What is a wavelet and why is it useful?

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Present at 2017 International Undergraduate

Summer Enrichment Program at UofA

July 17, 2017

How to represent data economically?

- In today's world, most data and signals are in digital format: digital TV, movies, images, songs,...
- How to represent data effectively (as few numbers as possible)?
- How to detect the sharp changes in data?

Record information effectively

Given a particular signal to you:
$[-21, -22, -23, -23, -25, 38, 36, 34]$.
If you are allowed to send out only one number about this signal,

which number shall you choose?

Your answer(s):

Record information effectively

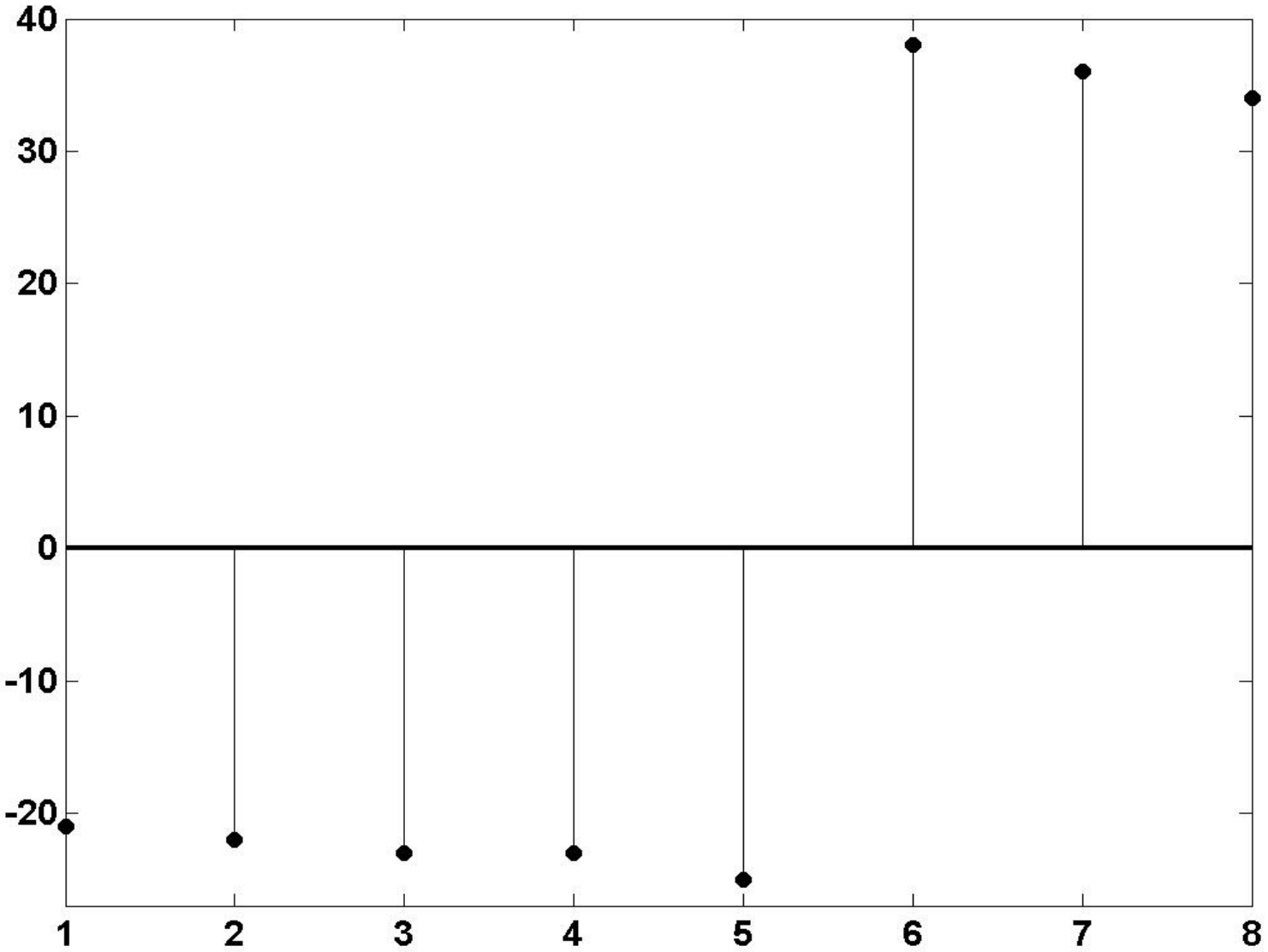
Given a particular signal to you:
$[-21, -22, -23, -23, -25, 38, 36, 34]$.
If you are allowed to send out only one number about this signal,

which number shall you choose?

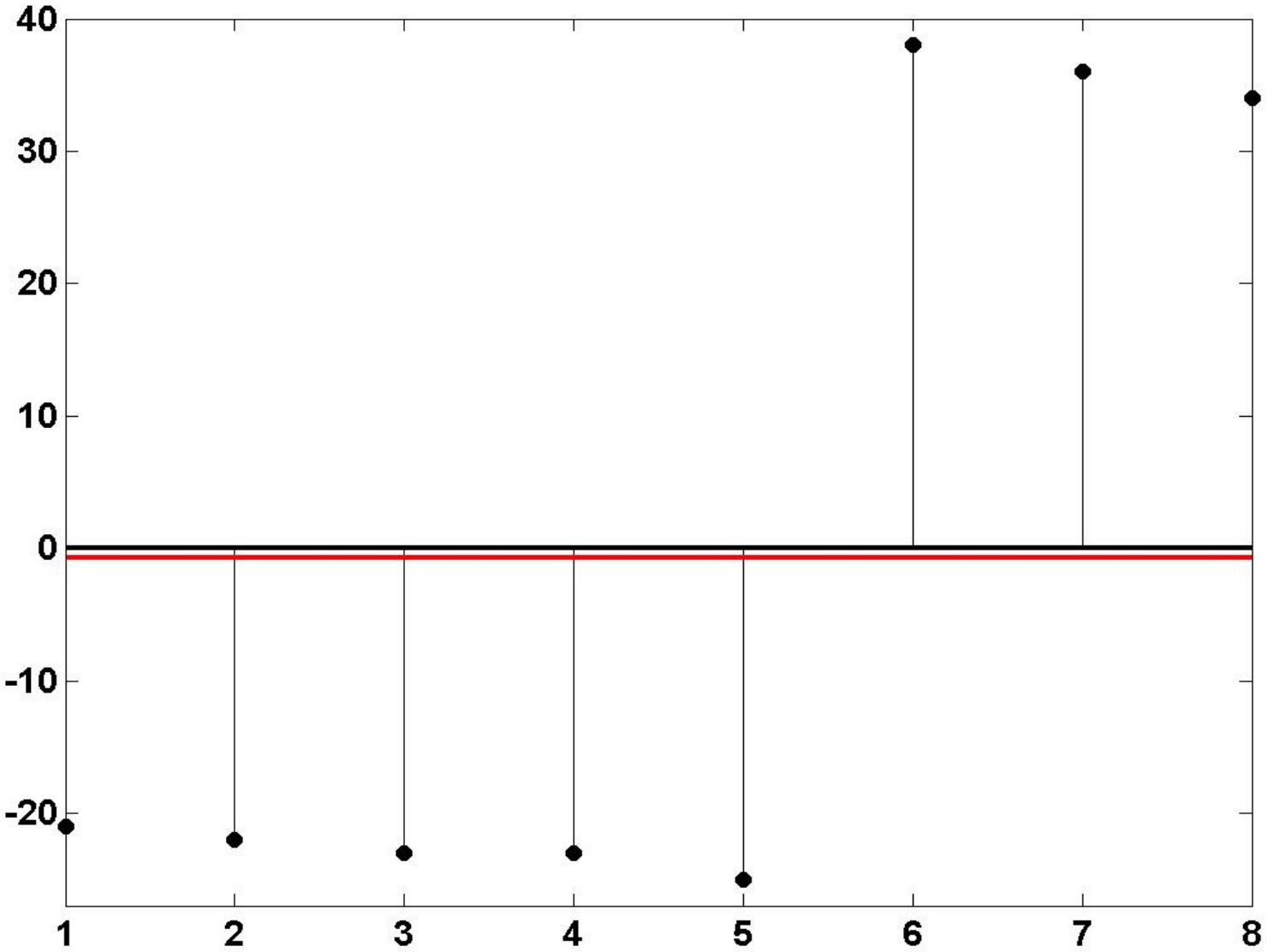
Your answer(s): Average

$$\frac{-21 - 22 - 23 - 23 - 25 + 38 + 36 + 34}{8} = -0.75.$$

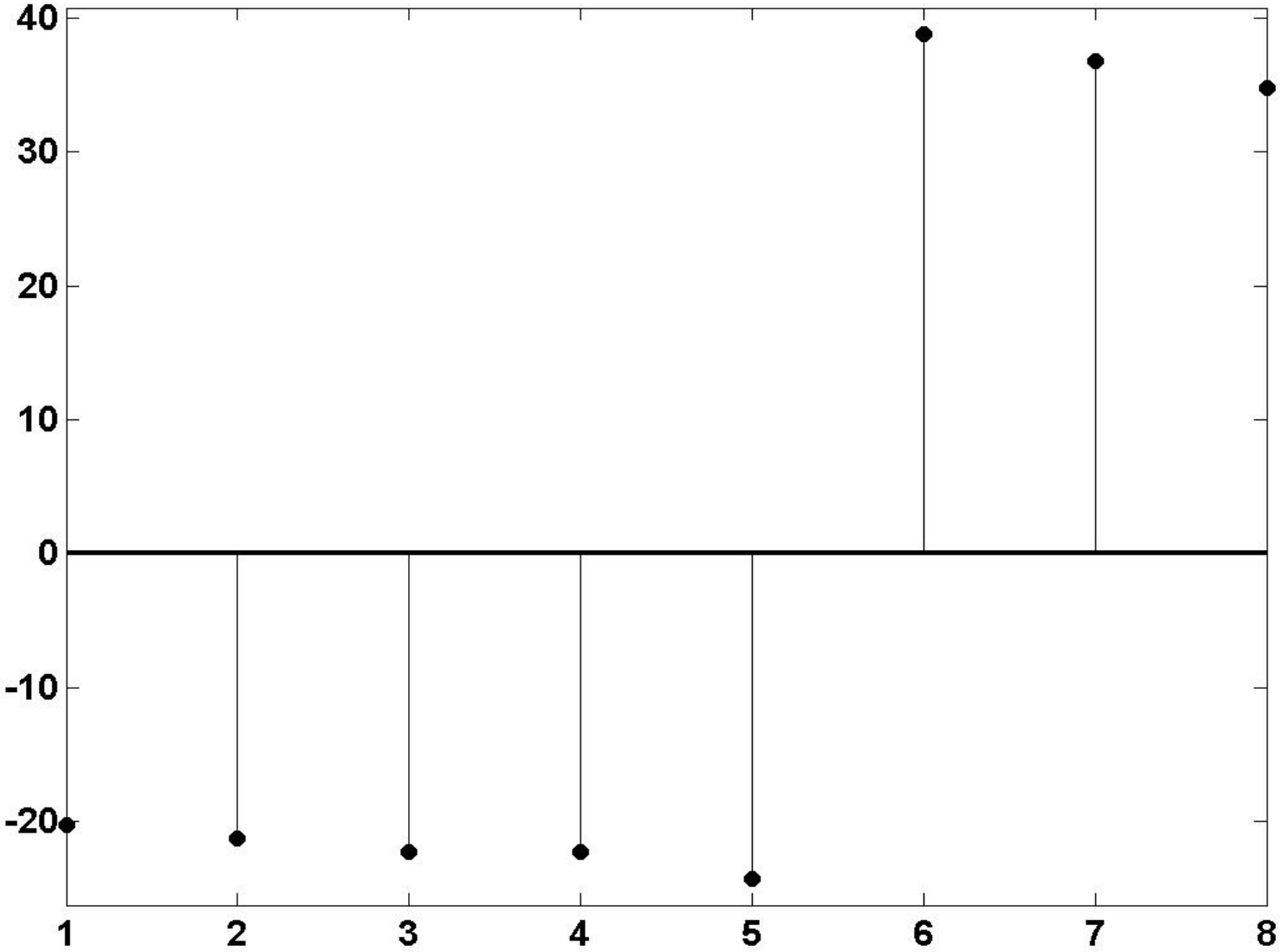
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



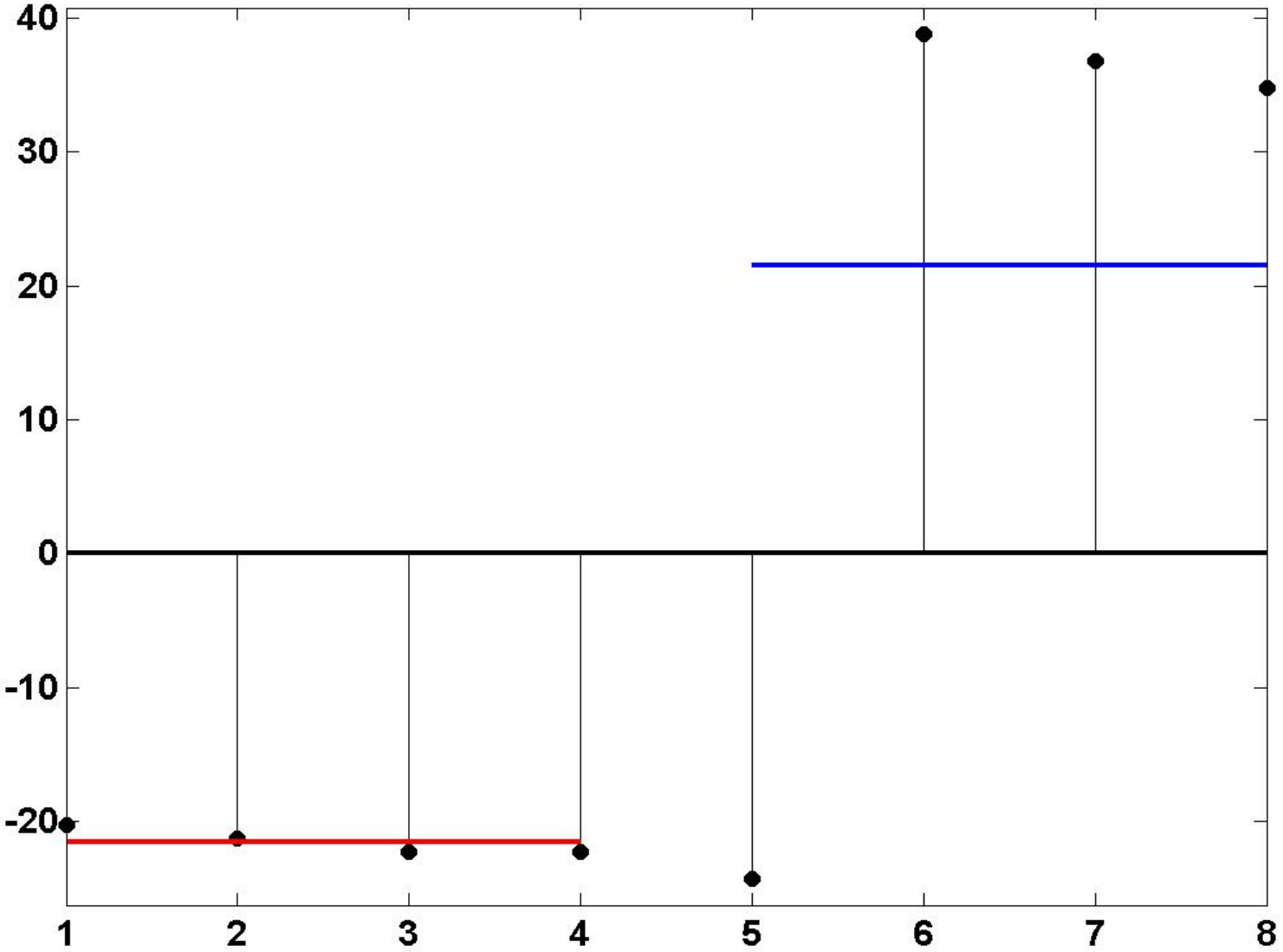
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



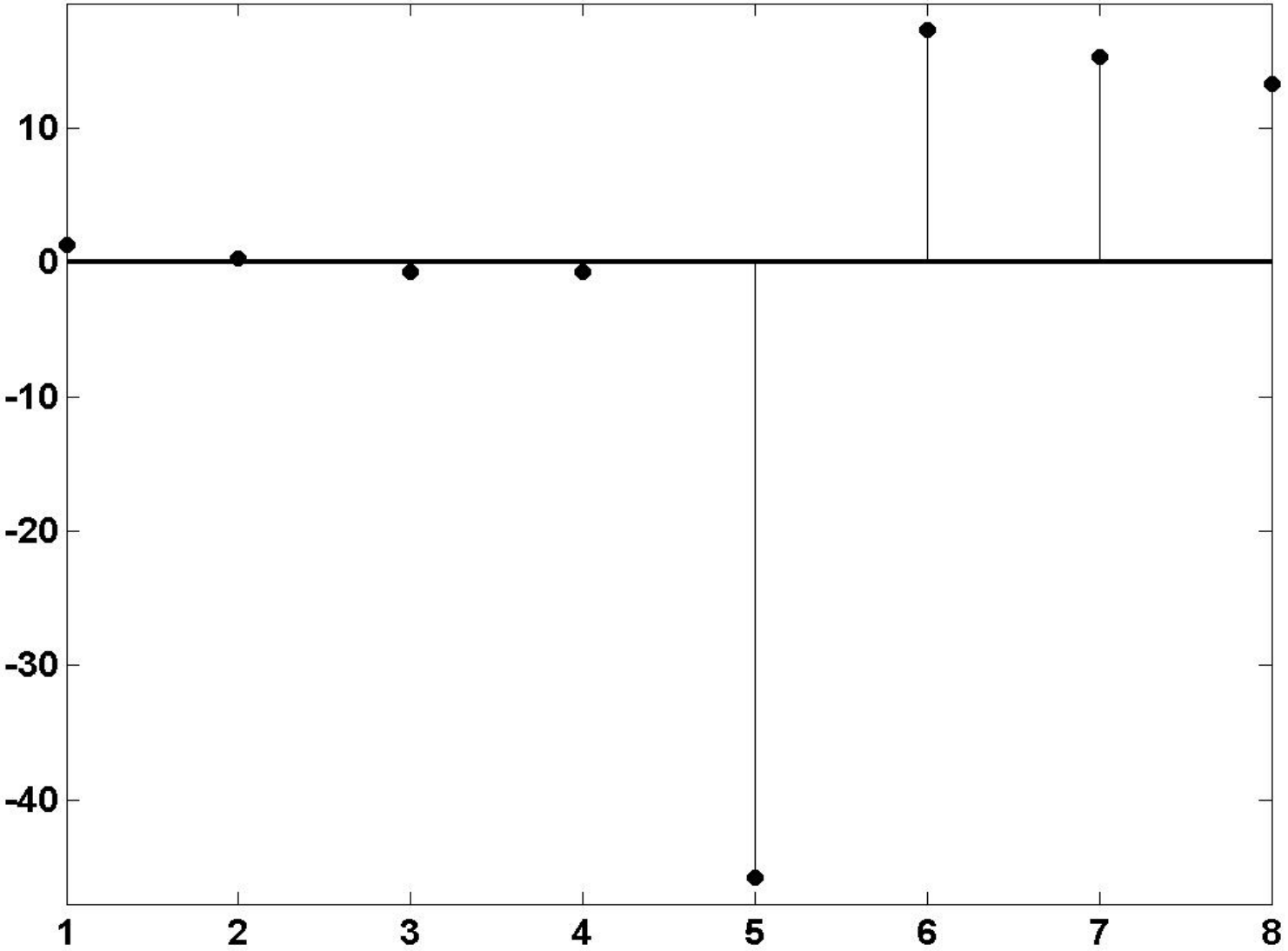
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



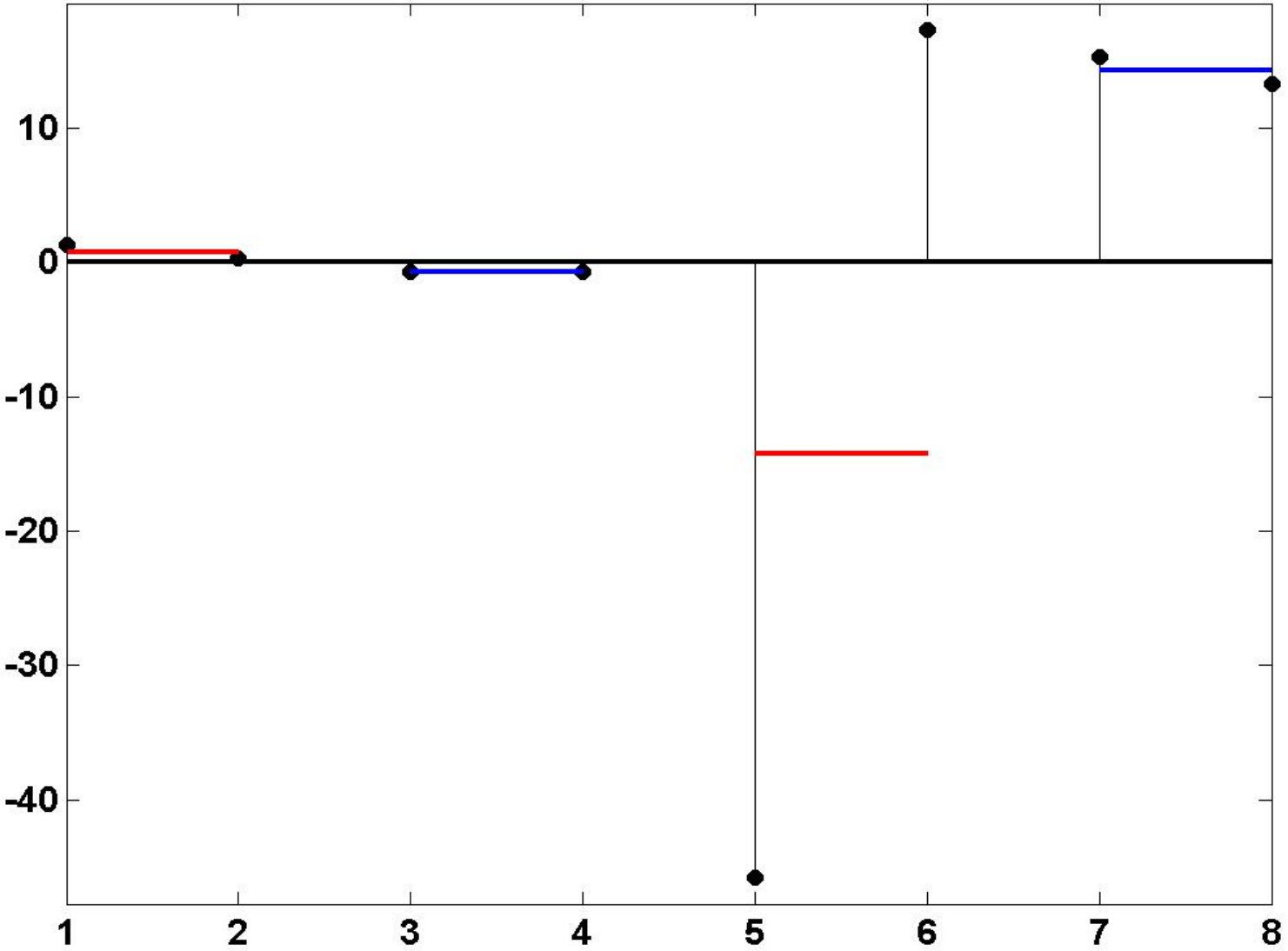
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



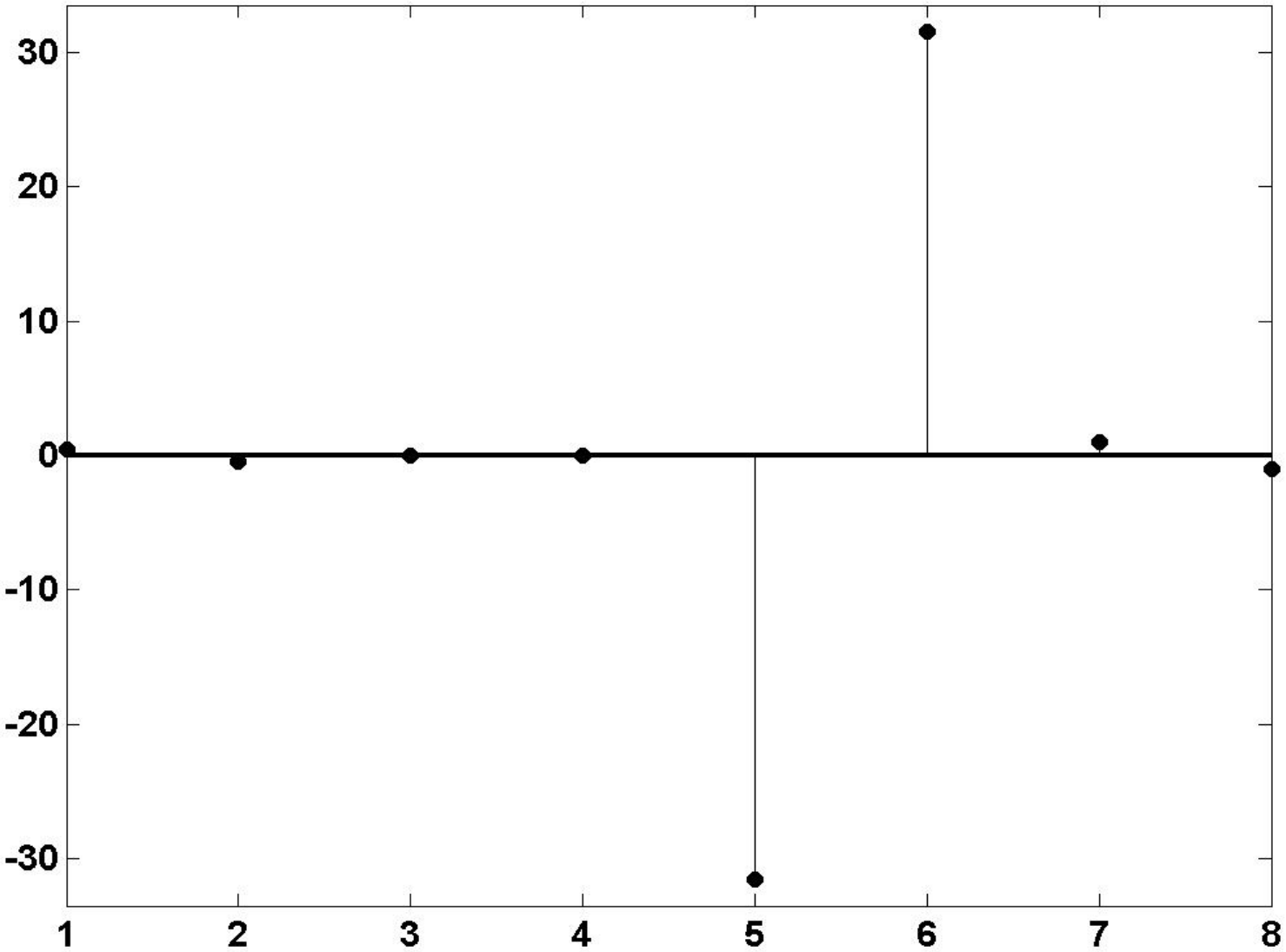
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



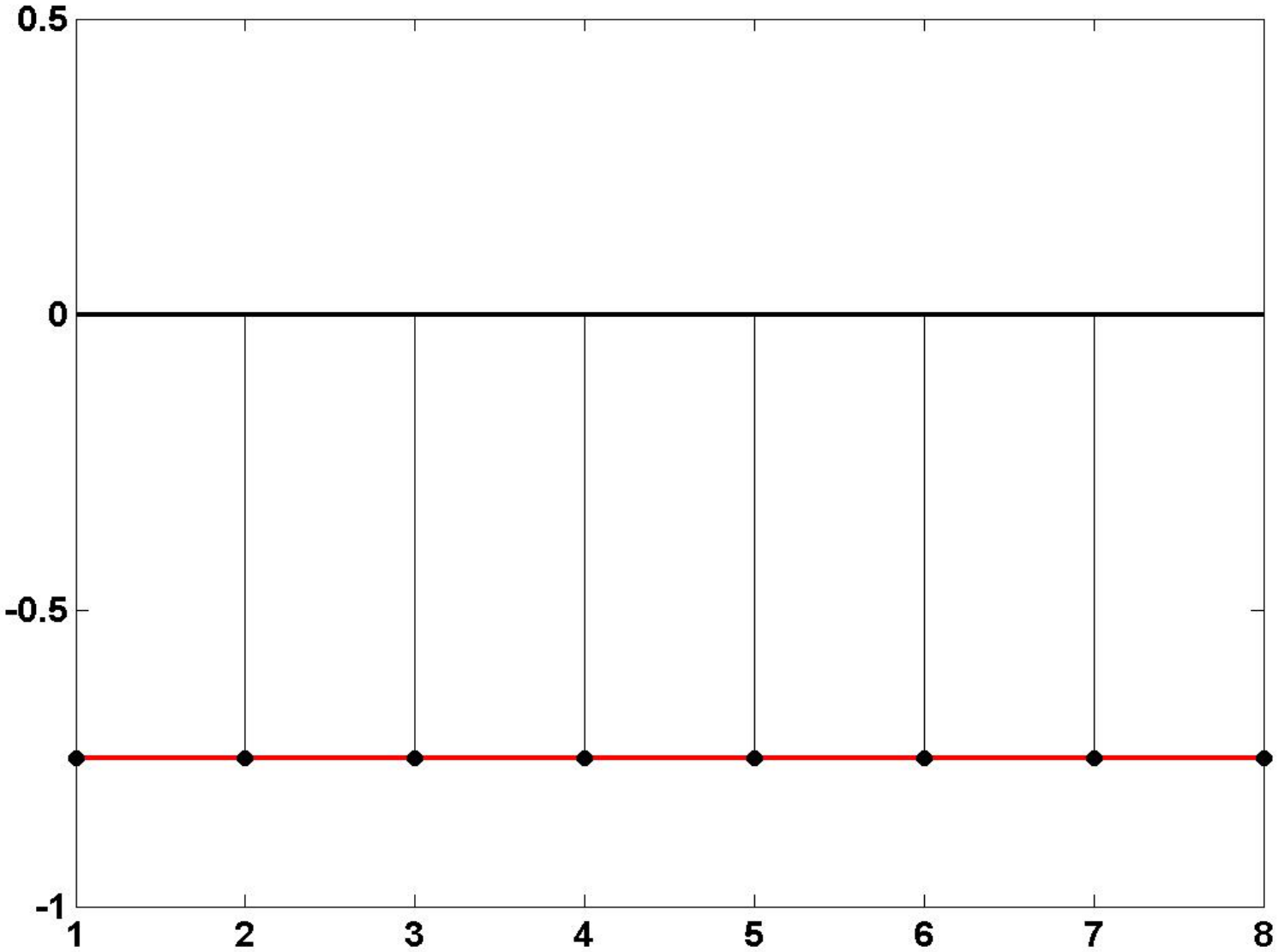
Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



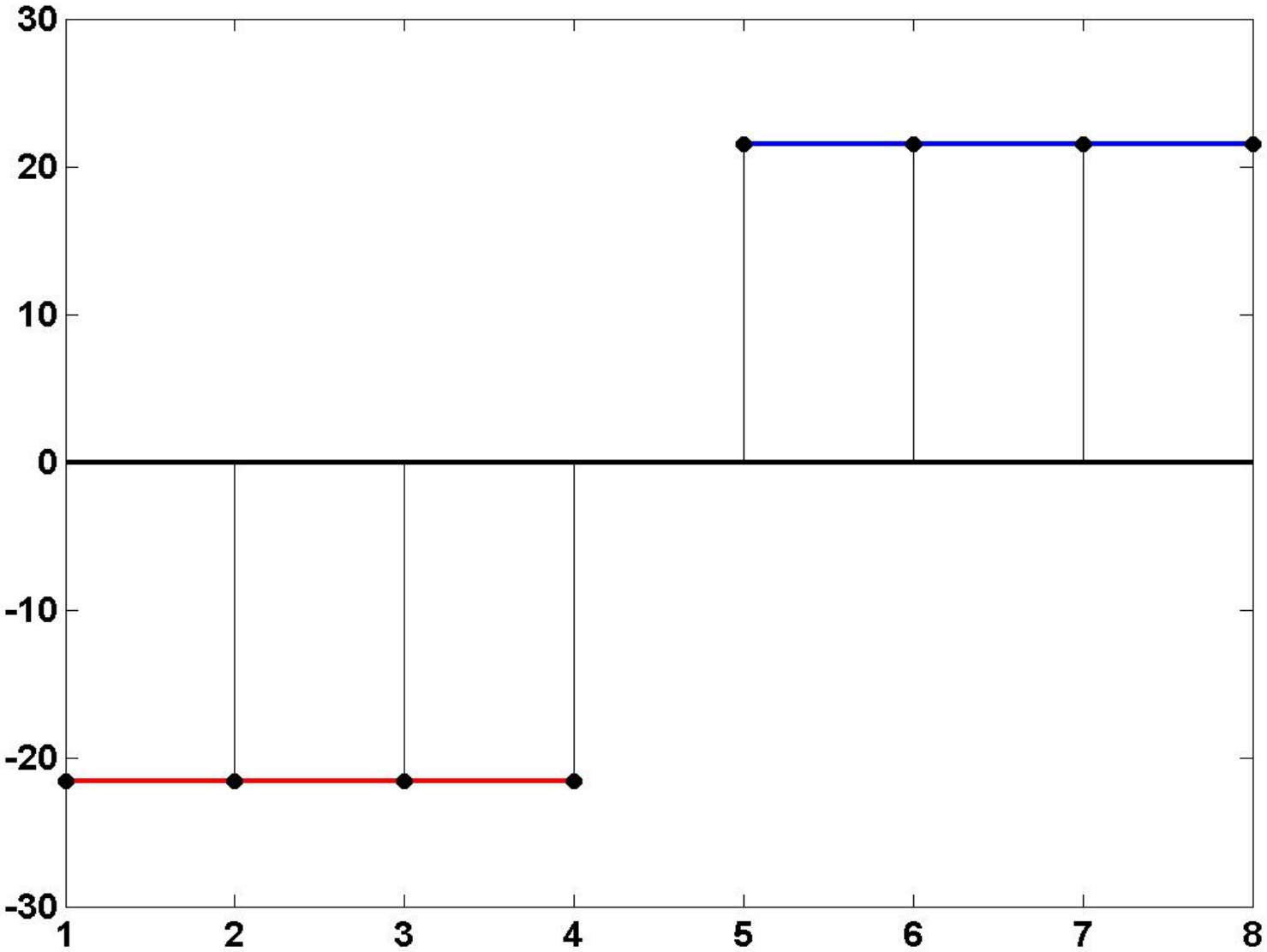
The idea of wavelets using numbers

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- Averages at level 1 (A1): -0.75,
- Average at level 2 (A2): -21.5, 21.5
- Averages at level 3 (A3): 0.75, -0.75, -14.25, 14.25.
- Averages at level 4 (A4): 0.5, -0.5, 0, 0, -31.5, 31.5, 1, -1.

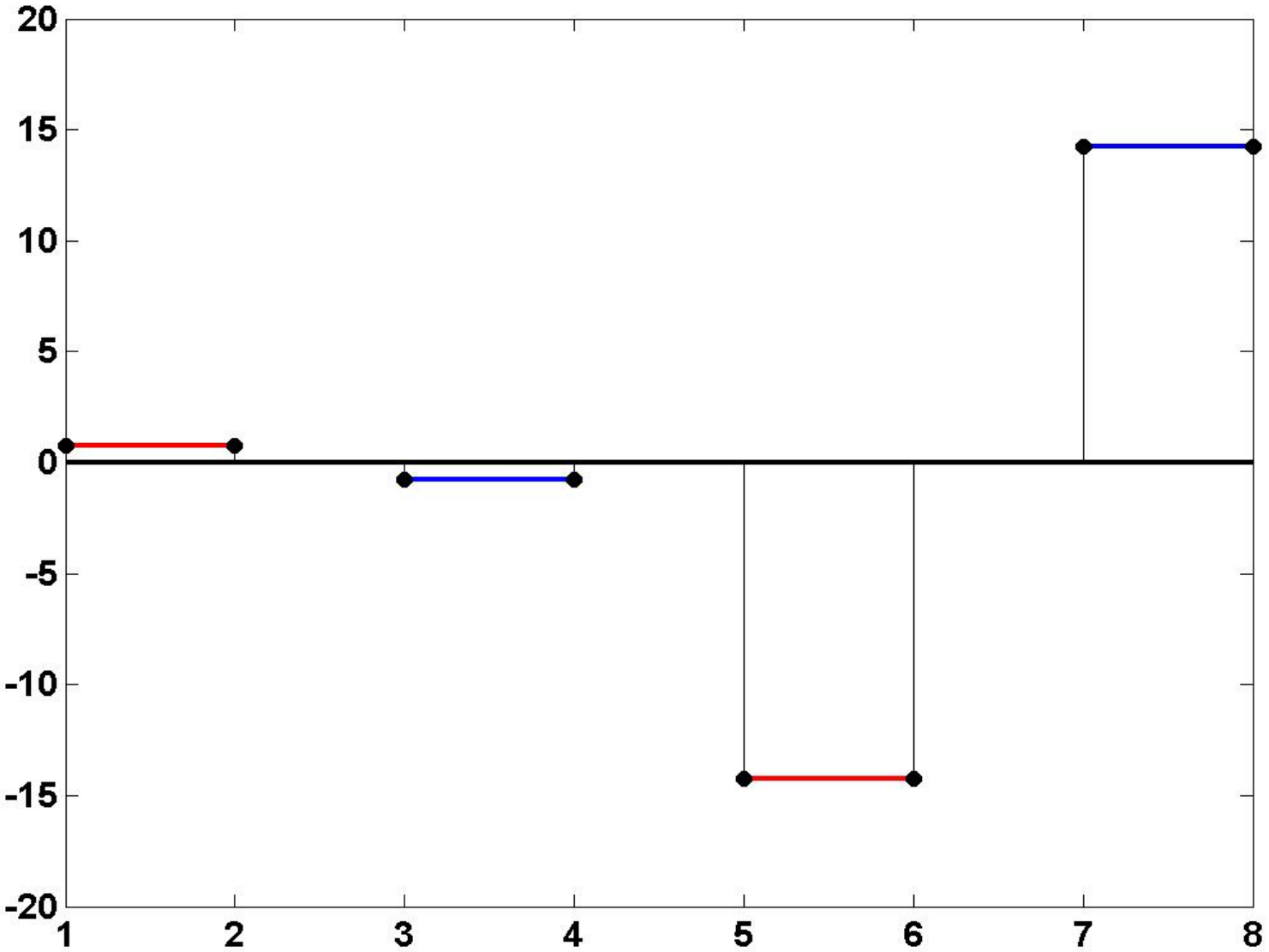
Graph of wavelet coefficients $A1$



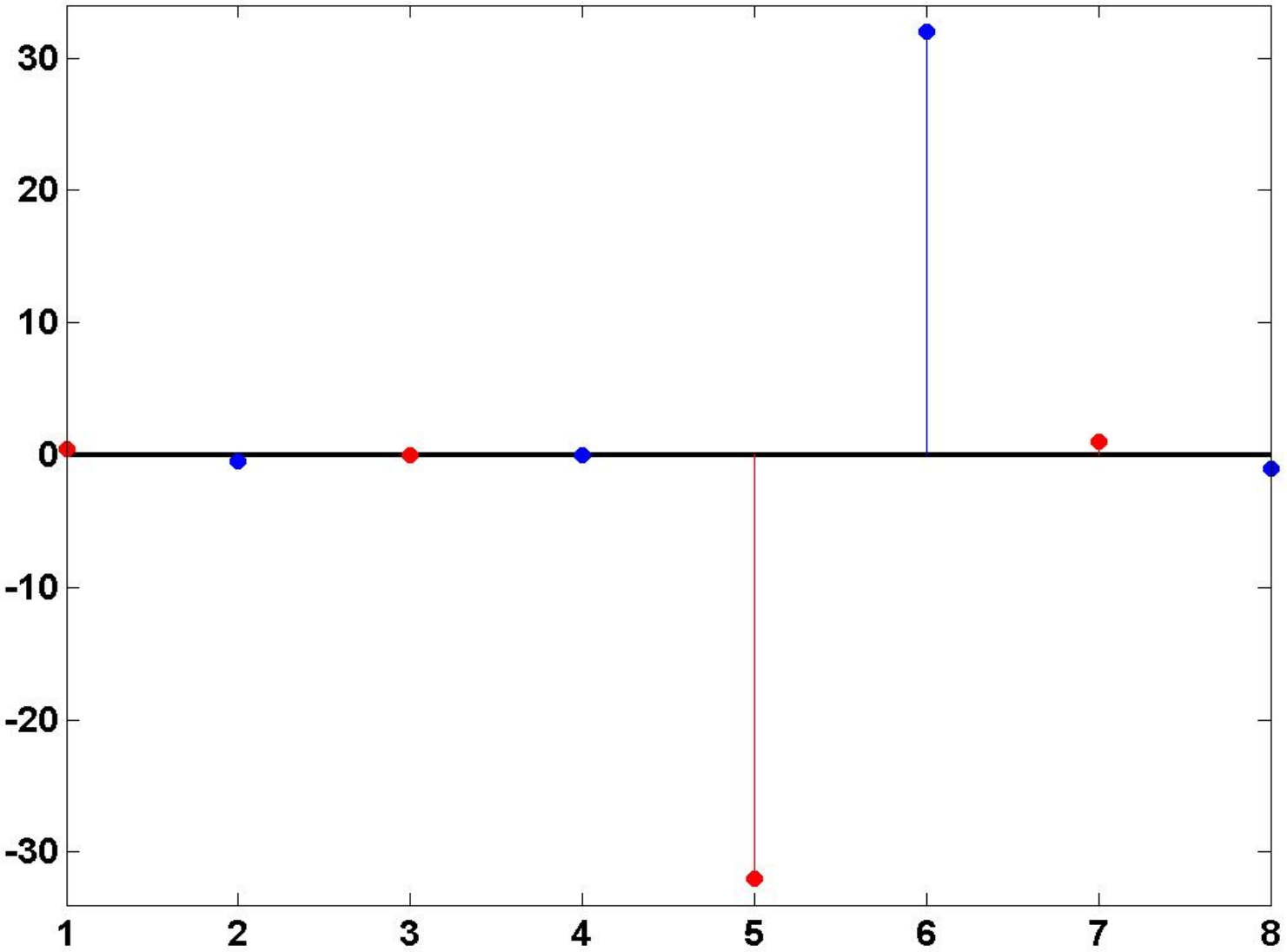
Graph of wavelet coefficients $A2$



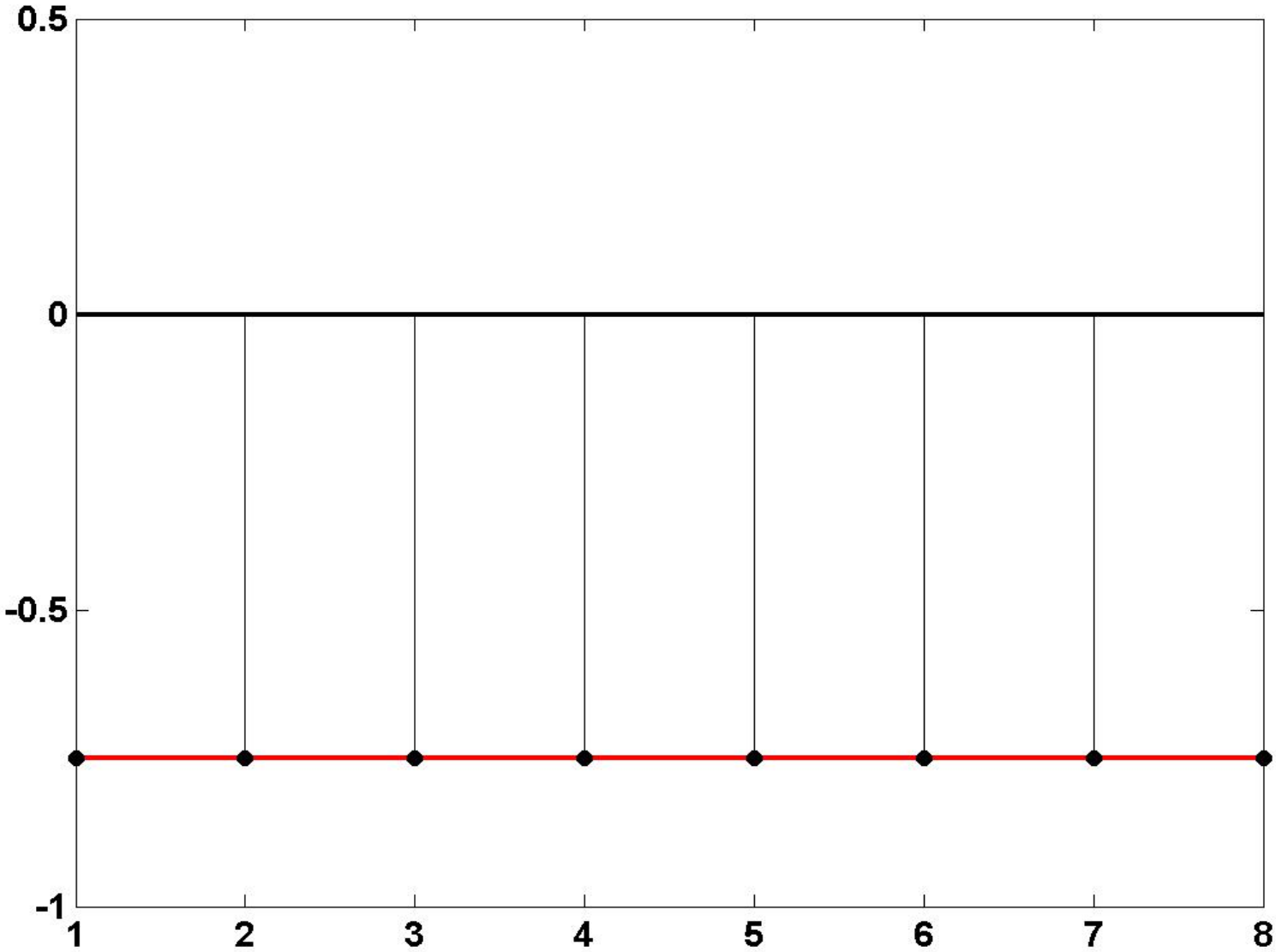
Graph of wavelet coefficients $A3$



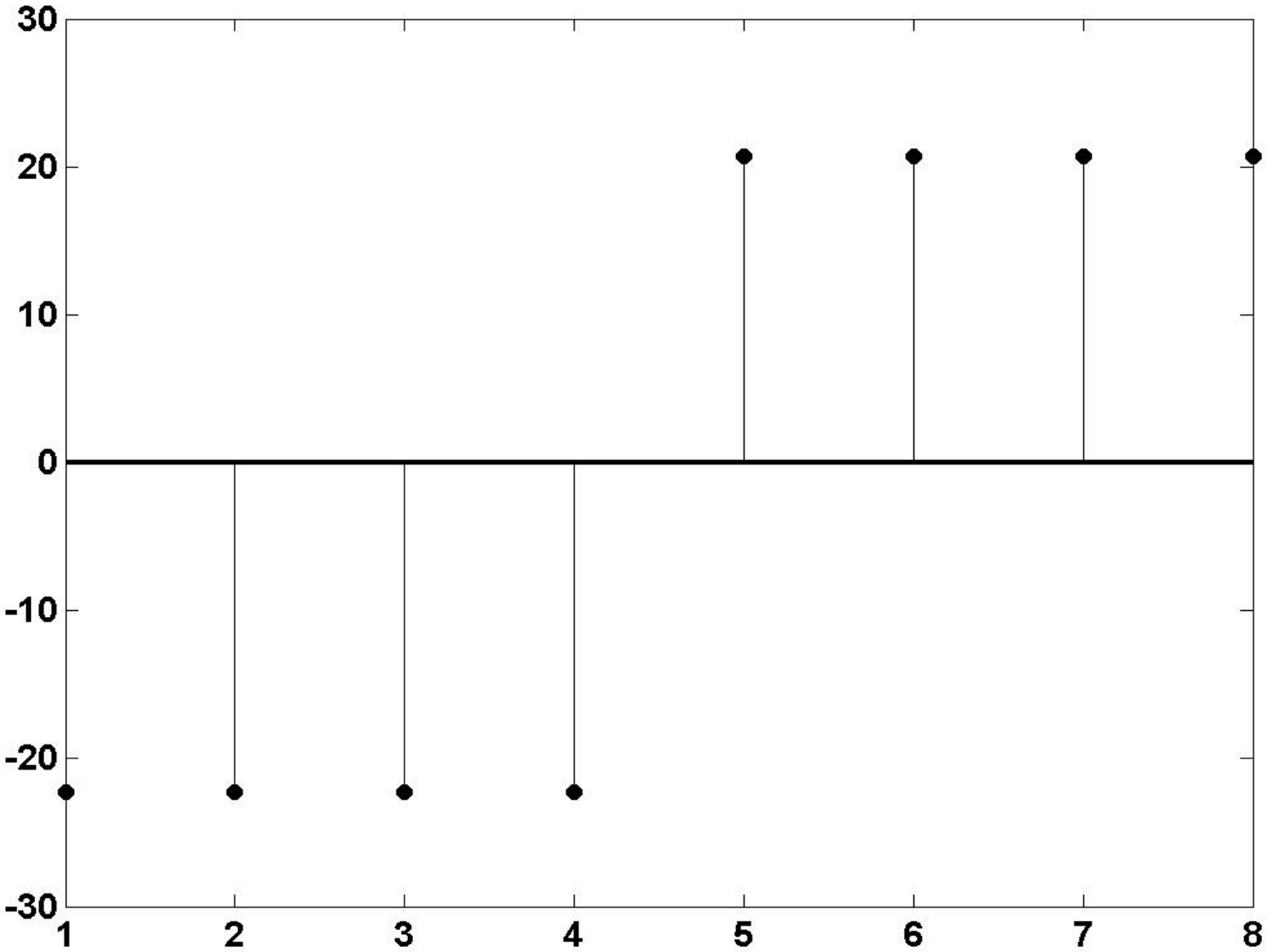
Graph of wavelet coefficients $A4$



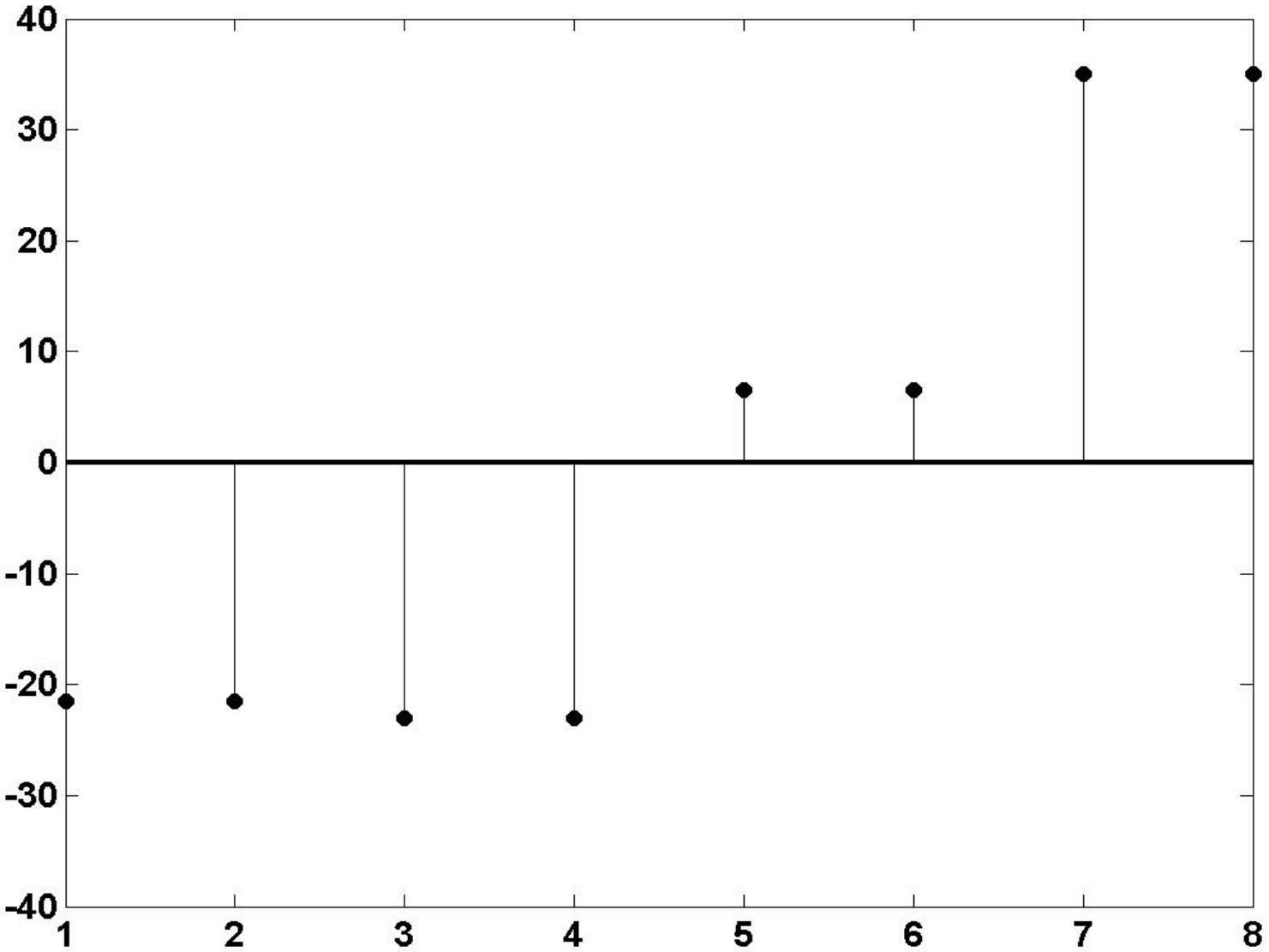
Reconstruction: $A1$ (1 number)



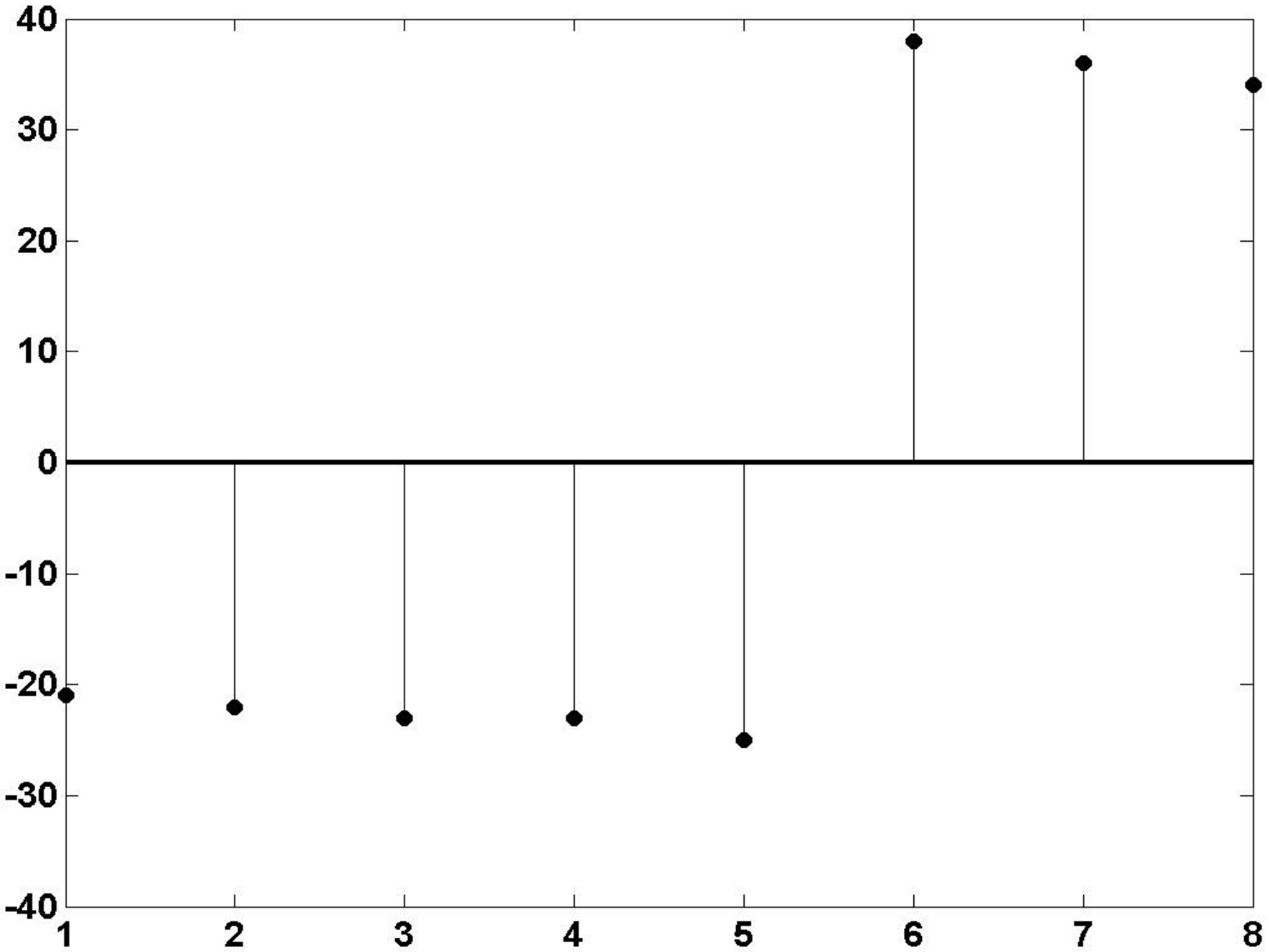
Reconstruction: $A1 + A2$ (2 numbers)



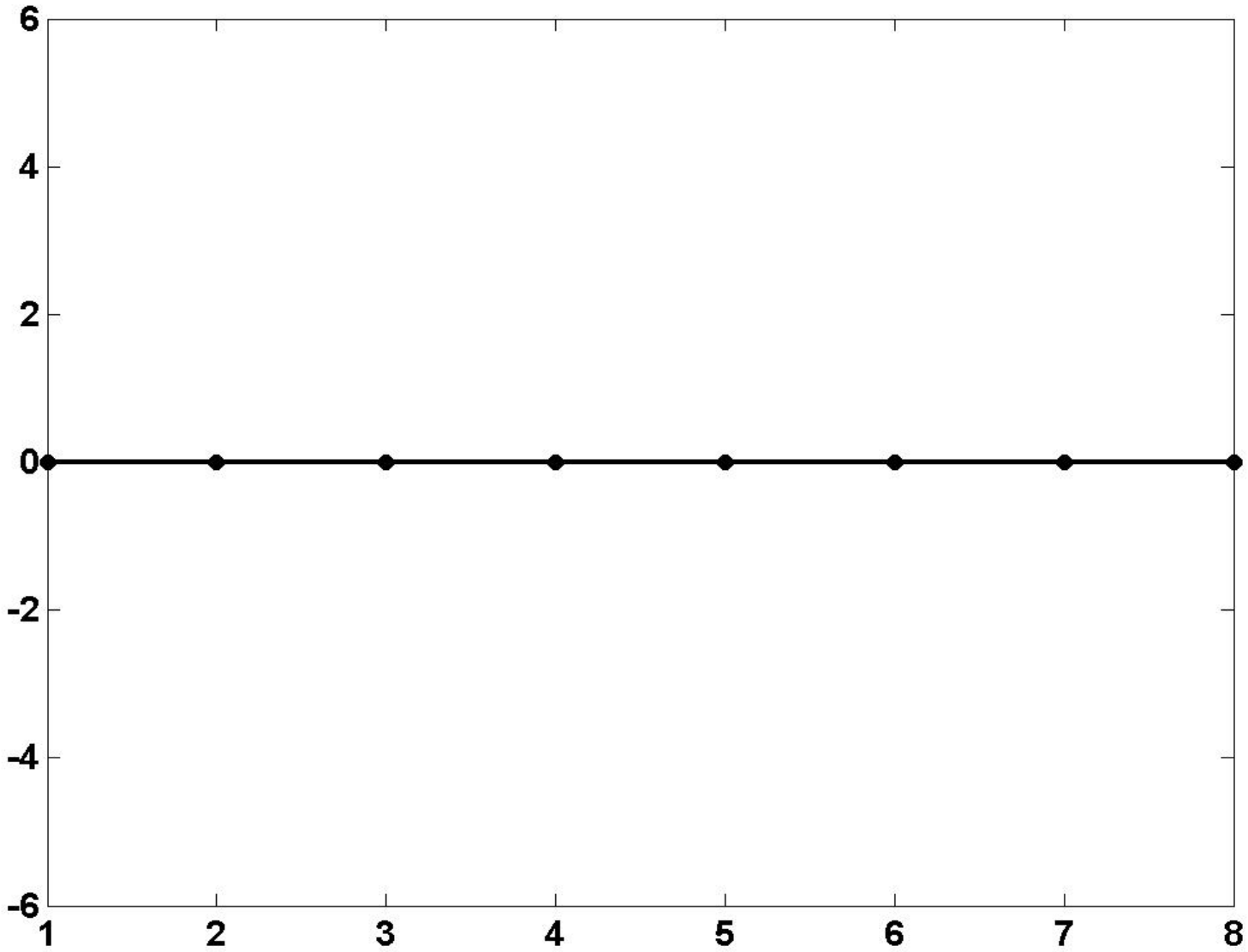
Reconstruction: $A1 + A2 + A3$ (4 numbers)



Reconstruction: $A1 + A2 + A3 + A4$ (8 numbers)



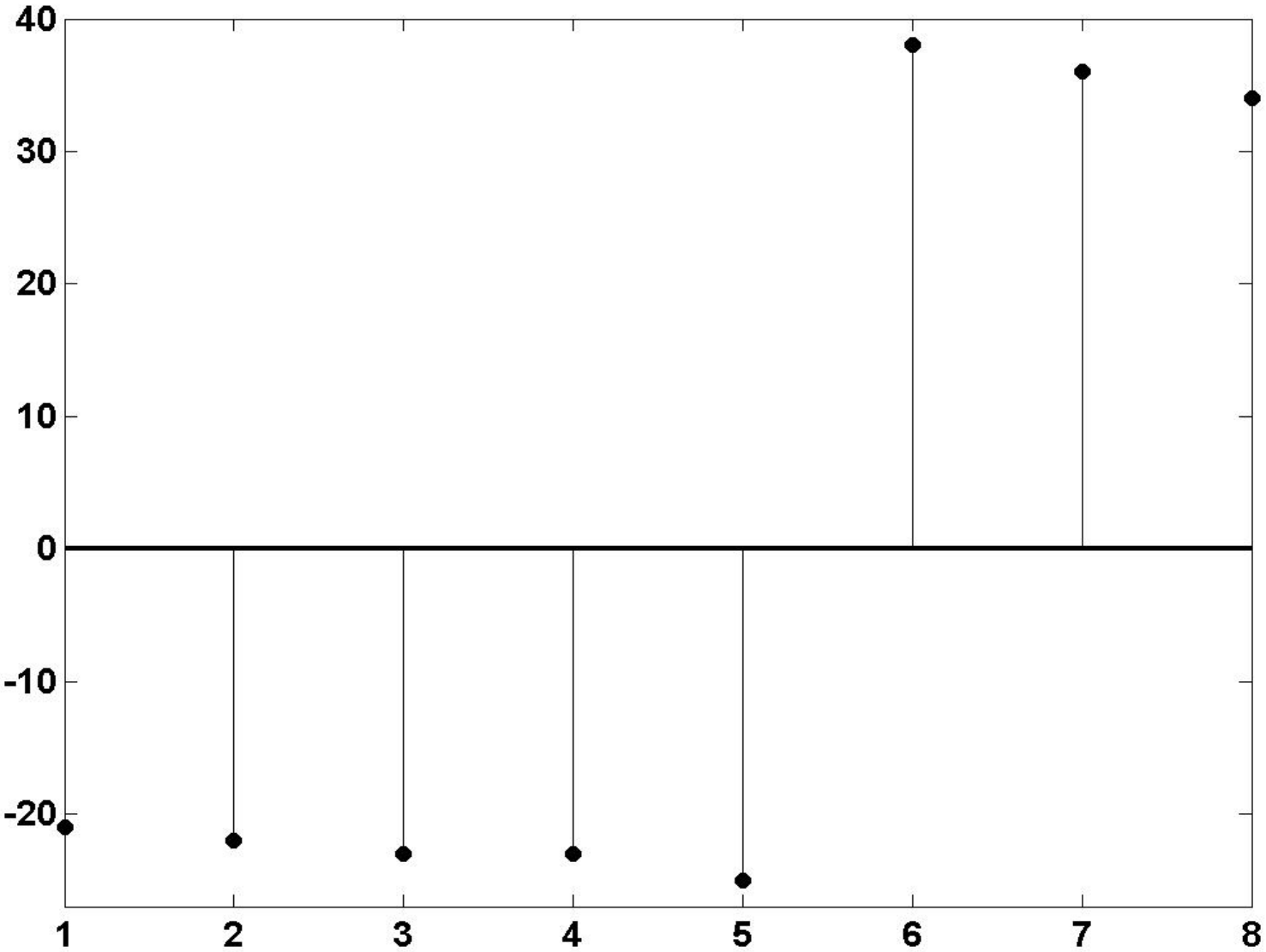
Comparison: Original–Reconstructed



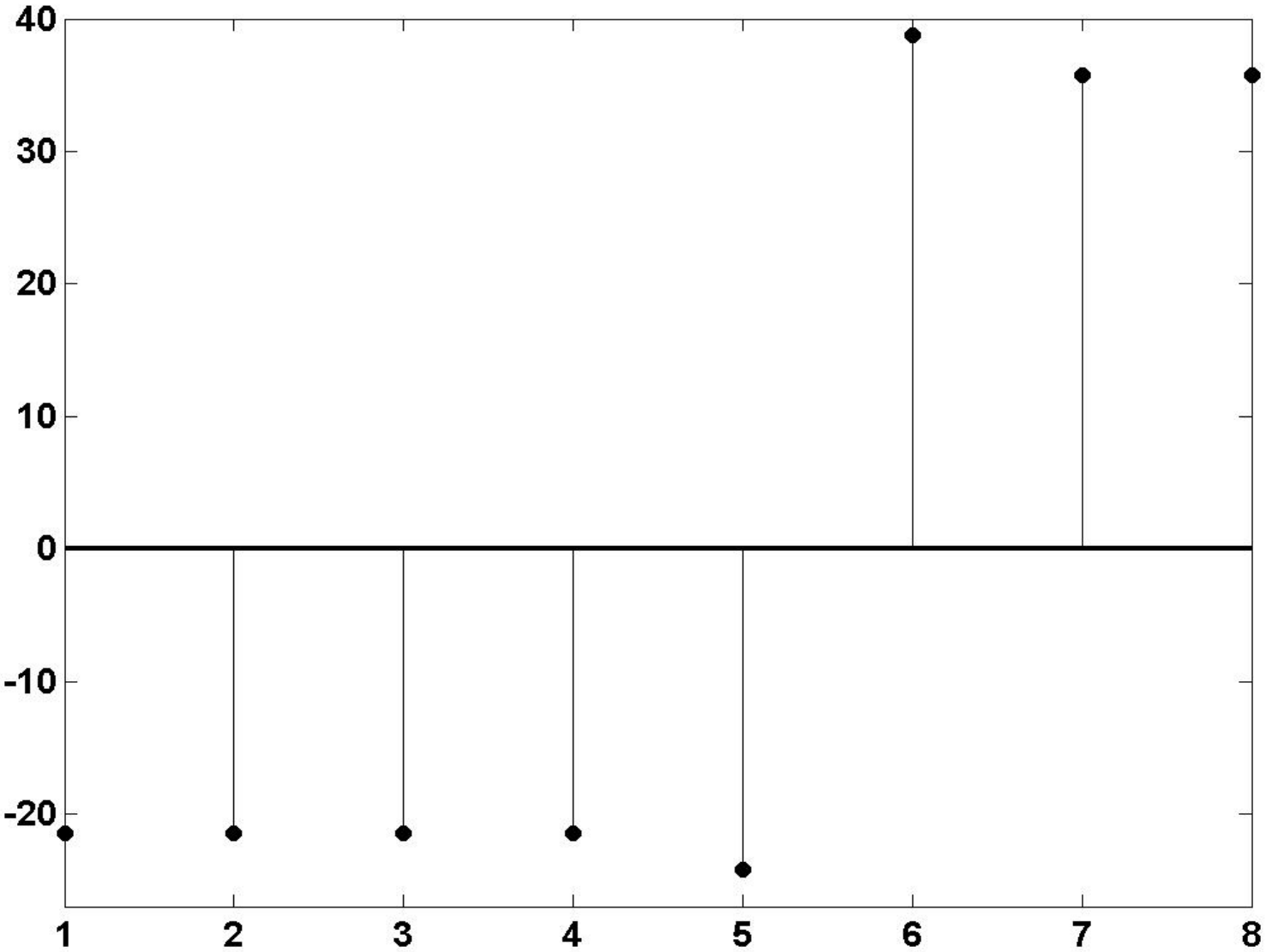
Why wavelets?

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- Averages at level 1 (A1): $\boxed{-0.75}$,
- Average at level 2 (A2): -21.5, 21.5
- Averages at level 3 (A3): $\boxed{0.75}$, -0.75, -14.25, 14.25.
- Averages at level 4 (A4): $\boxed{0.5}$, -0.5, $\boxed{0}$, 0, -31.5, 31.5, $\boxed{1}$, -1

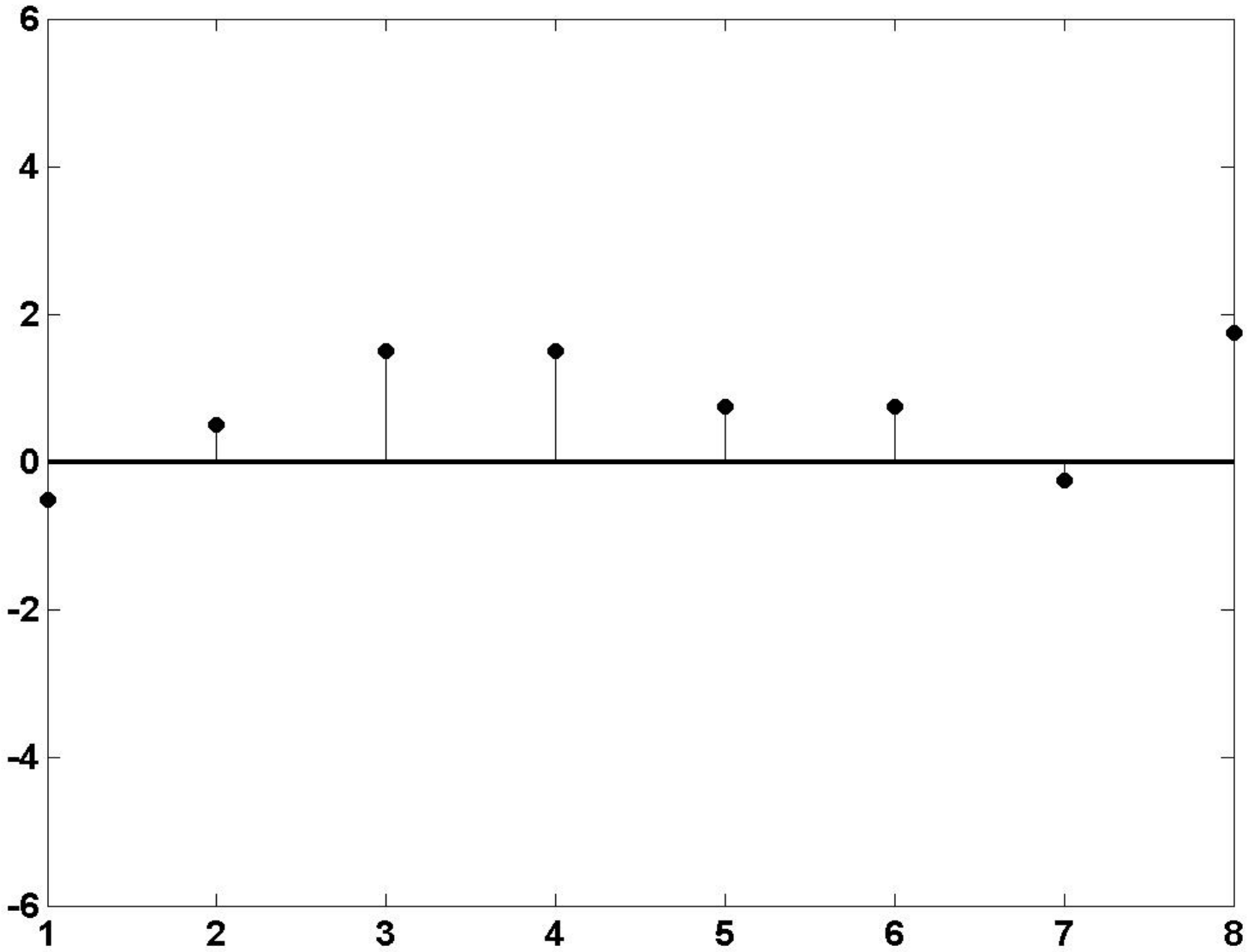
Comparison: Original



Reconstructed with 3 numbers by thresholding



Comparison: Original–Reconstructed



How to compute wavelet coefficients fast?

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- Averages at level 1 (A1): -0.75,
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Are we missing something for wavelets? or can we expect more from wavelets?

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For applications,
a fast computational algorithm
is highly demanded!

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- Averages: $[-22.25, 20.75]$. Differences: $[0.75, -14.25]$.

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- Compare:

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- Averages: $[-22.25, 20.75]$. Differences: $[0.75, -14.25]$.
- Averages: $[-0.75]$. Differences: $[-21.5]$.
- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.

Fast Wavelet Transform (FWT): Reconstruction

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- Averages: $[-22.25, 20.75]$. Differences: $[0.75, -14.25]$.
- Averages: $[-0.75]$. Differences: $[-21.5]$.
- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$

Fast Wavelet Transform (FWT): Reconstruction

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- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
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- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.

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- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
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- Averages: $[-0.75]$. Differences: $[-21.5]$.
- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.
- Add detail $[0.75, -0.75 - 14.25, 14.25] \Rightarrow [-21.5, -23, 6.5, 35]$.

Fast Wavelet Transform (FWT): Reconstruction

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- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- Averages: $[-22.25, 20.75]$. Differences: $[0.75, -14.25]$.
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- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \to \underline{[-0.75, -0.75]}$.
- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.
- Add detail $[0.75, -0.75 - 14.25, 14.25] \Rightarrow [-21.5, -23, 6.5, 35]$.
- Subdivide$[-21.5, -23, 6.5, 35] \to$
  $[-21.5, -21.5, -23, -23, 6.5, 6.5, 35, 35]$.

Fast Wavelet Transform (FWT): Reconstruction

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- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
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- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$
- Subdivide $[-22.25, 20.75] \to [-22.25, -22.25, 20.75, 20.75]$.
- Add detail $[0.75, -0.75 - 14.25, 14.25] \Rightarrow [-21.5, -23, 6.5, 35]$.
- Subdivide$[-21.5, -23, 6.5, 35] \to$ $[-21.5, -21.5, -23, -23, 6.5, 6.5, 35, 35]$.
- Add detail $[0.5, -0.5, 0, 0, -31.5, 31.5, 1, -1]$ to get $[-21, -22, -23, -23, -25, 38, 36, 34]$.