Dynamical Systems on Networks: Part II

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Outline

- A network as a directed graph, examples
- Dynamical systems on networks, examples
- Global-stability problems for network dynamics
- Kirchhoff Matrix-Tree Theorem
- A general global stability result
- Application I: flight formation control of drones
- Application II: global synchronization of coupled oscillators

A Network as a Directed Graph

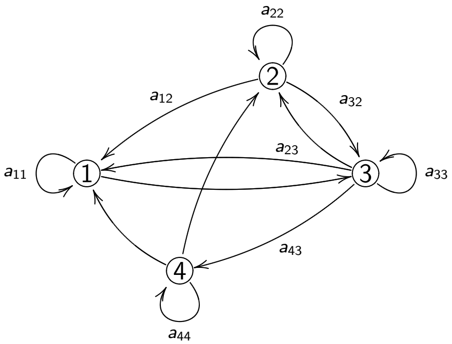
A directed graph $\mathcal{G} = (V, E, A)$

Vertex set: $V = \{1, 2, \cdots, n\}$

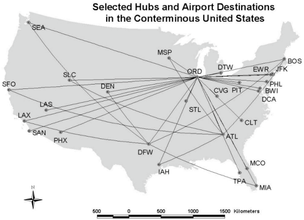
Directed edge: (i, j) from vertex i to j

Weights: $A = (a_{ij})$, $a_{ij} \neq 0 \iff (j, i)$ exists.

Given a nonnegative matrix, there corresponds a digraph $\mathcal{G}_A$, for which A is the weight matrix.



Examples of Networks



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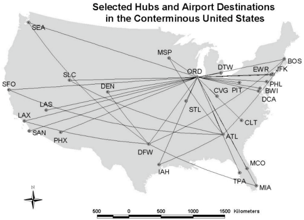

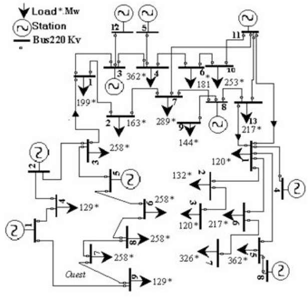

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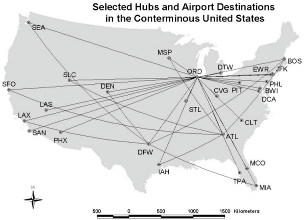

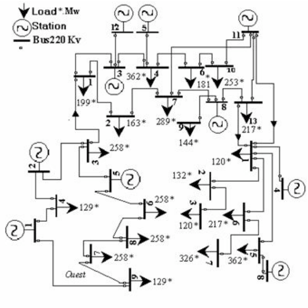

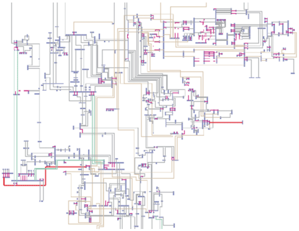

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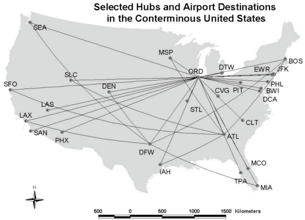

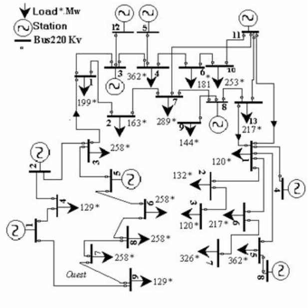

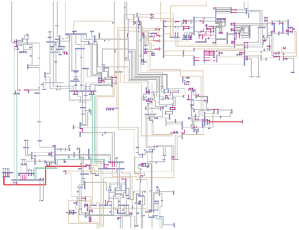

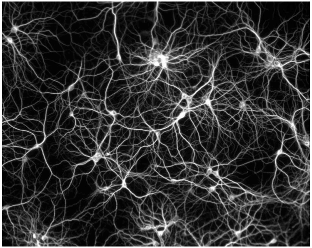

Dynamical Systems on Networks

Given a digraph $G = (V, E)$, a dynamical system can be defined over G.

Vertex dynamics: $u_i' = f_i(t, u_i), \; i = 1, \cdots, n.$
$u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$.

Connections: $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$ influence of j on i
$g_{ij} \equiv 0 \iff (j, i)$ does not exist.

Coupled system over G:

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \quad i = 1, 2, \ldots, n.$$

Examples of Dynamical Systems on Networks

- **Coupled Oscillators:**

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij} (\dot{x}_i - \dot{x}_j) = 0,$$

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- **Dispersal of a single species among n patches**

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \qquad i = 1, 2, \ldots, n.$$

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- **An n-patch predator-prey model**

$$\begin{aligned} x_i' &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \\ y_i' &= y_i(-\gamma_i - \delta_i y_i + \epsilon_i x_i), \end{aligned} \qquad i = 1, 2, \ldots, n.$$

Examples of Dynamical Systems on Networks cont'ed

- **Cellular Neural Network and Lattice Dynamical Systems**

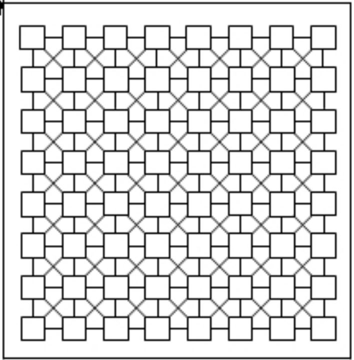

Figure 1: Cellular Neural Network 2D example

Examples of Dynamical Systems on Networks cont'ed

- **A Delayed Hopfield-Cohen-Grossberg Model of Neural Networks**

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{i=1}^{n} J_{ij} f\left(u_j(t-\tau)\right), \quad 1 \leq i \leq n$$

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- **An Epidemic Model in Heterogeneous Populations**

$$\begin{aligned} S_i' &= \Lambda_i - d_i^S S_i - \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j), \\ E_i' &= \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j) - (d_i^E + \epsilon_i) E_i, \quad i = 1, 2, \cdots, n. \\ I_i' &= \epsilon_i E_i - (d_i^I + \gamma_i) I_i. \end{aligned}$$

Research Questions

Assume: Independent vertex dynamics are simple or identical

Investigate: If, what, how complex dynamic behaviours emerge through network interactions.

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- Stability and control

Global Stability in Network Dynamics

Given a coupled system over a digraph G:

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \quad i = 1, 2, \ldots, n. \tag{1}$$

Assume: Each vertex $u_i' = f_i(t, u_i)$ is globally stable, as insured by a global Lyapunov function V_i.

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Question: Under what conditions on the underlying network and coupling is the coupled system globally stable?

Of significance in disease control, stability of eco-systems, power distribution grids etc.

Main Result

Theorem [Z. Shuai and ML, 2009] Assume

(1) There exist $F_{ij}(t, u_i, u_j)$ such that

$$\dot{V}_i(u) \le \sum_{j=1}^{n} a_{ij} F_{ij}(t, u_i, u_j), \quad t > 0,\ u_i \in D_i,\ u_j \in D_j, j = 1, \cdots, n. \tag{2}$$

(2) Along each directed cycle $\mathcal{C}$ of G,

$$\sum_{(r,s)\in E(\mathcal{C})} F_{rs}(t, u_r, u_s) \le 0, \quad t > 0,\ u_r \in D_r,\ u_s \in D_s. \tag{3}$$

Then there exist constants $c_i \ge 0$ such that $V(u) = \sum_{i=1}^{n} c_i V_i(u)$ satisfies

$$\dot{V}(u) \le 0, \quad u \in D_1 \times \cdots \times D_n.$$

Kirchhoff Matrix-Tree Theorem

Let (G, A) be a weighted digraph with weight matrix $A = (a_{ij})$.
The Laplacian matrix of graph G is

$$L = \begin{bmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k\neq n} a_{nk} \end{bmatrix}.$$

Let c_i be the cofactor of the i-th diagonal element of L.

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Theorem [Kirchhoff (1847)] Assume $n \geq 2$. Then

$$c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} w(\mathcal{T}), \qquad i = 1, 2, \ldots, n, \tag{4}$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ rooted at vertex i, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$.

Reordering of a Double Sum

Proposition [Tree-Cycle-Identity, Z. Shuai and ML 2009] Let c_i be given by the Matrix-Tree Theorem. Then the following identity holds.

$$\sum_{i,j=1}^{n} c_i \, a_{ij} \, F_{ij}(x_i, x_j) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x_r, x_s), \qquad (5)$$

where $F_{ij}(x_i, x_j), 1 \leq i, j \leq n$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs $\mathcal{Q}$ of $(\mathcal{G}, A)$, $w(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $\mathcal{C}_{\mathcal{Q}}$ denotes the oriented cycle of $\mathcal{Q}$.

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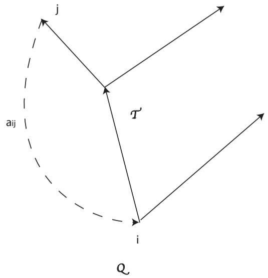
Proof: Note $w(\mathcal{T})\, a_{ij} = w(\mathcal{Q})$,

where $\mathcal{Q}$ is the unicyclic graph obtained by adding an arc (j, i) to $\mathcal{T}$.



Proof of Main Theorem

$$
\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} c_i \dot{V}_i \leq \sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(t, u_i, u_j) \quad \text{(assumption (1))} \\
&= \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(t, u_r, u_s) \quad \text{(Proposition)} \\
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Our Theorem offers a systematic way to construct global Lyapunov functions for the coupled system, using individual Lyapunov functions for the vertex dynamics.

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Is the theorem any good?

Application I: A Network of Coupled Oscillators

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0, \tag{6}$$

or in systems

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^{n} \epsilon_{ij}(y_i - y_j). \end{aligned} \tag{7}$$

Each vertex dynamics is given by a damped nonlinear oscillator

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) = 0.$$

Assume that the damping $\alpha_i \geq 0$ and the potential energy $F_i(x_i) = \int^{x_i} f_i(s)ds$ has a strictly global minimum at $x_i = x_i^*$. Then $x = x_i^*$ is globally stable (using the Lyapunov function)

$$V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}.$$

Application I: A Network of Coupled Oscillators

Theorem Assume $\alpha_k > 0$ for some k and digraph $\mathcal{G}$ is strongly connected. Then $E^*(x_1^*, 0, x_2^*, 0, \cdots, x_n^*, 0)$ is globally asymptotically stable in $\mathbb{R}^{2n}$.

Proof. $V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}$

$$
\begin{aligned}
\dot{V}_i &= -\alpha_i y_i^2 - \sum_{j=1}^{n} \epsilon_{ij}(y_i - y_j) y_i \\
&\leq \sum_{j=1}^{n} \epsilon_{ij}[-\frac{1}{2}(y_i - y_j)^2 - \frac{1}{2} y_i^2 + \frac{1}{2} y_j^2] \\
&\leq \sum_{j=1}^{n} \epsilon_{ij} F_{ij}(y_i, y_j)
\end{aligned}
$$

where

$$
F_{ij}(y_i, y_j) = -\frac{1}{2} y_i^2 + \frac{1}{2} y_j^2 .
$$

Application II: A Single Species Model with Dispersal

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \quad i = 1, 2, \ldots, n. \tag{8}$$

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Theorem [Z. Shuai and ML (2009)] Assume

(1) matrix (d_{ij}) is irreducible;

(2) $f_i'(x_i) \leq 0, x_i > 0, i = 1, 2, \ldots, n;\ \exists\ k,\ f_k'(x_k) \not\equiv 0$ in any open interval of $\mathbb{R}^+$;

(3) system (8) is uniformly persistent;

(4) solutions of (8) are uniformly bounded.

Then system (8) has a globally asymptotically stable positive equilibrium E^*.

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Note: Lu and Tacheuchi (1993) proved the result under the assumption $f_i'(x_i) < 0,\ x_i > 0$ for all i, using the theory of monotone dynamical systems.

Application III: An n-Patch Predator-Prey Model

$$
\begin{aligned}
x_i' &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \\
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Theorem [Z. Shuai and ML (2009)] Assume that (d_{ij}) is irreducible, and that $\exists\ k$ such that $b_k \delta_k > 0$. Then the positive equilibrium E^*, whenever it exists, is unique and globally asymptotically stable in the positive cone $\mathbb{R}_+^{2n}$.

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Kuang and Tacheuchi (1994) proved the two-patch case.

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Kuang and Tacheuchi (1994) proved the two-patch case.

$$V_i(x_i, y_i) = \epsilon_i(x_i - x_i^* \ln x_i) + e_i(y_i - y_i^* \ln y_i)$$

Application IV: A Multi-group Delayed Epidemic Model

$$\begin{aligned} S_i' &= \Lambda_i - d_i^S S_i - \sum_{j=1}^{n} \beta_{ij} S_i I_j(t-\tau_j), \\ I_i' &= \sum_{j=1}^{n} \beta_{ij} S_i I_j(t-\tau_j) - (d_i^I + \gamma_i) I_i, \end{aligned} \qquad i = 1, 2, \cdots, n. \tag{10}$$

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When $n = 1$, C. McCluskey proved the global stability with Lyapunov function

$$
V_i = (S_i - S_i^* + S_i^* \ln \frac{S_i}{S_i^*}) + (I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}) +
\sum_{j=1}^{n} \beta_{ji} S_i^* \int_0^{\tau_j} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.
$$

A video on Youtube
https://www.youtube.com/watch?v=QmWD76jwjbQ

GRASP Lab, University of Pennsylvania

Each robotic agent has position vector $r_i = (x_i, y_i) \in \mathbb{R}^2$ and velocity vector $v_i = \dot{r}_i = (\dot{x}_i, \dot{y}_i)$.

The system's evolution is governed by Newton's equation

$$\begin{aligned} \dot{r}_i &= v_i, \\ \dot{v}_i &= u_i, \end{aligned} \qquad i = 1, \cdots, n. \tag{11}$$

Here

- $u_i,\ i = 1, \cdots, n$, define the control protocol
- Formation control is achieved through communications among agents
- Network represents the communication graph (topology)
- A complete communication graph is too costly.

Hierarchical Potential Clustering (HPC) Protocol

Proposed by J. Maidens and ML:

1) Divide the agents into clusters
2) Assign a leader to each cluster
3) Implement an artificial potential scheme (with a complete graph) within each cluster
4) Implement a velocity consensus scheme among the cluster leaders.

HPC Protocol: control within a cluster i

For $j \neq 1$, (i.e. r_{ij} is not a leader in cluster i)

$$u_{ij} = -\nabla_{r_{ij}} P_{ij} - \sum_k \frac{(\theta_{ij} - \theta_{ik})||v_{ij}||}{||r_{ij} - r_{ik}||} \hat{n}(ij),$$

where

$$P_{ij} = \sum_{k=1}^{n_i} P_{ij}^{ik}$$

controls distance of agents in the cluster and

$$\theta_{ij} = \tan^{-1}\left(\frac{\dot{y}_{ij}}{\dot{x}_{ij}}\right)$$

is the heading of agent (i, j).

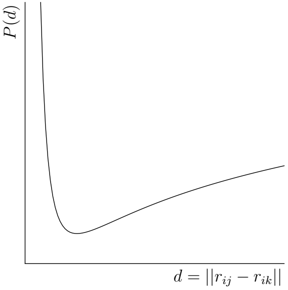

Figure: Potential function P_{ij}^{ik}.

HPC Protocol: control among leaders

For $j = 1$, (i.e., r_{i1} is the leader in cluster i), we add additional force to control there heading

$$u_{i1} = -\nabla_{r_{i1}} P_{i1} - \sum_{k} \frac{(\theta_{i1} - \theta_{ik})||v_{i1}||}{||r_{i1} - r_{ik}||} \hat{n}(i1) + \sum_{h \in N_i} b_{ih}(v_{h1} - v_{i1})$$

where matrix $B = (b_{ij})$ is any nonnegative irreducible matrix. The correspond communication graph $\mathcal{G}_B$ among leaders is strongly connected.

Formation Stabilization Problem

Definition A control protocol is said to solve the formation stabilization problem if solutions of (11) converge asymptotically to a state such that

(a) the relative positions of each agent (i, j) within a cluster are such that a local minimum of the total vertex potential P_{ij} is achieved,

(b) the headings of any two agents (i, j) and (h, k) satisfy $\theta_{ij} = \theta_{hk}$.

Main Result

Theorem (J. Maidens and ML, 2013)

Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph $\mathcal{G}_B$ is strongly connected.

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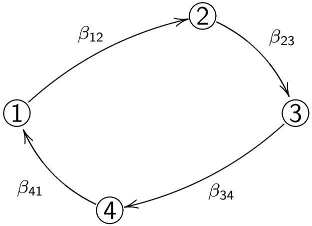
An example graph that is strongly connected:

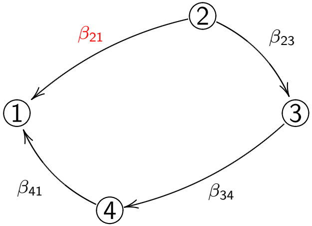


Main Result

Theorem (J. Maidens and ML, 2013)

Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph $\mathcal{G}_B$ is strongly connected.

An example graph that is not strongly connected:



Simulations

- Clustering without control protocol
  - Video 1
- Clustering without leader control
  - Video 2
  - Video 3
- Clustering with leader control
  - Video 4
  - Video 5

Synchronization

Synchronization of metronomes: a video

`https://www.youtube.com/watch?v=Aaxw4zbULMs`

Coupled Oscillators Revisited

Consider a system of coupled oscillators:

$$\ddot{x}_i + f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0,$$

Assume that $f_i(x_i)$ and $F_i(x_i) = \int_i^x f_i(t)dt$ satisfy

(C_1) $f_i(x_i)x_i > 0,\ x_i \neq 0, i = 1, 2, \cdots, n,$

(C_2) $F_i(x_i) \to \infty$ as $|x_i| \to \infty,\ i = 1, 2, \cdots, n.$

Both (C_1) and (C_2) are satisfied for $f_i(x_i) = x_i^3$.

Global Synchronization

Definition: System (29) is said to achieve global synchronization if, for every solution $x(t)$ of system (29) and all $1 \leq i, j \leq n$,

$$\dot{x}_i(t) - \dot{x}_j(t) = 0.$$

Question: Under what conditions of matrix $A = (a_{ij})$ does the system (29) achieves global synchronization?

A Theorem

Theorem (P. Du and ML 2015)

In system (29)*, suppose that the direct graph* $\mathcal{G}_A$ *is strongly connected, and assumptions* (C_1) *and* (C_2) *are satisfied. Then system* (29) *achieves global synchronization.*

For the proof, considering the equivalent system

$$\dot{x}_i = y_i$$

$$\dot{y}_i = -f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij}(y_j - y_i)$$

Using Lyapunov functions:

$$V_i = \frac{1}{2}y_i^2 + F_i(x_i),$$

and

$$V = \sum_{i=1}^{n} c_i V_i.$$