] = \frac{1}{1+r}(q_u f_u + q_d f_d),$$

where f is the random variable of the option payoff and E^Q denotes the expectation under the measure Q. In other words,

option value = expectation of the discounted payoff under a measure Q

(discounted because payoff is divided by $1+r$).

A crucial observation is that the probability measure Q used in the pricing formula does not equal the historical (from the model construction) probability measure because in general

$$q_u = \frac{1+r-d}{u-d} \neq p, \quad q_d = \frac{u-1-r}{u-d} \neq 1-p.$$

- If we calculate the expectation of the discounted stock price under Q, we obtain

$$\begin{aligned} & E^Q\left[\frac{\text{terminal value of stock}}{1+r}\right] \\ &= q_u \frac{uS_0}{1+r} + q_d \frac{dS_0}{1+r} \\ &= \frac{1+r-d}{u-d} \cdot \frac{uS_0}{1+r} + \frac{u-1-r}{u-d} \cdot \frac{dS_0}{1+r} \\ &= S_0, \end{aligned}$$

which shows that the expectation of the discounted terminal stock price under Q equals its initial value. Therefore, q_u and q_d are called risk-neutral probabilities and Q a risk-neutral probability measure.

- Remark. A financial market (like that we considered here) where every payoff can be replicated is called complete. It can be proved that a risk-neutral probability measure exists if there is no arbitrage in the market model and it is unique if the market is complete.

- The pricing formula we derived was based on a replication argument: we replicated the payoff of the derivative by investing in the stock and bank account. As a byproduct, we also saw the right number of stocks we need for the replication, which is $\Delta = \frac{f_u - f_d}{S_0(u-d)}$.

  This means that as a buyer of the option, we can "neutralize" the option by investing $-\Delta$ in the stock. Conversely, as a writer (= seller) of the option, we can buy Δ units of the stock to hedge against our risk. Consequently, this is called a replicating strategy or hedging strategy.

2. Two-period binomial model

- We can extend the model of Section 1 by adding a second period. We then have a tree of the form

time 0		time 1		time 2	probability
				$u^2 S_0$	p^2
stock:			$\nearrow_{p}$		
		uS_0			
	$\nearrow_{p}$		$\searrow_{1-p}$		
S_0				udS_0	$2p(1-p)$
	$\searrow_{1-p}$		$\nearrow_{p}$		
		dS_0			
			$\searrow_{1-p}$		
				$d^2 S_0$	$(1-p)^2$

bank account:

$$1 \quad \rightarrow \quad 1+r \quad \rightarrow \quad (1+r)^2$$

- Trading is now also possible at the intermediate time 1. We still assume $d < 1+r < u$.

- Let us consider a European call option with maturity 2. It has the following payoff at time 2:

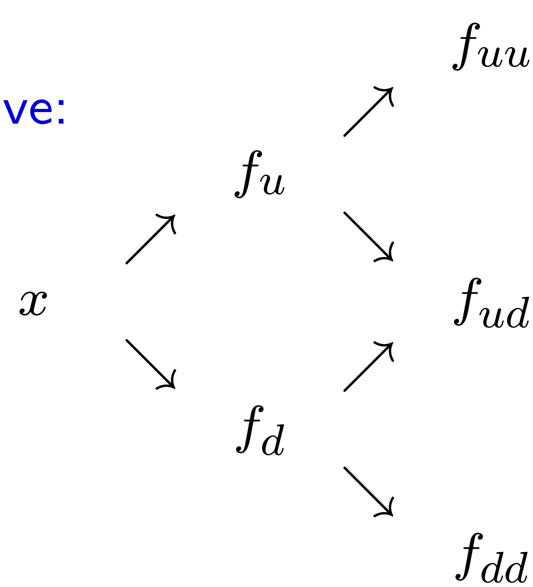
$$
\begin{cases}
f_{uu} = \max\{u^2S_0 - K, 0\} & \text{if stock } = u^2S_0 \\
f_{ud} = \max\{udS_0 - K, 0\} & \text{if stock } = udS_0 \\
f_{dd} = \max\{d^2S_0 - K, 0\} & \text{if stock } = d^2S_0
\end{cases}
$$

- By a replication argument similarly to that in Section 1 applied to both trading periods, the price of the derivative equals

$$
\frac{1}{(1+r)^2}\Big(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}\Big),
$$

where $q = \frac{1+r-d}{u-d}$. Indeed, we have

derivative:



Applying the reasoning of Section 1 to each branch of the tree gives

$$f_u = \frac{1}{1+r}\Big(qf_{uu} + (1-q)f_{ud}\Big),$$
$$f_d = \frac{1}{1+r}\Big(qf_{ud} + (1-q)f_{dd}\Big),$$
$$x = \frac{1}{1+r}\Big(qf_u + (1-q)f_d\Big)$$
$$= \frac{1}{(1+r)^2}\Big(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}\Big).$$

This means that the price equals $\frac{1}{(1+r)^2}E^Q[f]$, where $f = \max\{S_2 - K, 0\}$ is the option payoff and Q is the probability measure with probabilities q^2, $2q(1-q)$, $(1-q)^2$ corresponding to the different states u^2S_0, udS_0, d^2S_0, respectively, of the stock at time 2.

- One can also show that

$$\frac{1}{(1+r)^2}\Big(q^2u^2S_0 + 2q(1-q)udS_0 + (1-q)^2d^2S_0\Big)$$

equals S_0 so that Q is a risk-neutral measure.

3. Multiperiod binomial model

- We can further extend the model to n periods so that we have

	time n	probability
stock:		
	$u^n S_0$	p^n
	$u^{n-1} d S_0$	$np^{n-1}(1-p)$
	$\vdots$	$\vdots$
	$u^{n-j} d^j S_0$	$\binom{n}{j} p^{n-j}(1-p)^j$
	$\vdots$	$\vdots$
	$u d^{n-1} S_0$	$np(1-p)^{n-1}$
	$d^n S_0$	$(1-p)^n$
bank account:		
	$(1+r)^n$	

- A European call with maturity n and strike K has the payoff $\max\{S_n - K, 0\}$, which means

$$
\begin{cases}
f_{u^n} = \max\{u^n S_0 - K, 0\} & \text{if } S_n = u^n S_0 \\
\quad \vdots & \quad \vdots \\
f_{u^{n-j}d^j} = \max\{u^{n-j} d^j S_0 - K, 0\} & \text{if } S_n = u^{n-j} d^j S_0 \\
\quad \vdots & \quad \vdots \\
f_{d^n} = \max\{d^n S_0 - K, 0\} & \text{if } S_n = d^n S_0
\end{cases}
$$

- Extending the pattern of the two-period, the fair price of the option is given by

$$\frac{1}{(1+r)^n}\Big(q^n f_{u^n} + \cdots + \binom{n}{j} q^{n-j}(1-q)^j f_{u^{n-j}d^j} + \cdots + (1-q)^n f_{d^n}\Big),$$

  where

$$q = \frac{1+r-d}{u-d}.$$

- Associating to q the corresponding measure Q, we can write the option price as

$$\frac{1}{(1+r)^n} E^Q[f] = \frac{1}{(1+r)^n} E^Q[\max\{S_n - K, 0\}],$$

  where we emphasize that it is the expectation under Q and not under the historical probability.

- Remark: Under the historical probability, S_n is related to a binomial distribution with parameters p and n. Under the probability measure Q, S_n is still related to a binomial distribution but with parameters $q = \frac{1+r-d}{u-d}$ and n. So for the option pricing, we just change the parameters of the distribution of S_n and take then expectations of discounted values.

Pricing a call option by writing a MATLAB function **callnperiod.m**:

```
function price = callnperiod(u,d,r,S0,K,n)
% calculate the price of a call option with ...
  strike K in an n period binomial model
if d<1+r && 1+r<u
    price=0;
    q = (1+r-d)/(u-d);
  for j=0:n
      price = price + ...
        nchoosek(n,j)*q^(n-j)*(1-q)^j*...
          max(u^(n-j)*d^j*S0-K,0)/(1+r)^n;
  end
else
error('wrong parameters')
end
```

```
% plot the call price in dependence of the ...
  strike price
K = 0:0.05:10;
price = callnperiod(1.2,.95,.05,1,K,20);
plot(K,price,'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
```

```
ylabel('option price','fontsize',14);
axis([0 10 0 1]) % choosing suitable range ...
  for axes
```

4. Transition to continuous time

- The binomial model can be used as approximation for a model with continuous trading possibilities on some time interval $[0, T]$.

To show convergence, one lets tend the number n of periods to infinity and, simultaneously, the length of each period tend to zero. This means that one makes specific choices for u, d and r depending on n; for details, please see the Appendix.

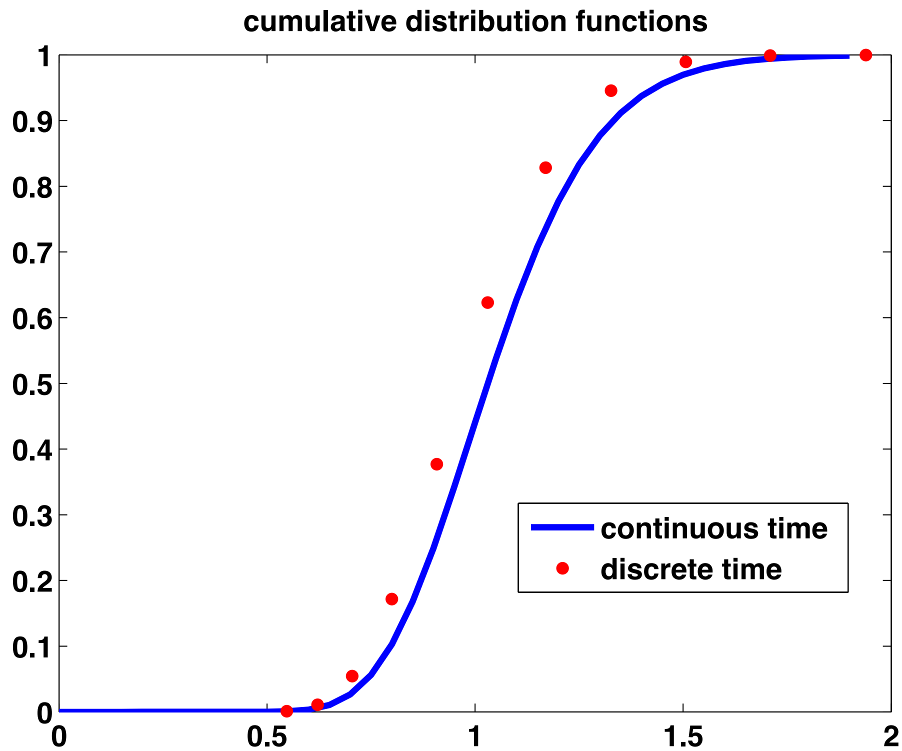
- The resulting continuous-time model has a bank account whose value at time T equals $\exp(\rho T)$ and a stock whose price at time T is given by

$$S_T = S_0 \exp\left((\mu - \sigma^2/2)T + \sigma\sqrt{T}N\right),$$

where ρ, μ and $\sigma > 0$ are constants and N is a standard normally distributed random variable.

```
function convergenceS(mu,sigma,T,n)
% compares the cumulative distribution ...
  function of S_n in a binomial model with ...
  that of the corresponding log-normal ...
  distribution

u = exp((mu-sigma^2/2)*T/n+sigma*(T/n)^.5);
% appropriate choice of u
d = exp((mu-sigma^2/2)*T/n-sigma*(T/n)^.5);
% appropriate choice of d
j=0:n;
```

```
Sn = d.^(n-j).*u.^j; % S_n for p = 1/2
bin=binocdf(0:n,n,1/2); % cumulative ...
  distribution function of S_n
points = 0:.05:Sn(n+1); % choose equidistant ...
  points for plot
lognorm=logncdf(points,(mu-sigma^2/2)*T,...
    sigma*T^.5); % cumulative distribution ...
      function of S
plot(points,lognorm,Sn,bin,'r.','LineWidth',...
    3,'MarkerSize',18) % r = red, . = point
set(gca,'fontsize',14,'FontWeight','bold');
title('cumulative distribution functions');
legend('continuous time','discrete ...
  time','location','best');
```



- Similarly to the binomial model, the price of a European call option with strike K and maturity T is given by

$$\frac{1}{\exp(\rho T)} E^{Q}[\max\{S_T - K, 0\}]$$

for some probability measure Q. This probability measure is such that

$$S_T = S_0 \exp\Big((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\Big)$$

for a random variable $\tilde{N}$ that is normally distributed under Q. Using this fact, we can rewrite the price of the European call as

$$c = S_0 \Phi(d_1) - K\exp(-\rho T)\Phi\Big(d_1 - \sigma\sqrt{T}\Big), \quad (\star)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\,\mathrm{d}u$$

is the standard-normal distribution function and $d_1 = \frac{\log\frac{S_0}{K} + \rho T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$.

Comments:

- $(\star)$ is the famous Black-Scholes formula. Note c depends on S_0, K, ρ, σ and T, but not on μ.

- In continuous time, the underlying process of the stock price dynamics is related to a Brownian motion.

- The partial derivatives of the Black-Scholes formula ($\star$) with respect to its parameters are called Greeks.

  1. Delta $= \frac{\partial c}{\partial S_0} = \Phi(d_1) \in (0,1)$ is the amount of the risky asset held in the replicating portfolio.

  2. Gamma $= \frac{\partial^2 c}{\partial S_0^2} = \Phi'(d_1)\frac{1}{S_0\sigma\sqrt{T}} > 0$;
     if Gamma is big, frequent adjustments of the replicating portfolio are necessary.

  3. Theta $= -\frac{\partial c}{\partial T}$
     $= -\frac{S_0\sigma\Phi'(d_1)}{2\sqrt{T}} - K\rho\exp(-\rho T)\Phi\left(d_1 - \sigma\sqrt{T}\right)$
     < 0.

  4. Rho $= \frac{\partial c}{\partial \rho} = KT\exp(-\rho T)\Phi\left(d_1 - \sigma\sqrt{T}\right) > 0$.

  5. Vega $= \frac{\partial c}{\partial \sigma} = S_0\sqrt{T}\Phi'(d_1) > 0$.

- The principle of valuation under Q holds generally. The price of a derivative with payoff $f(S_T)$ is

$$\exp(-\rho T)\, E^Q[f(S_T)]$$
$$= \mathrm{e}^{-\rho T} E^Q\Big[f\Big(S_0 \exp\big((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\big)\Big)\Big]$$

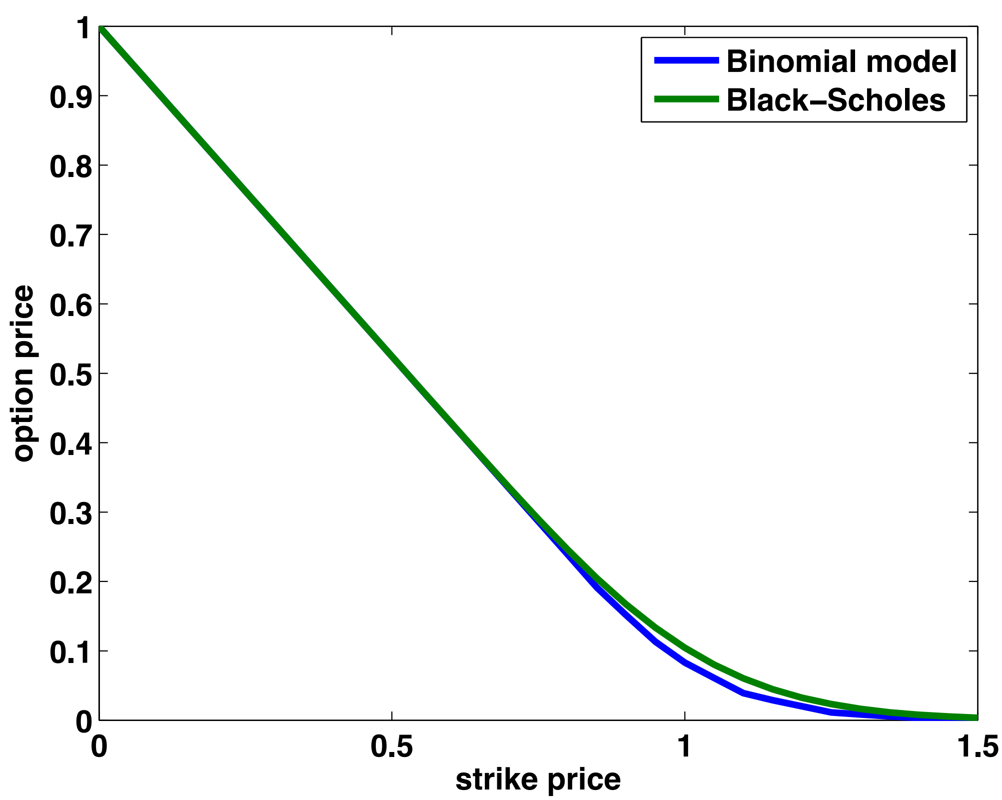
for a normally distributed $\tilde{N}$ under Q.

Comparison of Black-Scholes with Binomial model:

```
function [priceBin,priceBS] = ...
  compareCall(rho,mu,sigma,T,K,n)
% compares the call option price in a ...
  binomial model with the continuous-time ...
  analogue from the Black-Scholes formula
u = exp((mu-sigma^2/2)*T/n+sigma*(T/n)^.5); ...
  % appropriate choice of u
d = exp((mu-sigma^2/2)*T/n-sigma*(T/n)^.5); ...
  % appropriate choice of d
r = rho*T/n; % appropriate choice of r
priceBin = callnperiod(u,d,r,1,K,n);
d1 = (log(1./K) + rho*T)/sigma/T^.5 + ...
  sigma*T^.5/2;
priceBS = normcdf(d1) - ...
  K.*exp(-rho*T).*normcdf(d1-sigma*T^.5);

% or, alternatively, by applying the ...
  Financial Toolbox, we could use
% priceBS = blsprice(1,K,rho,T,sigma);
```

Plot the comparison of the Call option prices using the script `compareCallPlot.m`:

```
% plot comparison of Call option prices in ...
  binomial model and Black—Scholes model
K=0:.05:1.5;
[a,b] = compareCall(.05,.5,.2,1,K,10);
plot(K,a,K,b,'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
ylabel('option price','fontsize',14);
legend('Binomial model','Black—Scholes')
```



5. Implied volatility

- The value of σ is hard to determine $\rightarrow$ idea: find σ by inverting the Black-Scholes formula and using the market price of the option.

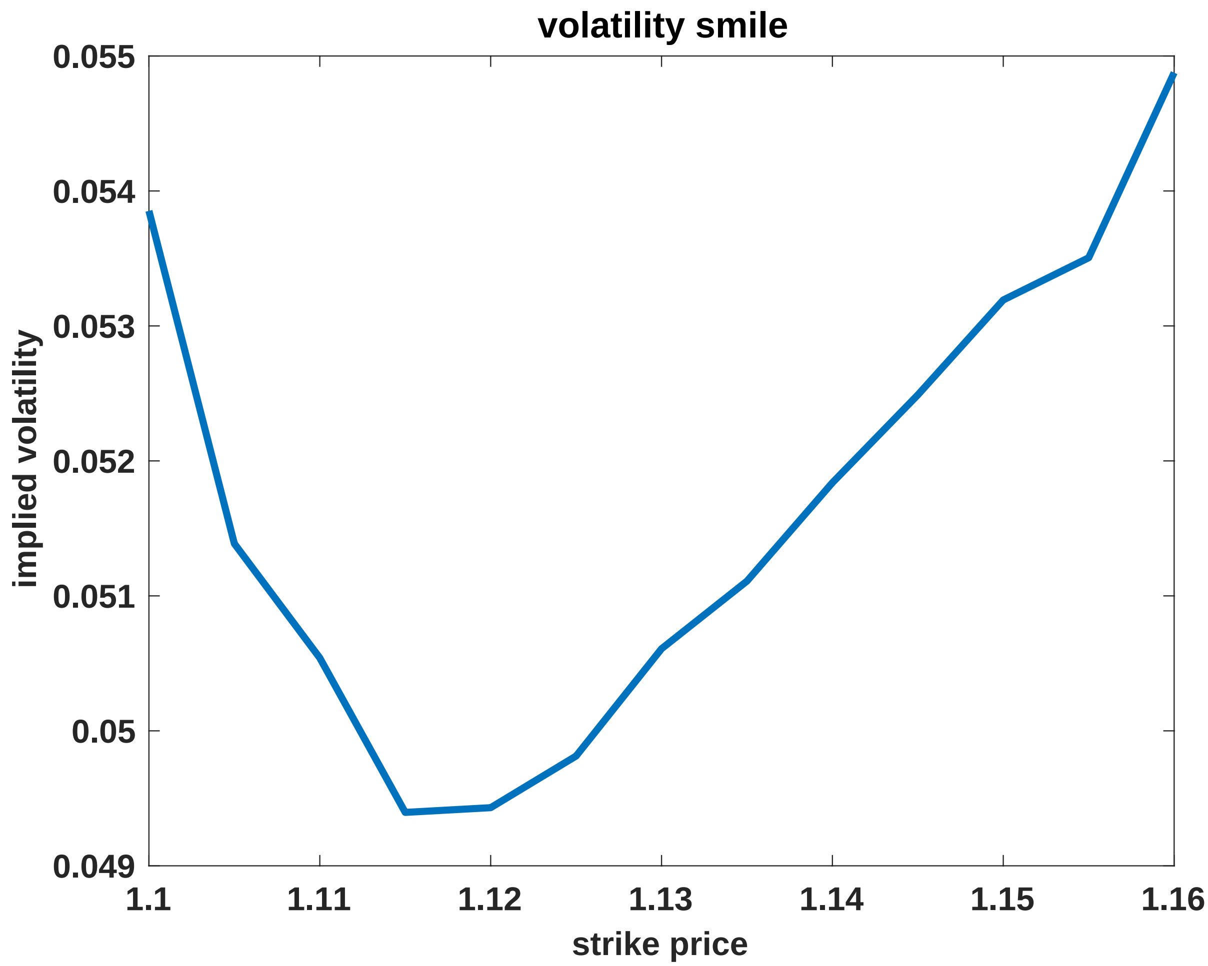
- The implied volatility σ_{impl} is defined as the unique σ such $c_{\text{BS}}(\sigma) = c_{\text{market}}$, where c_{market} is the market price of the option and c_{BS} is the value of the Black-Scholes formula $(\star)$ depending on σ.

- If the Black-Scholes model is correct, σ_{impl} does not depend on K, S_0, T and ρ. But in reality, one sees a strong dependence on K (volatility smile/skew).

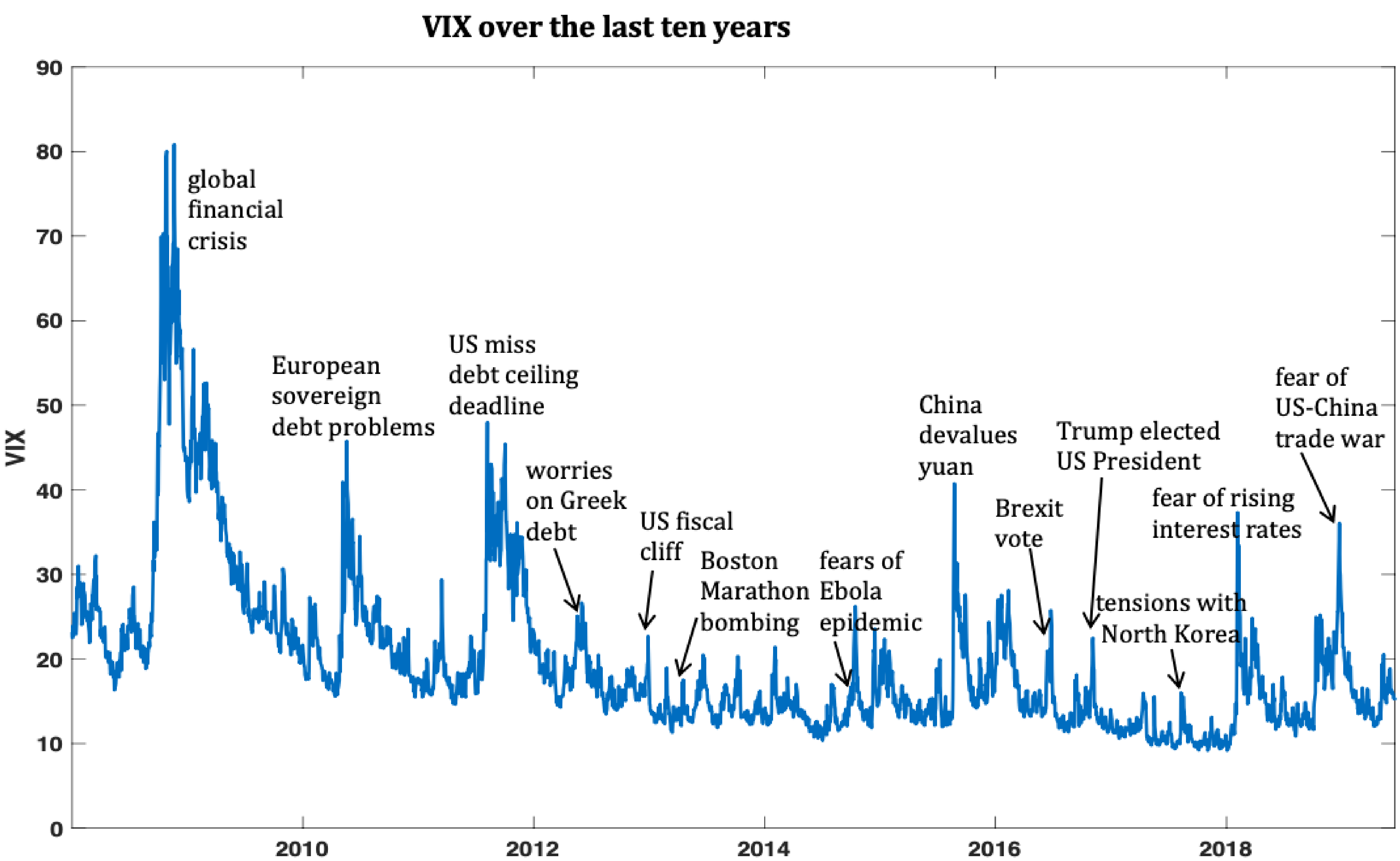
```
% The financial toolbox has the function  ...
  blsimpv to calculate implied volatility: ...
  blsimpv(Current price of Stock S_0,  ...
  Strike K, Interest rate rho, Time to ...
  maturity T, Option price)

>> blsimpv(100, 95, 0, 0.25, 10)

ans =
    0.3722
```

We now calculate the implied volatility on EUR/USD Call options, writing a script `volaEURUSD.m`. The resulting plot shows a volatility smile.

```
% volaEURUSD.m needs financial toolbox
%
% Implied volatility of currency options.  ...
  We have the following data: 1 EUR = 1.1197 ...
  USD (June 1st, 2019); the matrix A gives ...
  the prices (in USD) of call options with ...
  maturity end of September [such data can ...
  be found at http://www.cmegroup.com/]
 A = [1.10 0.0258; 1.105 0.0218; 1.11 ...
   0.0184; 1.115 0.0152; 1.12 0.0126; 1.125 ...
   0.0104; 1.13 0.0086; 1.135 0.007; 1.14 ...
   0.0057; 1.145 0.0046; 1.15 0.0037; 1.155 ...
   0.0029; 1.16 0.0024];

% calculate the implied volatilities:
A(:,3) = blsimpv(1.1197,A(:,1),0,4/12,A(:,2));
% from June 1st until end of September = 4 ...
  months
% A(:,2) means all numbers of the 2nd column
plot(A(:,1),A(:,3),'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
ylabel('implied volatility','fontsize',14);
xlim([A(1,1),A(end,1)])
title('volatility smile')
```



There exist indices which measure the implied volatility. A popular measure is VIX, which reflects the implied volatility of options on the stock index S&P 500. VIX is often referred to as “fear index”, because a high level of VIX means a lot of uncertainty in the market; see the development of VIX on the next page.


VIX over the last ten years
VIX
90
80
70
60
50
40
30
20
10
0
2010
2012
2014
2016
2018
global financial crisis
European sovereign debt problems
US miss debt ceiling deadline
worries on Greek debt
US fiscal cliff
Boston Marathon bombing
fears of Ebola epidemic
China devalues yuan
Brexit vote
Trump elected US President
tensions with North Korea
fear of rising interest rates
fear of US-China trade war

Appendix: additional explanations and proofs to Section 4

A.1 Choice in the continuous-time model

In the continuous-time situation, we model the terminal value of the bank account as $B_T = \exp(\rho T)$ and the terminal value of the stock as

$$S_T = S_0 \exp\Big((\mu - \sigma^2/2)T + \sigma\sqrt{T}N\Big), \qquad (1)$$

where ρ, μ and $\sigma > 0$ are constants and N is a standard normally distributed random variable.

A.2 Explanations behind choice

The reason behind these choices is as follows. In continuous time, the bank account models continuous interest, which means

$$\mathrm{d}B_t = \rho B_t \, \mathrm{d}t.$$

We can interpret this as that the infinitesimal change $\mathrm{d}B_t$ in the bank account is equal to the continuous interest rate ρ times the capital B_t. This equation is equivalent to $\frac{\mathrm{d}B_t}{\mathrm{d}t} = \rho B_t$, which yields $B_T = \exp(\rho T)$ using that $B_0 = 1$.

To explain the form (1) of the stock price, we can say that on average (which means in expectation) the stock should have a similar growth form than the bank account. Hence, $E[S_T] = S_0 \exp(\mu T)$ for some constant μ (typically μ will be bigger than ρ to compensate for the risk in the stock), using that S starts at S_0 and not necessarily at 1, in contrast to the bank account. Now, S_T will not just be equal to the deterministic value $S_0 \exp(\mu T)$, but will also reflect some random factor because we do not know future prices. Hence, S_T is of the form

$$S_T = S_0 \exp(\mu T) \times (\text{positive random factor}). \quad (2)$$

The reason for this positive random factor is related to the so-called Brownian motion. At the moment, you should just accept that we can model it with a normally distributed random variable, but because it should be positive, we take the exponential of this normally distributed random variable so that

$$\text{positive random factor} = \exp(cN) \quad (3)$$

where c is some constant and N is a standard normally distributed random variable. The bigger T, the longer the time horizon is and more uncertain S_T is. Therefore, c should depend on T, and we will again see later that the right form is $c = \sigma\sqrt{T}$, hence it grows like square root in T times some constant σ, which gives us how big the fluctuation in S_T is. Combining this with (2) and (3), we get

$$S_T = S_0 \exp\left(\mu T + \sigma\sqrt{T}N\right) \tag{4}$$

for some normally distributed N. Recall we wanted to have $E[S_T] = S_0 \exp(\mu T)$ so that μ has the interpretation of the mean growth rate, but we can calculate

$$\begin{aligned} & E\Big[S_0 \exp\left(\mu T + \sigma\sqrt{T}N\right)\Big] \\ & = S_0 \exp(\mu T) E\Big[\exp\left(\sigma\sqrt{T}N\right)\Big] \\ & = S_0 \exp\left(\mu T + \sigma^2 T/2\right), \end{aligned}$$

using the formula that $E[\exp(\alpha N)] = \exp(\alpha^2/2)$ for any constant α and standard normally distributed N. Therefore, to get $E[S_T] = S_0 \exp(\mu T)$, we need to divide (3) by $\exp(\sigma^2 T/2)$, which leads to (1).

A.3 Convergence proofs

We show now that under suitable choices of r_n, d_n and u_n, the terminal values of the bank account and stock in the binomial model converge to $B_T = \exp(\rho T)$ and S_T given in (1).

Proposition 1 *For $r_n = \rho T/n$, we have*

$$\lim_{n\to\infty}(1+r_n)^n = \exp(\rho T).$$

Proof.

$$\begin{aligned}\lim_{n\to\infty}(1+\rho T/n)^n &= \exp\Big(\ln\Big(\lim_{n\to\infty}(1+\rho T/n)^n\Big)\Big)\\ &= \exp\Big(\lim_{n\to\infty}\ln(1+\rho T/n)^n\Big)\\ &= \exp\Big(\lim_{n\to\infty} n\ln(1+\rho T/n)\Big)\\ &= \exp\left(\lim_{n\to\infty}\frac{\ln(1+\rho T/n)}{1/n}\right),\end{aligned}$$

which equals

$$\begin{aligned}\exp\left(\lim_{n\to\infty}\frac{\ln(1+\rho T/n)}{1/n}\right) &\overset{(*)}{=} \exp\left(\lim_{s\searrow 0}\frac{\ln(1+\rho T s)}{s}\right)\\ &\overset{(**)}{=} \exp\left(\lim_{s\searrow 0}\frac{\rho T}{1+\rho T s}\right)\\ &= \exp(\rho T)\end{aligned}$$

($*$) set $s = 1/n$, then $n \to \infty \iff s \searrow 0$
($**$) L'Hôpital's rule using $\frac{d}{ds} \ln(1 + \rho T s) = \frac{\rho T}{1+\rho T s}$ ∎

Proposition 2 *Set $p = 1/2$ and define*

$$u_n = \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right),$$
$$d_n = \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$$

then S_n in the n-period binomial model converges to S_T in (1).

Proof. If S_n reflects j times u_n and $n-j$ times d_n, we have

$$\begin{aligned} S_n &= S_0 u_n^j d_n^{n-j} \\ &= S_0 \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n}j + \sigma\sqrt{\frac{T}{n}}j\right) \\ &\quad \times \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n}(n-j) - \sigma\sqrt{\frac{T}{n}}(n-j)\right) \\ &= S_0 \exp\left(\left(\mu - \sigma^2/2\right)T + \sigma\sqrt{T}\frac{2j-n}{\sqrt{n}}\right). \end{aligned}$$

Comparing this with (1), it remains to show that $\frac{2j-n}{\sqrt{n}}$ converges to a standard normally distributed random variable. Define random variables X_i by

$$X_i = \begin{cases} 1, & \text{if we have } u_n \text{ in period } i \\ -1, & \text{if we have } d_n \text{ in period } i \end{cases} \tag{5}$$

and note that if we have j times u_n and $n-j$ times d_n, then

$$\sum_{i=1}^{n} X_i = j + (n-j)(-1) = 2j - n.$$

Therefore, we can write

$$\frac{2j-n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i. \tag{6}$$

We now apply the Central Limit Theorem, which says that for independent and identically distributed random variables $X_1, X_2, \ldots$ with mean $\mu = E[X_i]$ and finite variance $\sigma^2 = \mathrm{Var}(X_i)$,

$$\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right) \tag{7}$$

converges (in distribution) to a standard normally distributed random variable. In our case of X_i given by

(5) with equal probability 1/2 for the two cases (because $p = 1/2$ by assumption), we have

$$\mu = E[X_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0,$$
$$\sigma^2 = \mathrm{Var}(X_i) = E\Big[X_i^2\Big] = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot (-1)^2 = 1.$$

Therefore, (7) simplifies in our case to $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. Because of (6), this shows that $\frac{2j-n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ converges (in distribution) to a standard normally distributed random variable. ∎

A.4 Derivation of the Black-Scholes formula

Similarly to the binomial model, the price for a payoff f in the Black-Scholes model is given by $\frac{1}{\mathrm{e}^{\rho T}} E^Q[f]$ where the terminal value of the stock price is

$$S_T = S_0 \exp\Big((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\Big)$$

with $\tilde{N}$ standard normally distributed under Q. In the case of a call option with strike price K, the price equals

$$
\begin{aligned}
c &= \frac{1}{\mathrm{e}^{\rho T}} E^{Q}[\max\{S_T - K, 0\}] \\
&= \frac{1}{\mathrm{e}^{\rho T}} E^{Q}\Big[\max\Big\{S_0 \mathrm{e}^{(\rho-\sigma^2/2)T+\sigma\sqrt{T}\tilde{N}} - K, 0\Big\}\Big] \\
&= \frac{1}{\mathrm{e}^{\rho T}} \int_{-\infty}^{\infty} \max\Big\{S_0 \mathrm{e}^{(\rho-\sigma^2/2)T+\sigma\sqrt{T}x} - K, 0\Big\} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \max\Big\{S_0 \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T}, 0\Big\} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x.
\end{aligned}
$$

Now, we use the equivalences

$$
\begin{aligned}
& S_0 \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T} \geq 0 \\
& \iff S_0 \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} \geq K\mathrm{e}^{-\rho T} \\
& \iff \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} \geq \frac{K}{S_0}\mathrm{e}^{-\rho T} \\
& \iff -\sigma^2 T/2 + \sigma\sqrt{T}x \geq \log\left(\frac{K}{S_0}\right) - \rho T \\
& \iff x \geq \frac{\log(K/S_0) - \rho T}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2.
\end{aligned}
$$

Therefore, defining $d = \frac{\log(K/S_0)-\rho T}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2$ allows us to write

$$c = \int_{-\infty}^{\infty} \max\left\{S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T}, 0\right\} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$

$$= \int_{d}^{\infty} \left(S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T}\right) \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$

$$= \int_{d}^{\infty} S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$

$$- \int_{d}^{\infty} K\mathrm{e}^{-\rho T} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x.$$

For the first term, we calculate

$$\int_{d}^{\infty} S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$

$$= S_0 \mathrm{e}^{-\sigma^2 T/2} \int_{d}^{\infty} \frac{\mathrm{e}^{\sigma\sqrt{T}x - x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$

$$= S_0 \mathrm{e}^{-\sigma^2 T/2} \int_{d}^{\infty} \frac{\mathrm{e}^{-(x-\sigma\sqrt{T})^2/2}\mathrm{e}^{\sigma^2 T/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$

$$= S_0 \int_{d-\sigma\sqrt{T}}^{\infty} \frac{\mathrm{e}^{-y^2}}{\sqrt{2\pi}}\,\mathrm{d}y$$

$$= S_0\Big(1 - \Phi\Big(d - \sigma\sqrt{T}\Big)\Big)$$

$$= S_0\Phi\Big(-d + \sigma\sqrt{T}\Big).$$

For the second term, we have

$$\int_d^\infty K\mathrm{e}^{-\rho T}\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = K\mathrm{e}^{-\rho T}\int_d^\infty \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= K\mathrm{e}^{-\rho T}\Big(1-\Phi(d)\Big)$$
$$= K\mathrm{e}^{-\rho T}\Phi(-d).$$

Defining

$$d_1 = -d + \sigma\sqrt{T} = \frac{\log(K/S_0)+\rho T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

we obtain

$$c = \int_d^\infty S_0\mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x}\ \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$-\int_d^\infty K\mathrm{e}^{-\rho T}\ \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= S_0\Phi(d_1) - K\mathrm{e}^{-\rho T}\Phi\Big(d_1-\sigma\sqrt{T}\Big),$$

which is the Black-Scholes formula.