Applications of Wavelets and Framelets

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Outline of Tutorial

- Wavelets in the function setting.
- Some applications of wavelets and framelets
- Tensor product wavelets and framelets
- Image processing using complex tight framelets.
- Subdivision schemes in computer graphics.

Declaration: Some figures and graphs in this talk are from various sources from Internet, or from published papers, or produced by `matlab`, `maple`, or `C` programming. [Details and sources of all graphs can be provided upon request of the audience.]

What Is a Wavelet in the Function Setting?

- Let $\phi = (\phi_1, \dots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \dots, \psi_s)^\mathsf{T}$ in $L_2(\mathbb{R})$.
- A system is derived from ϕ, ψ via dilates and integer shifts:

$$\mathrm{AS}_0(\phi;\psi) := \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi_{j;k} := 2^{j/2}\psi(2^j \cdot - k) \ : \ j \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}\}.$$

- $\{\phi;\psi\}$ is called an orthogonal wavelet in $L_2(\mathbb{R})$ if $\mathrm{AS}_0(\phi;\psi)$ is an orthonormal basis of $L_2(\mathbb{R})$.
- $\{\phi;\psi\}$ is a tight framelet in $L_2(\mathbb{R})$ if

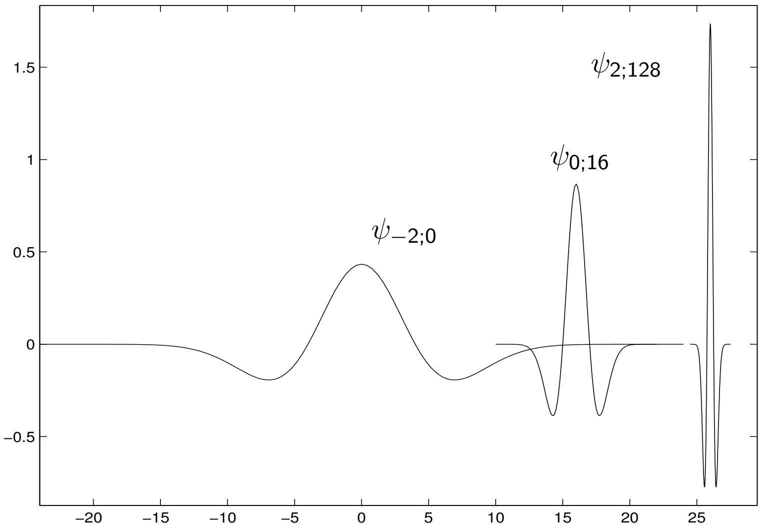
$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} \|\langle f, \phi(\cdot - k)\rangle\|^2_{l_2} + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \|\langle f, \psi_{j;k}\rangle\|^2_{l_2}, \ f \in L_2(\mathbb{R}).$$

- Orthogonal wavelet and tight framelet representation:

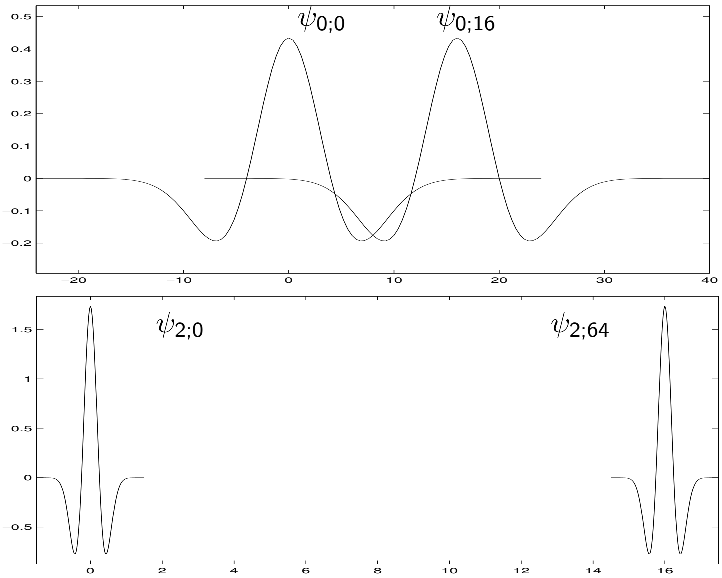
$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j;k}\rangle \psi_{j;k}, \quad f \in L_2(\mathbb{R}),$$

where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}^\mathsf{T} dx$ is the inner product.

Dilates of a Wavelet



Integer Shifts of a Wavelet



Why Wavelets?

A wavelet ψ often has

1. compact support $\Rightarrow$ good spatial localization.
2. high smoothness/regularity $\Rightarrow$ good frequency localization.
3. high vanishing moments $\Rightarrow$ multiscale sparse representation.
4. associated filter banks $\Rightarrow$ fast wavelet transform to compute coefficients $\langle f, \psi_{j;k}\rangle$ through filter banks.
5. singularity detecting/locating and good approximation property.
6. close relations to windowed and fast Fourier transform.

Explanation:

- Vanishing moments: $\langle x^j, \psi(x)\rangle = 0$ for $j = 0, \dots, N$.
- $\text{supp}\psi_{j;k} = 2^{-j}k + 2^{-j}\text{supp}\psi \approx 2^{-j}k$ when $j \to \infty$.
- $\langle f, \psi_{j;k}\rangle = \langle f - P, \psi_{j;k}\rangle \approx 0$ if $f \approx$ a polynomial P on $\text{supp}\psi_{j;k}$.
- If $\langle f, \psi_{j;\mathrm{k}}\rangle$ is large, then the singularity is around $2^{-j}k$.

Tight Framelets or Orthogonal Wavelets

Theorem: Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_s)^\mathsf{T}$ in $L_2(\mathbb{R})$. $\{\phi; \psi\}$ is a tight framelet (or orthogonal wavelet) in $L_2(\mathbb{R})$ $\Longleftrightarrow$

1. $\lim_{j\to\infty} \|\widehat{\phi}(2^{-j}\xi)\|_{l_2}^2 = 1$;
2. there exist $r \times r$ matrix $\widehat{a}$ and $s \times r$ matrix $\widehat{b}$ of 2π-periodic measurable functions in $L_\infty(\mathbb{T})$ such that

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \quad \text{i.e.,} \quad \phi = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2\cdot -k),$$

$$\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi), \quad \text{i.e.,} \quad \psi = 2\sum_{k\in\mathbb{Z}} b(k)\phi(2\cdot -k),$$

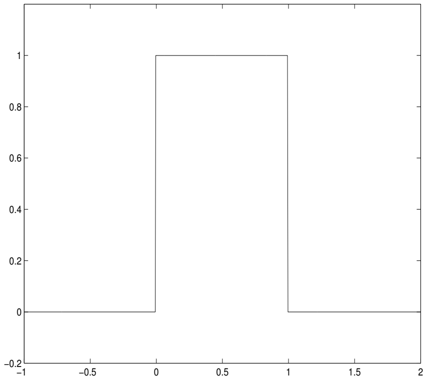
   and $\{\widehat{a}; \widehat{b}\}$ is a tight framelet filter bank:

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{a}(\xi+\pi) \\ \widehat{b}(\xi) & \widehat{b}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{a}(\xi)}^\mathsf{T} & \overline{\widehat{b}(\xi)}^\mathsf{T} \\ \overline{\widehat{a}(\xi+\pi)}^\mathsf{T} & \overline{\widehat{b}(\xi+\pi)}^\mathsf{T} \end{bmatrix} = I_{2r}, \qquad a.e.\, \xi \in \mathbb{R}.$$

3. $s = r$ and $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal system in $L_2(\mathbb{R})$,

where $\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x}dx$ and $\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}$.

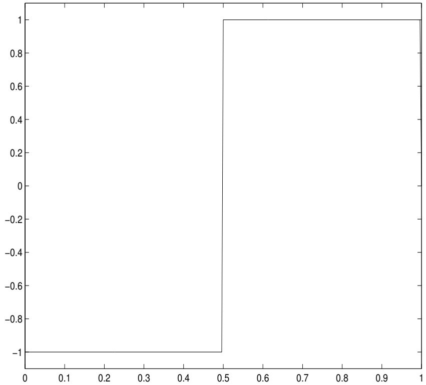
Example: Haar Orthogonal Wavelet $\{\phi;\psi\}$



Refinable function

$$\phi = \chi_{[0,1]}$$

$$\phi = \phi(2\cdot) + \phi(2\cdot -1)$$



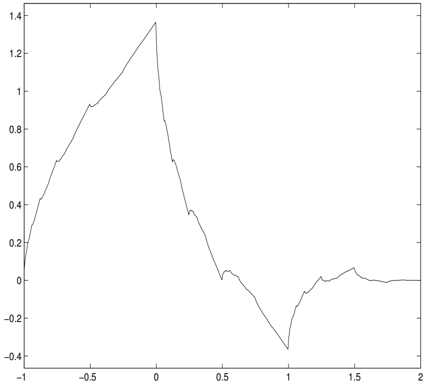
Wavelet

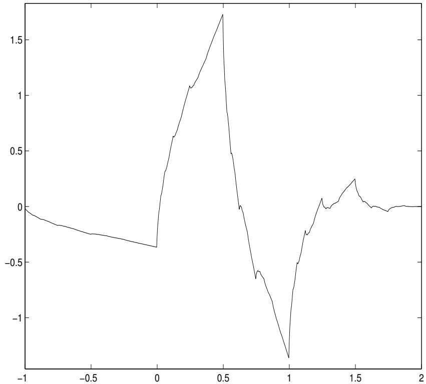
$$\psi := \chi_{[1/2,1]} - \chi_{[0,1/2]}.$$

$$\psi = \phi(2\cdot -1) - \phi(2\cdot).$$

ϕ and ψ have explicit expressions and ϕ is the B-spline of order 1.

Example: Daubechies Orthogonal Wavelet $\{\phi; \psi\}$





$$\phi = \frac{1+\sqrt{3}}{4}\phi(2\cdot) + \frac{3+\sqrt{3}}{4}\phi(2\cdot-1) + \frac{3-\sqrt{3}}{4}\phi(2\cdot-2) + \frac{1-\sqrt{3}}{4}\phi(2\cdot-3).$$

$$\psi = \frac{1-\sqrt{3}}{4}\phi(2\cdot) - \frac{3-\sqrt{3}}{4}\phi(2\cdot-1) + \frac{3+\sqrt{3}}{4}\phi(2\cdot-2) - \frac{1+\sqrt{3}}{4}\phi(2\cdot-3).$$

The functions ϕ and ψ do not have explicit expressions.

Tensor Product (Separable) Tight Framelet

- Let $\{a; b_1, \dots, b_s\}$ be a 1D tight framelet filter bank.
- If $s = 1$, $\{a; b_1\}$ is called an orthonormal wavelet filter bank.
- Tensor product filters:
  $[u_1 \otimes \cdots \otimes u_d](k_1, \dots, k_d) = u_1(k_1) \cdots u_d(k_d)$.
- Tensor product tight framelet filter bank:
  $\{a; b_1, \dots, b_s\} \otimes \cdots \otimes \{a; b_1, \dots, b_s\}$.
- Tensor product functions:
  $[f_1 \otimes \cdots \otimes f_d](x_1, \dots, x_d) = f_1(x_1) \cdots f_d(x_d)$.
- Tensor product tight framelet:
  $\{\phi; \psi^1, \dots, \psi^s\} \otimes \cdots \otimes \{\phi; \psi^1, \dots, \psi^s\}$.
- Advantages: fast and simple algorithm.

Tree Structure and Sparsity of Wavelet Coefficients

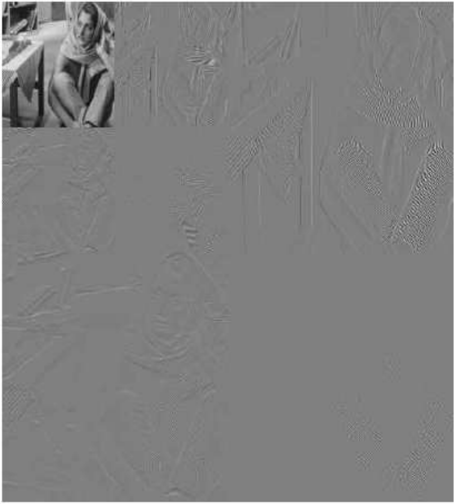


Image Compression Using Orthogonal Wavelets

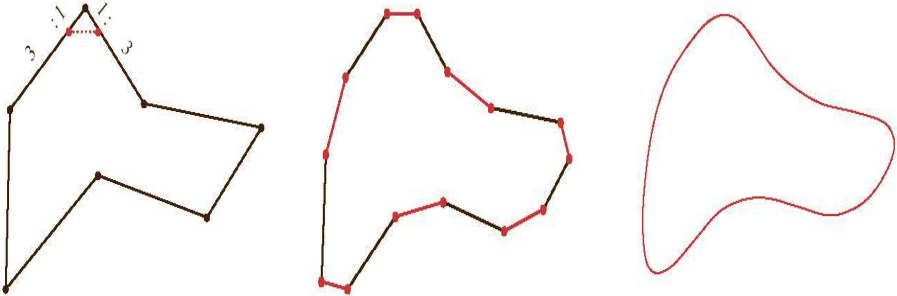
Original Lena image and reconstructed Lena images with compression ratios 32 and 128 using SPIHT.

Large coefficients are recorded with priority and tree structure is used.

Image Denoising Using Orthogonal Wavelets

Wavelet-shrinkage from statistics: small coefficients are set to 0.

Curve Modeling: Corner Cutting Subdivision Scheme

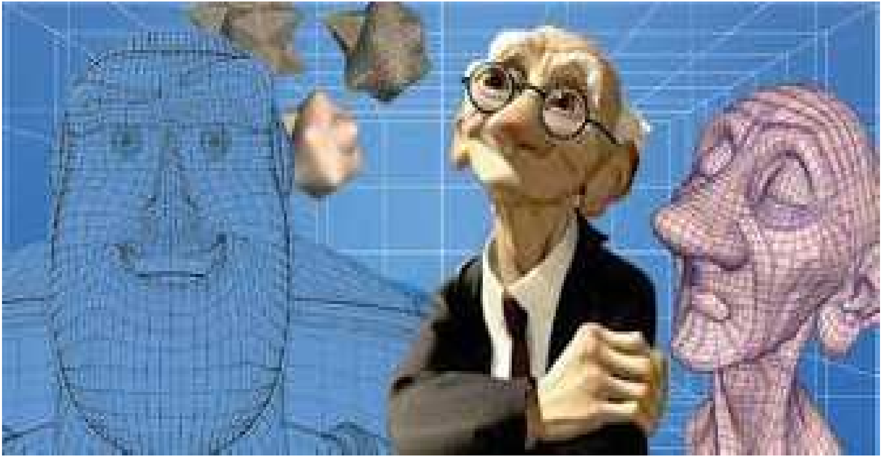


Initial control polygon v, iterated once $\mathcal{S}_a v$, iterated 5 times $\mathcal{S}_a^5 v$, where $a = \{\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\}_{[0,3]}$ is the B-spline filter of order 3.

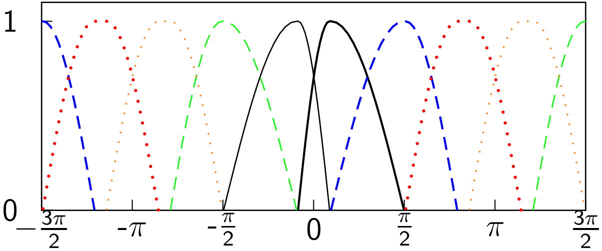
Surface Modeling by Subdivision scheme

Initial mesh v, iterated once $\mathcal{S}_{a,\mathrm{M}}v$, iterated twice $\mathcal{S}^2_{a,\mathrm{M}}v$.

Subdivision Surfaces Used in Animated Movies



Bandlimited Complex Tight Framelets TP-$\mathbb{C}\mathrm{TF}_6$

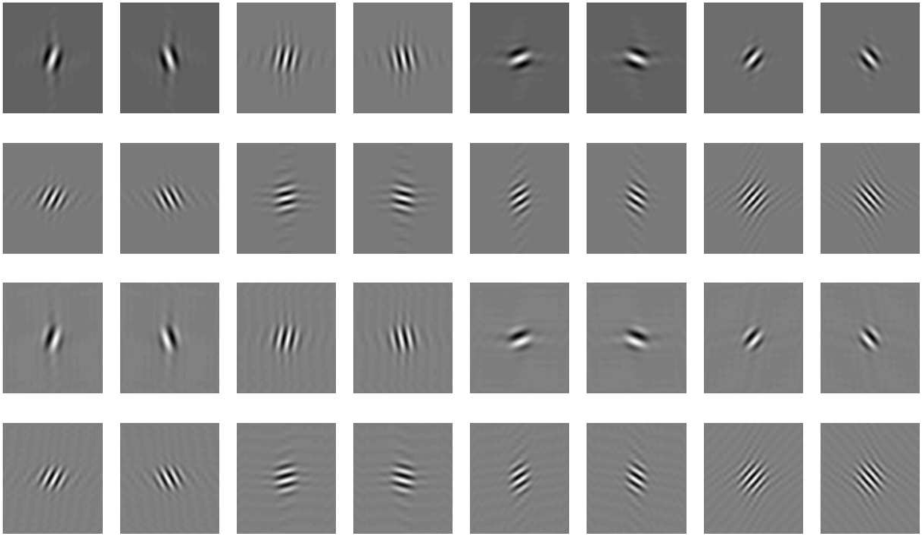


- Tight framelet filter bank $\mathbb{C}\mathrm{TF}_6 := \{a^+, a^-; b_1^+, b_2^+, b_1^-, b_2^-\}$: black lines for $\widehat{a^+}$ and $\widehat{a^-}$; dashed lines for $\widehat{b_1^+}$ and $\widehat{b_1^-}$; dotted lines for $\widehat{b_2^+}$ and $\widehat{b_2^-}$: $a^- := \overline{a^+}$, $b_1^- := \overline{b_1^+}$, $b_2^- := \overline{b_2^+}$, and

$$\widehat{a^+} := \chi_{[0,c];\varepsilon,\varepsilon}, \quad \widehat{b_1^+} := \chi_{[c_1,c_2];\varepsilon,\varepsilon}, \quad \widehat{b_1^+} := \chi_{[c_2,\pi];\varepsilon,\varepsilon}.$$

- The tensor product tight framelet TP-$\mathbb{C}\mathrm{TF}_6 := \otimes^d \mathbb{C}\mathrm{TF}_6$.
- Take advantages of wavelets and Discrete Cosine Transform.

Two-dimensional TP-$\mathbb{C}$TF$_6$ (14 directions)



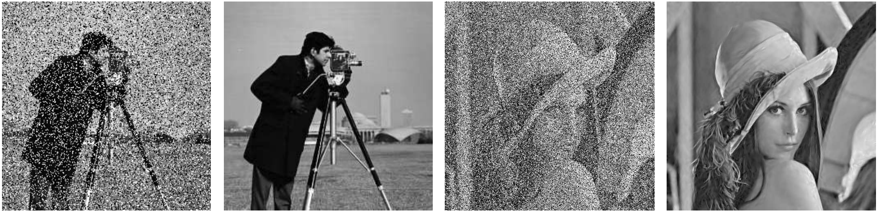
Denoising Comparison for Barbara Image

	DTCWT	TP-$\mathbb{C}$TF$_6$	UDWT	TV	Shearlet
Redundancy $\rightarrow$	4	10.7	13	N/A	49
$\sigma = 10$	33.52	**34.14**	32.64	31.57	33.69
$\sigma = 15$	31.38	**32.02**	30.30	28.99	31.61
$\sigma = 20$	29.87	**30.49**	28.70	27.28	30.10
$\sigma = 25$	28.70	**29.31**	27.50	26.06	28.93
$\sigma = 30$	27.77	**28.34**	26.56	25.17	27.97

DTCWT=Dual Tree Complex Wavelet Transform.
TP-$\mathbb{C}$TF$_6$=Han and Zhao, SIAM J. Imag. Sci. 7 (2014), 997–1034.
UDWT=Undecimated Discrete Wavelet Transform.
TV=Rudin-Osher-Fatemi (ROF) model using higher-order scheme.
Shearlet=shearlet frames in W. Lim, IEEE T. Image Process., 2013.

Measure of performance: $\text{PSNR} = 10 \log_{10} \frac{255^2}{MSE}$.
The larger PSNR value the better performance.

Remove Mixed Gaussian and Impulse Noises



Gaussian and Pepper–and–Salt impulse noise. Cameraman: $\sigma = 0$, $p = 0.3$, PSNR $= 32.50$. Lena: $\sigma = 15$, $p = 0.5$, PSNR $= 30.95$.

Gaussian and Random-valued impulse noises: Barbara: $\sigma = 30$, $p = 0.2$, PSNR $= 25.93$. Peppers: $\sigma = 20$, $p = 0.1$, PSNR $= 27.31$.

Remove Gaussian & Pepper-and-Salt Noise

	AOP	TP-$\mathbb{C}$TF$_6$	AOP	TP-$\mathbb{C}$TF$_6$	AOP	TP-$\mathbb{C}$TF$_6$
σ p	256×256 Cameraman		256×256 House		256×256 Peppers	
5 0.1	31.09	**32.97**(1.88)	36.35	**38.16**(1.81)	31.29	**32.11**(0.82)
5 0.3	29.02	**31.12**(2.10)	34.38	**36.23**(1.85)	28.79	**29.55**(0.76)
15 0.1	27.44	**29.24**(1.80)	29.32	**32.83**(3.51)	27.42	**28.85**(1.43)
15 0.3	26.45	**27.75**(1.31)	29.22	**31.90**(2.68)	26.46	**27.37**(0.91)
σ p	256×256 Cameraman		256×256 House		256×256 Peppers	
5 0.1	36.40	**37.65**(1.25)	29.39	**34.52**(5.13)	33.80	**35.53**(1.73)
5 0.3	34.74	**36.33**(1.59)	27.43	**33.68**(6.25)	31.66	**33.68**(2.02)
15 0.1	29.39	**33.12**(3.73)	26.14	**30.63**(4.49)	28.48	**30.73**(2.25)
15 0.3	29.16	**31.89**(2.74)	25.25	**29.48**(4.23)	27.96	**29.68**(1.72)

AOP, TV-based, SIAM J. Imaging, 5 (2013),1227–1245.

TP-$\mathbb{C}$TF$_6$, Shen/Han/Braverman, J. Math. Imaging Vis., 54 (2016), 64–77.

Image Inpainting Using TP-$\mathbb{C}$TF$_6$

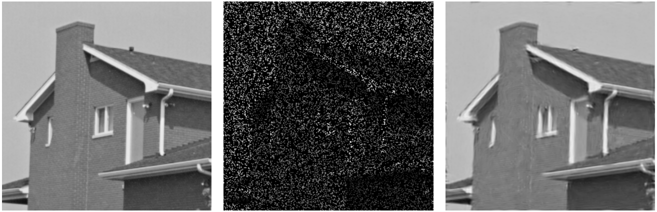


Figure: 80% missing pixels. Recovered by our algorithm: PSNR=31.67.

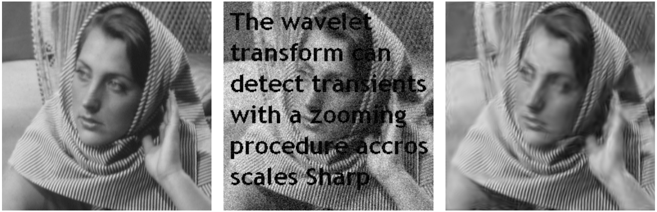
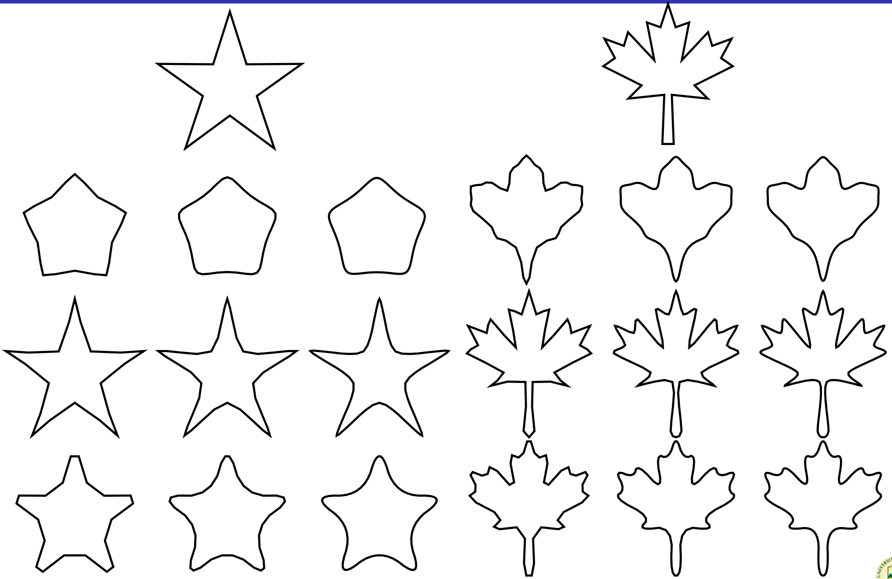
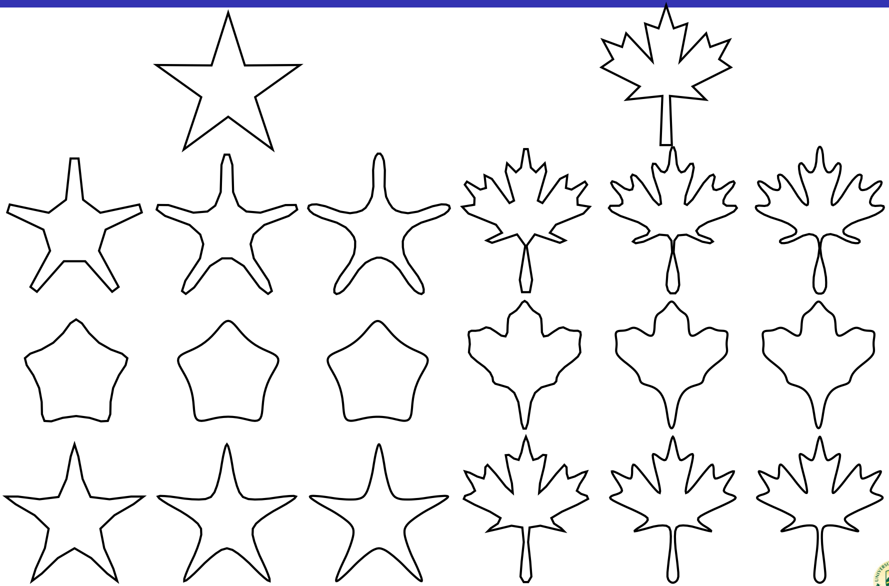


Figure: Corrupted by text with $\sigma = 20$. Recovered with PSNR= 28.93.

Examples of Subdivision Curve



Examples of Subdivision Curve



Subdivision Schemes

- A dilation matrix M is a $d \times d$ integer matrix such that all the eigenvalues of M are greater than one in modulus.
- Examples of dilation matrices: $2I_d$ (dyadic), $3I_d$ (ternary),

$$\mathrm{M}_{\sqrt{2}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathrm{N}_{\sqrt{2}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathrm{M}_{\sqrt{3}} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

- $\mathrm{M}_{\sqrt{2}}$ and $\mathrm{N}_{\sqrt{2}}$ are called the quincunx dilation matrices inducing the quincunx lattice

$$\mathrm{M}_{\sqrt{2}}\mathbb{Z}^2 = \mathrm{N}_{\sqrt{2}}\mathbb{Z}^2 = \{(j,k) \in \mathbb{Z}^2 \ : \ j+k \text{ is even}\}.$$

- The subdivision operator $\mathcal{S}_{a,\mathrm{M}} : l(\mathbb{Z}^d) \to l(\mathbb{Z}^d)$ is

$$[\mathcal{S}_{a,\mathrm{M}} v](n) := |\det(\mathrm{M})| \sum_{k\in\mathbb{Z}^d} v(k)a(n-\mathrm{M}k),$$

where $v = \{v(k)\}_{k\in\mathbb{Z}^d} \in l(\mathbb{Z}^d)$.

Subdivision Triplets: Symmetry is Necessary

- A symmetry group G is a finite set of $d \times d$ integer matrices with determinants ± 1 forming a group under matrix multiplication.
- A mask/filter $a = \{a(k)\}_{k \in \mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{R}$ is G-symmetric with symmetry center c_a if

$$a(E(k - c_a) + c_a) = a(k), \qquad \forall\, k \in \mathbb{Z}^d, E \in G.$$

- A dilation matrix M is compatible with G if

$$\mathrm{M} E \mathrm{M}^{-1} \in G, \qquad \forall\, E \in G.$$

- (a, M, G) is called a subdivision triplet if M is compatible with G and the mask a is G-symmetric.

Subdivision Schemes Using Triplet (a, M, G)

- Subdivision scheme: calculate $v_n := \mathcal{S}^n_{a,\mathrm{M}} v$ for $n \in \mathbb{N}$ and attach the value $v_n(k)$ at the point $\mathrm{M}^{-n}(k - c_a)$, $k \in \mathbb{Z}^d$.
- The subdivision scheme converges if $\{v_n\}_{n=1}^{\infty}$ converges to a continuous function v_∞ for every bounded initial control mesh v.
- If the symmetry center $c_a = 0$, it is called a primal subdivision scheme; otherwise, it is called a dual subdivision scheme.
- Proposition: For a subdivision triplet (a, M, G) with symmetry center c_a, if $\widehat{a}(0) = 1$ with $\widehat{a}(\xi) := \sum_{k \in \mathbb{Z}^d} a(k) e^{-ik \cdot \xi}$, then

$$\phi(E(\cdot - c_\phi) + c_\phi) = \phi \quad \forall\, E \in G \text{ with } c_\phi := (\mathrm{M} - I_d)^{-1} c_a,$$

where ϕ is the M-refinable (or basis) function associated with the mask/filter a defined by $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}((\mathrm{M}^{\mathsf{T}})^{-j}\xi), \xi \in \mathbb{R}^d$.

Important Dilation Matrices

- Two important symmetry groups:

$$D_4 := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\},$$

$$D_6 := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right\}.$$

- D_4 for the quadrilateral mesh and D_6 for the triangular mesh.
- N is G-equivalent to M if $N = EMF$ for some $E, F \in G$.
- $\mathsf{N}_{\sqrt{2}}$ is D_4-equivalent to $\mathsf{M}_{\sqrt{2}}$.
- Theorem: For a 2×2 real-valued matrix M,
  1. if M is compatible with the symmetry group D_4, then M must be D_4-equivalent to either cI_2 or $c\mathsf{M}_{\sqrt{2}}$ for some $c \in \mathbb{R}$.
  2. if M is compatible with the symmetry group D_6, then M must be D_6-equivalent to either cI_2 or $c\mathsf{M}_{\sqrt{3}}$ for some $c \in \mathbb{R}$.

Quad and Triangular Meshes

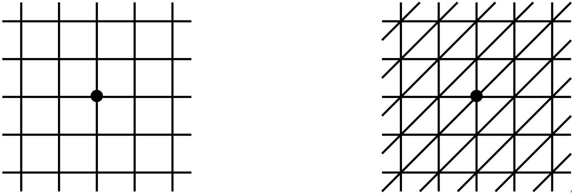


Figure: The quadrilateral mesh $\mathbb{Z}^2_Q$ (left) and the triangular mesh $\mathbb{Z}^2_T$ (right).

Definition of Linear-phase Moments

Interpolation: $[\mathcal{S}_{a,\mathrm{M}}v](\mathrm{M}k) = v(k)$ for all $k \in \mathbb{Z}$ and $v \in l(\mathbb{Z}^d)$ $\Longleftrightarrow$

$$a(0) = |\det(\mathrm{M})|^{-1}, \quad a(\mathrm{M}k) = 0, \quad \forall\, k \in \mathbb{Z}^d \backslash \{0\}.$$

Interpolation on Polynomials: $[\mathcal{S}_{a,\mathrm{M}}\mathrm{p}](\mathrm{M}k) = \mathrm{p}(k - \mathrm{M}^{-1}c)$ for all $k \in \mathbb{Z}$ and all polynomials p with $\deg(\mathrm{p}) < m$ $\Longleftrightarrow$

- a has linear-phase moments with phase c:

$$\widehat{a}(\xi) = e^{-ic\cdot\xi} + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0;$$

  Define $\mathrm{lpm}(a) = m$ with the highest possible m.

- a has order m sum rules:

$$\widehat{a}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^m), \quad \xi \to 0, \omega \in \Omega_{\mathrm{M}} \backslash \{0\},$$

  where $\Omega_{\mathrm{M}} := [0,1)^d \cap [(\mathrm{M}^{\mathsf{T}})^{-1}\mathbb{Z}^d]$. Define $\mathrm{sr}(a, \mathrm{M}) = m$ with the highest possible m.

Note: If a has symmetry with symmetry center c_a, then $c = c_a$.

Importance of Linear-phase Moments

- $\{a; b_1, \dots, b_s\}$ is called a tight M-framelet filter bank if

$$|\widehat{a}(\xi)|^2 + |\widehat{b_1}(\xi)|^2 + \cdots + |\widehat{b_s}(\xi)|^2 = 1,$$

$$\overline{\widehat{a}(\xi)}\widehat{a}(\xi + 2\pi\omega) + \sum_{\ell=1}^{s} \overline{\widehat{b_\ell}(\xi)}\widehat{b_\ell}(\xi + 2\pi\omega) = 0, \quad \omega \in \Omega_{\mathsf{M}} \backslash \{0\}.$$

- Called an orthogonal M-wavelet filter bank if $s = |\det(\mathsf{M})| - 1$.
- If $|\det(\mathsf{M})| = 2$, then $s = 1$, $\Omega_{\mathsf{M}} = \{0, \omega\}$, and $\{a; b\}$ is an orthogonal M-wavelet filter bank $\Longleftrightarrow$ for some $\gamma \in \mathbb{Z}^d \backslash [\mathsf{M}\mathbb{Z}^d]$,

$$|\widehat{a}(\xi)|^2 + |\widehat{a}(\xi + 2\pi\omega)|^2 = 1, \quad \widehat{b}(\xi) = e^{-i\gamma\cdot\xi}\overline{\widehat{a}(\xi + 2\pi\omega)}.$$

- A filter b has n vanishing moments if $\widehat{b}(\xi) = \mathscr{O}(\|\xi\|^n)$ as $\xi \to 0$. We define $\operatorname{vm}(b) := n$ with the highest n.
- Theorem: If $\{a; b_1, \dots, b_s\}$ is a tight M-framelet filter bank and a has symmetry with symmetry center c_a, then

$$\min(\operatorname{vm}(b_1), \dots, \operatorname{vm}(b_s)) = \min(\operatorname{sr}(a), \tfrac{1}{2}\operatorname{lpm}(a)).$$

Tight Framelets and Wavelets

- A function ψ has n vanishing moments if $\widehat{\psi}(\xi) = \mathscr{O}(\|\xi\|^n)$ as $\xi \to 0$. We define $\mathrm{vm}(\psi) := n$ with the largest n.
- Theorem: If $\{a; b_1, \ldots, b_s\}$ is a tight M-framelet filter bank with $\widehat{a}(0) = 1$, let $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}((\mathsf{M}^\mathsf{T})^{-j}\xi)$, $\widehat{\psi^\ell}(\mathsf{M}^\mathsf{T}\xi) := \widehat{b_\ell}(\xi)\widehat{\phi}(\xi)$. Then $\{\phi; \psi^1, \ldots, \psi^s\}$ is a tight framelet in $L_2(\mathbb{R}^d)$: $f \in L_2(\mathbb{R}^d)$,

$$\|f\|^2_{L_2(\mathbb{R}^d)} = \sum_{k \in \mathbb{Z}^d} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty} \sum_{\ell=1}^{s} \sum_{k \in \mathbb{Z}^d} |\langle f, |\det(\mathsf{M})|^{j/2}\psi^\ell(\mathsf{M}^j \cdot - k)\rangle|^2.$$

- $\mathrm{vm}(\psi^\ell) = \mathrm{vm}(b_\ell)$ for all $\ell = 1, \ldots, s$.
- It is a challenging problem to construct multivariate wavelets or tight framelets with symmetry and high vanishing moments.

Fourier Transform

- For a function f on $\mathbb{R}^d$, its Fourier transform is defined to be

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx, \quad \xi \in \mathbb{R}^d.$$

- For a sequence $a : \mathbb{Z}^d \to \mathbb{C}$, its Fourier series is

$$\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}^d} a(k) e^{-ik\cdot\xi}, \quad \xi \in \mathbb{R}^d.$$

Cascade Algorithms

- How to solve the refinement equation:

$$\phi = |\det(\mathrm{M})| \sum_{k \in \mathbb{Z}^d} a(k)\phi(\mathrm{M} \cdot -k),$$

where the mask $a : \mathbb{Z}^d \to \mathbb{R}$ is finitely supported, equivalently,

$$\widehat{\phi}(\xi) = \widehat{a}((\mathrm{M}^\mathsf{T})^{-1}\xi)\widehat{\phi}((\mathrm{M}^\mathsf{T})^{-1}\xi).$$

- Cascade algorithm: The cascade operator $\mathcal{R}$ is defined to be

$$\mathcal{R}_{a,\mathrm{M}} f := |\det(\mathrm{M})| \sum_{k \in \mathbb{Z}^d} a(k)\phi(\mathrm{M} \cdot -k).$$

- ϕ is a fixed point of $\mathcal{R}_{a,\mathrm{M}}$ by $\phi = \mathcal{R}_{a,\mathrm{M}}\phi$.
- $\{f_n := \mathcal{R}^n_{a,\mathrm{M}} f\}_{n \in \mathbb{N}}$ of functions is called a cascade algorithm.
- The cascade algorithm converges if for every compactly supported eligible initial function f, there exists a continuous function f_∞ such that $\lim_{n\to\infty} \|f_n - f_\infty\|_{C(\mathbb{R}^d)} = 0$.

Cascade Algorithm and Subdivision Schemes

- Cascade algorithm: the iterative sequence $\{f_n := \mathcal{R}^n_{a,\mathsf{M}} f\}_{n\in\mathbb{N}}$ of functions.
- Subdivision scheme: calculate $v_n := \mathcal{S}^n_{a,\mathsf{M}} v$ for $n \in \mathbb{N}$ and attach the value $v_n(k)$ at the point $\mathsf{M}^{-n}(k - c_a)$, $k \in \mathbb{Z}^d$.
- Relation:

$$f_n = \mathcal{R}^n_{a,\mathsf{M}} f = \sum_{k\in\mathbb{Z}^d} [\mathcal{S}^n_{a,\mathsf{M}}\boldsymbol{\delta}](k) f(\mathsf{M}^n \cdot - k),$$

  where $\boldsymbol{\delta}$ is the Dirac sequence such that $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for all $k \neq 0$.
- Let h be the hat function (in 1d, $h = \max(1 - |x|, 0)$). Then connecting points of v_n be flat pieces to form a function g_n is equivalent to (assume $c_a = 0$)

$$g_n = \mathcal{R}^n_{a,\mathsf{M}} f \quad \text{with} \quad f := \sum_{k\in\mathbb{Z}^d} v(k) h(\cdot - k).$$

Role of a Dilation Matrix

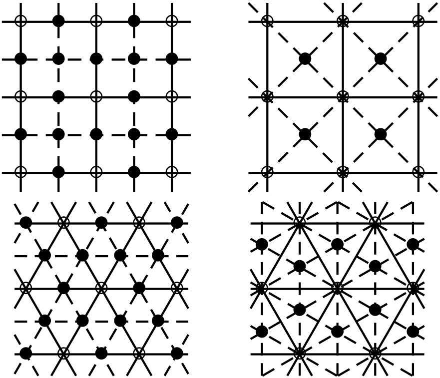


Figure: $\circ$ represents vertices in the coarse mesh $\mathbb{Z}^2$ and $\bullet$ represents new vertices in the refinement mesh $\mathsf{M}^{-1}\mathbb{Z}^2$. The M-refinement of the reference mesh $\mathbb{Z}^2$, from left to right, are for subdivision triplets $(a, 2I_2, D_4)$, $(a, \mathsf{M}_{\sqrt{2}}, D_4)$, $(a, 2I_2, D_6)$, and $(a, \mathsf{M}_{\sqrt{3}}, D_6)$, where $\mathsf{M}_{\sqrt{2}}$ and $\mathsf{M}_{\sqrt{3}}$.

Implemented by Convolution

- Subdivision scheme: calculate $v_n := \mathcal{S}^n_{a,\mathsf{M}} v$ for $n \in \mathbb{N}$ and attach the value $v_n(k)$ at the point $\mathsf{M}^{-n}(k - c_a)$, $k \in \mathbb{Z}^d$.
- For $\beta, \gamma \in \mathbb{Z}^d$,

$$
\begin{aligned}
[\mathcal{S}_{a,\mathsf{M}} v](\gamma + \mathsf{M}\beta) &= |\det(\mathsf{M})| \sum_{k \in \mathbb{Z}^d} v(k) a(\gamma + \mathsf{M}\beta - Mk) \\
&= |\det(\mathsf{M})| [v * a^{[\gamma:\mathsf{M}]}](\beta),
\end{aligned}
$$

- where the *coset mask* $a^{[\gamma:\mathsf{M}]}$ of the mask a is defined to be

$$
a^{[\gamma:\mathsf{M}]}(k) := a(\gamma + \mathsf{M}k), \qquad k, \gamma \in \mathbb{Z}^d.
$$

- Local averaging: $|\det(\mathsf{M})| \sum_{k \in \mathbb{Z}^d} a^{[\gamma:\mathsf{M}]}(k) = 1$ for all $\gamma \in \mathbb{Z}^d$.
- The value $[\mathcal{S}_{a,\mathsf{M}} v](\gamma + \mathsf{M}\beta) = \langle v(\beta + \cdot), |\det(\mathsf{M})| \overline{a^{[\gamma:\mathsf{M}]}(-\cdot)} \rangle$, is put at $\beta + \mathsf{M}^{-1}\gamma - \mathsf{M}^{-1}c_a$.
- $\mathsf{M}^{-1}\gamma$-stencil of the mask a: $\{|\det(\mathsf{M})| \overline{a(\gamma - \mathsf{M}k)}\}_{k \in \mathbb{Z}^d}$.

1D Subdivision Triplets

- For a finitely supported sequence $a : \mathbb{Z} \to \mathbb{R}$, we define

$$\mathrm{a}(z) := \sum_{k \in \mathbb{Z}} a(k) z^k, \quad z \in \mathbb{C} \backslash \{0\}.$$

- Let M be an integer greater than one.
- Subdivision operator: $[\mathcal{S}_{a,\mathrm{M}} \mathrm{v}](z) = \mathrm{M}\mathrm{v}(z^2)\mathrm{a}(z)$.
- a has order n sum rules if and only if

$$\mathrm{a}(z) = (1 + z + \cdots + z^{\mathrm{M}-1})^n \mathrm{b}(z)$$

  for some Laurent polynomial b.
- a has order n linear-phase moments if and only if

$$a(z) = z^c + \mathscr{O}(|z-1|^n), \qquad z \to 1.$$

- a is interpolatory with respect to M if

$$a(0) = \frac{1}{\mathrm{M}}, \quad a(\mathrm{M}k) = 0, \quad \forall\, k \in \mathbb{Z} \backslash \{0\}.$$

1D Subdivision Triplet

The triplet $(a, 2, \{-1, 1\})$ is a primal subdivision triplet with

$$a = \frac{1}{2}\{w_3, w_2, w_1, \underline{w_0}, w_1, w_2, w_3\}_{[-3,3]},$$

where

$$w_0 = \tfrac{3+t}{4}, \quad w_1 = \tfrac{8+t}{16}, \quad w_2 = \tfrac{1-t}{8}, \quad w_3 = -\tfrac{t}{16} \quad \text{with} \quad t \in \mathbb{R}.$$

If $t = -\frac{1}{2}$, then $a = a_6^B(\cdot - 3)$ and $\mathrm{sr}(a, 2) = 6$, $\mathrm{lpm}(a) = 2$ and $\mathrm{sm}_p(a, 2) = 5 + 1/p$ for all $1 \leqslant p \leqslant \infty$. If $t \neq -1/2$, then $\mathrm{sr}(a, 2) = 4$. $\mathrm{sm}_\infty(a, 2) = 3 - \log_2(1 + t)$ provided $t > -1/2$. We only have $\mathrm{sm}_\infty(a, 2) \geqslant 3 - \log_2 |t|$ for $t \leqslant -1/2$. When $t = 0$, $a = a_4^B(\cdot - 2)$ is the centered B-spline filter of order 4 with $\mathrm{sr}(a, 2) = 4$ and $\mathrm{lpm}(a) = 2$. When $t = 1$, a is an interpolatory 2-wavelet filter with $\mathrm{sr}(a, 2) = 4$ and $\mathrm{lpm}(a) = 4$.

Subdivision Stencils

Figure: The 0-stencil (left) and the $\frac{1}{2}$-stencil (right) of the primal subdivision scheme. It is an interpolatory 2-wavelet filter if $w_2 = \frac{1-t}{8} = 0$ (i.e. $t = 1$). Since $\mathsf{M} = 2$, each line segment (with endpoints $\circ$) in the coarse mesh $\mathbb{Z}$ is equally split into two line segments with one new vertex ($\bullet$) in the middle.

1D Subdivision Triplet

The triplet $(a, 2, \{-1, 1\})$ is a dual subdivision triplet with

$$a = \frac{1}{2}\{w_2, w_1, \underline{w_0}, w_0, w_1, w_2\}_{[-2,3]},$$

where

$$w_0 = \tfrac{12+3t}{16}, \quad w_1 = \tfrac{8-3t}{32}, \quad w_2 = -\tfrac{3t}{32} \qquad \text{with} \quad t \in \mathbb{R}.$$

If $t = -\frac{2}{3}$, then $a = a_5^B(\cdot - 2)$ and $\mathrm{sr}(a, 2) = 5, \mathrm{lpm}(a) = 2$ and $\mathrm{sm}_p(a, 2) = 4 + 1/p$ for all $1 \leqslant p \leqslant \infty$. $\mathrm{sr}(a, 2) = 3$ and $\mathrm{sm}_\infty(a, 2) = 4 - \log_2(4 + 3t)$ provided $t > -2/3$. We only have $\mathrm{sm}_\infty(a, 2) \geqslant 1 - \log_2(3|t|)$ for $t \leqslant -2/3$. When $t = 0$, $a = a_3^B(\cdot - 1)$ is the shifted B-spline filter of order 3 with $\mathrm{sr}(a, 2) = 3$ and $\mathrm{lpm}(a) = 2$. When $t = 1$, $\mathrm{sr}(a, 2) = 3$ and $\mathrm{lpm}(a) = 4$.

Subdivision Stencils

Figure: The 0-stencil (left) and the $\frac{1}{2}$-stencil (right) of the dual subdivision scheme. The $\frac{1}{2}$-stencil is the same as the 0-stencil. The value $[\mathcal{S}_{a,2}v](k)$ for $k \in \mathbb{Z}$ is attached to the center $\frac{k-1}{2}$ of the line segment $[k-1, k]$ instead of the vertex $\frac{k}{2}$. Since $\mathsf{M} = 2$, each line segment is equally split into two.

1D Subdivision Triplet

The triplet $(a, 3, \{-1, 1\})$ is a primal subdivision triplet with

$$a = \frac{1}{3}\{w_5, w_4, w_3, w_2, w_1, \underline{w_0}, w_1, w_2, w_3, w_4, w_5\}_{[-5,5]},$$

where

$$\begin{array}{lll} w_0 = \frac{7-2t_1-8t_2}{9}, & w_1 = \frac{6-2t_1-5t_2}{9}, & w_2 = \frac{3+t_1+t_2}{9}, \\ w_3 = \frac{1+t_1+4t_2}{9}, & w_4 = \frac{t_1+3t_2}{9}, & w_5 = \frac{t_2}{9} \end{array} \quad \text{with} \quad t_1, t_2 \in \mathbb{R}.$$

If $t_1 = 2/9$ and $t_2 = 1/9$, then $\text{sr}(a, 3) = 5$ and $\text{sm}_p(a, 3) = 4 + 1/p$ for all $1 \leqslant p \leqslant \infty$ whose 3-refinable function is the B-spline of order 5.

$$\text{sm}_\infty(a, 2) \geqslant 2 - \log_3 \max(|1 - 2t_1 - 2t_2|, |2t_1|, |2t_2|).$$

If $t_1 = 7/9$ and $t_2 = -4/9$, then a is an interpolatory 3-wavelet filter with $\text{sr}(a, 3) = 4 = \text{lpm}(a)$ and $\text{sm}_\infty(a, 3) \geqslant \log_3 14 - 4 \approx 1.5978$. If $t_1 = 5/11$ and $t_2 = -4/11$, then a is an interpolatory 3-wavelet filter with $\text{sr}(a, 3) = 3 = \text{lpm}(a)$ and $\text{sm}_\infty(a, 3) \geqslant 2 + \log_3(11/10) > 2$.

1D Subdivision Triplet

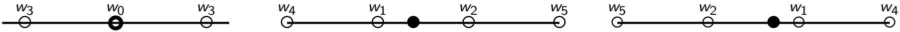
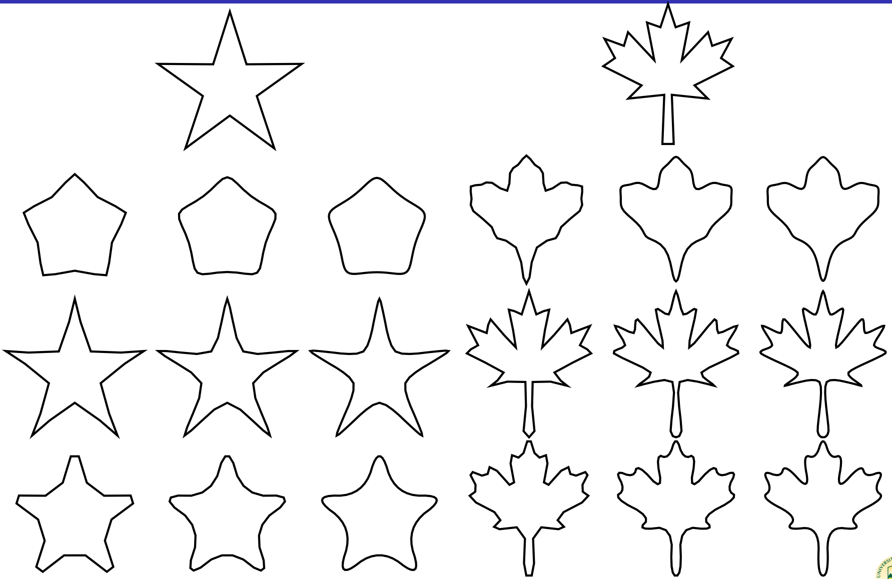
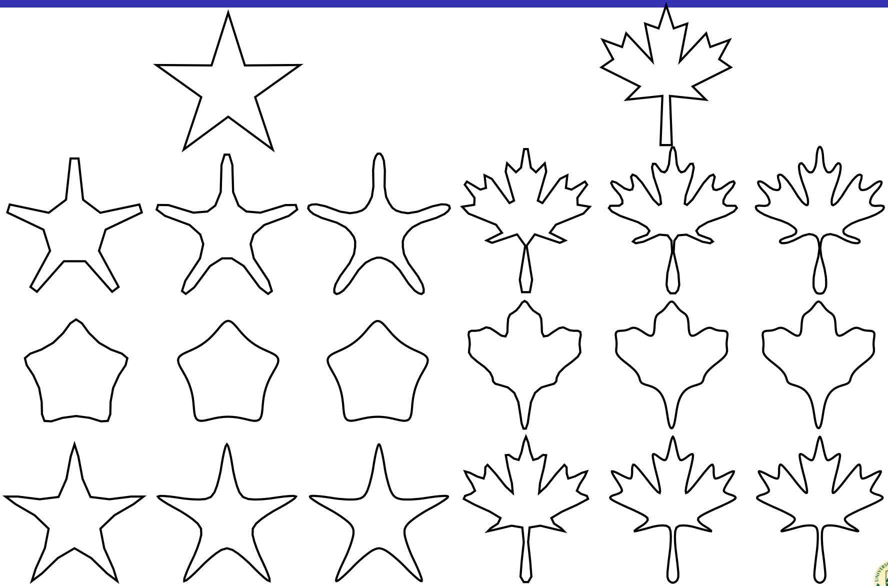


Figure: The 0-stencil (left), the $\frac{1}{3}$-stencil (middle), and $\frac{2}{3}$-stencil of the subdivision scheme. Due to symmetry, $\frac{2}{3}$-stencil is the same as the $\frac{1}{3}$-stencil. It is an interpolatory 3-wavelet filter if $w_3 = \frac{1+t_1+4t_2}{9} = 0$. Since $\mathsf{M} = 3$, each line segment (with endpoints $\circ$) is equally split into three line segments with two new inserted vertices ($\bullet$) at $\frac{1}{3} + \mathbb{Z}$ and $\frac{2}{3} + \mathbb{Z}$.

Examples of Subdivision Curve



Examples of Subdivision Curve



Masks Used

- Subdivision curves at levels $1, 2, 3$ with the initial control polygons at the first row.
- (1) uses the subdivision triplet $(a, 2, \{-1, 1\})$ with $a = a_4^B(\cdot - 2)$
- (2) uses interpolatory subdivision triplet $(a, 2, \{-1, 1\})$.
- (3) uses $(a, 2, \{-1, 1\})$ with $a = a_3^B(\cdot - 1)$.
- (4) the corner cutting scheme
- (5) uses $(a, 3, \{-1, 1\})$.
- (6) uses interpolatory $(a, 3, \{-1, 1\})$.