

# Dynamical Systems on Networks: Part II

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# Outline

- A network as a directed graph, examples
- Dynamical systems on networks, examples
- Global-stability problems for network dynamics
- Kirchhoff Matrix-Tree Theorem
- A general global stability result
- Application I: flight formation control of drones
- Application II: global synchronization of coupled oscillators

## A Network as a Directed Graph

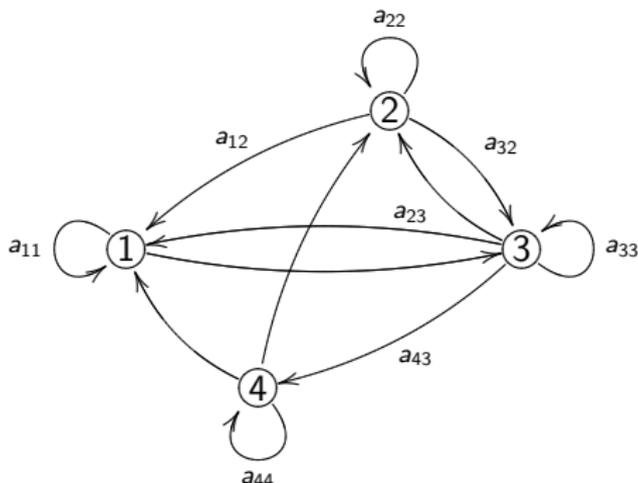
A directed graph  $\mathcal{G} = (V, E, A)$

Vertex set:  $V = \{1, 2, \dots, n\}$

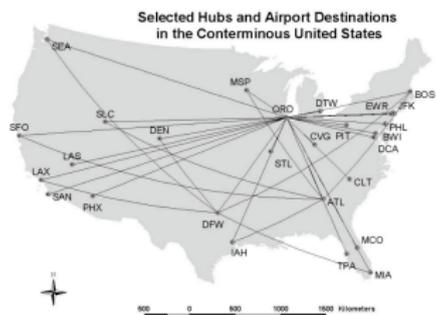
Directed edge:  $(i, j)$  from vertex  $i$  to  $j$

Weights:  $A = (a_{ij})$ ,  $a_{ij} \neq 0 \iff (j, i)$  exists.

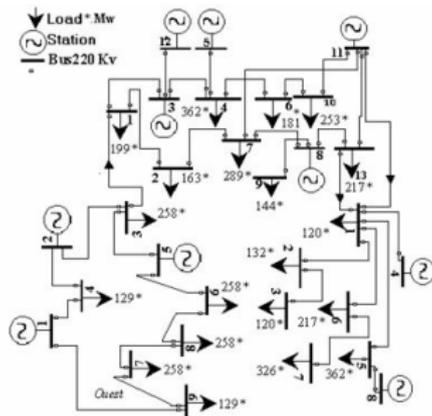
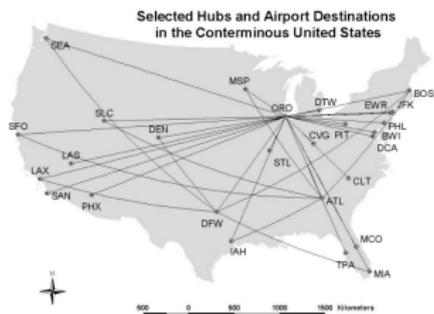
Given a nonnegative matrix, there corresponds a digraph  $\mathcal{G}_A$ , for which  $A$  is the weight matrix.



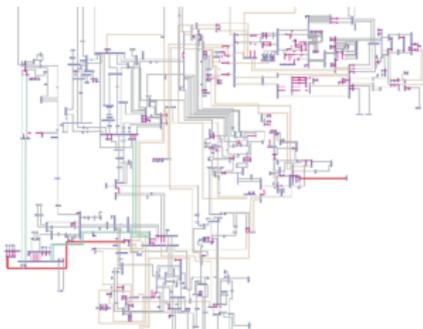
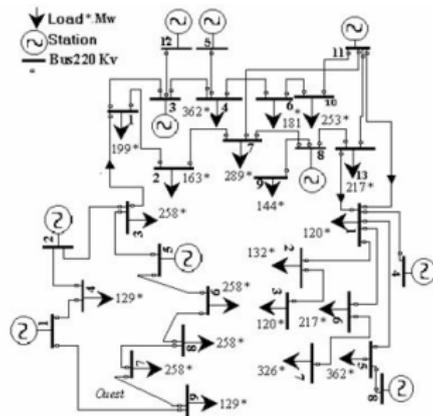
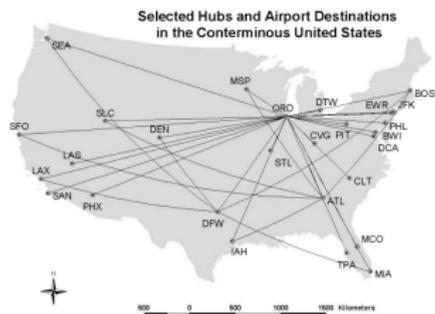
# Examples of Networks



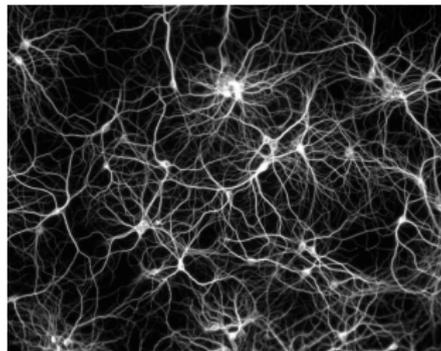
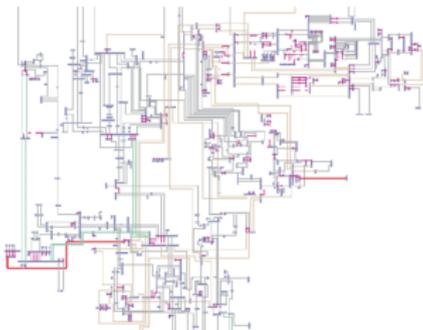
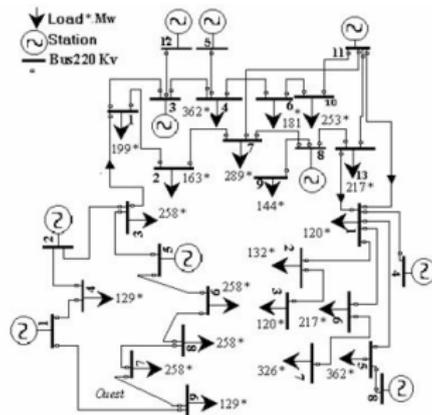
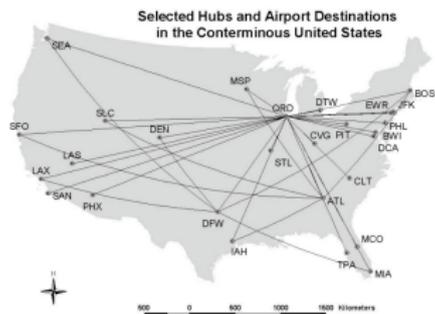
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# Dynamical Systems on Networks

Given a digraph  $G = (V, E)$ , a dynamical system can be defined over  $G$ .

**Vertex dynamics:**  $u'_i = f_i(t, u_i)$ ,  $i = 1, \dots, n$ .  
 $u_i \in \mathbb{R}^{m_i}$  and  $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i}$ .

**Connections:**  $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^{m_i}$  influence of  $j$  on  $i$   
 $g_{ij} \equiv 0 \iff (j, i)$  does not exist.

**Coupled system over  $G$ :**

$$u'_i = f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j), \quad i = 1, 2, \dots, n.$$

# Examples of Dynamical Systems on Networks

- **Coupled Oscillators:**

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^n \epsilon_{ij} (\dot{x}_i - \dot{x}_j) = 0,$$

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- **Dispersal of a single species among  $n$  patches**

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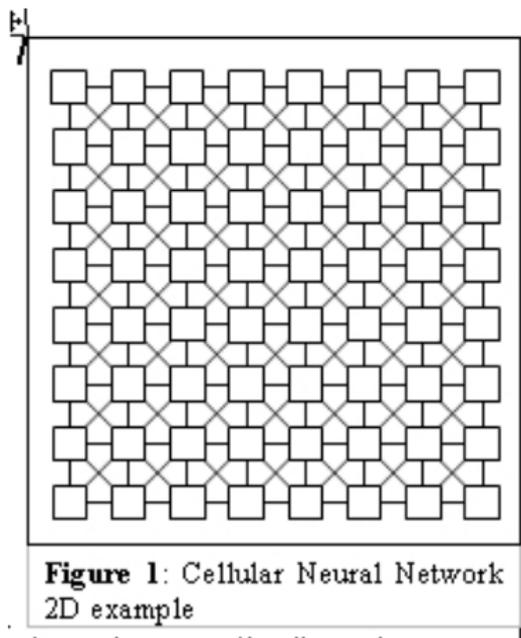
- **An  $n$ -patch predator-prey model**

$$x_i' = x_i (r_i - b_i x_i - e_i y_i) + \sum_{j=1}^n d_{ij} (x_j - \alpha_{ij} x_i), \quad i = 1, 2, \dots, n.$$

$$y_i' = y_i (-\gamma_i - \delta_i y_i + \epsilon_i x_i),$$

# Examples of Dynamical Systems on Networks cont'ed

- Cellular Neural Network and Lattice Dynamical Systems



# Examples of Dynamical Systems on Networks cont'ed

- **A Delayed Hopfield-Cohen-Grossberg Model of Neural Networks**

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{j=1}^n J_{ij} f(u_j(t - \tau)), \quad 1 \leq i \leq n$$

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- **An Epidemic Model in Heterogeneous Populations**

$$S'_i = \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j),$$

$$E'_i = \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) - (d_i^E + \epsilon_i) E_i, \quad i = 1, 2, \dots, n.$$

$$I'_i = \epsilon_i E_i - (d_i^I + \gamma_i) I_i.$$

# Research Questions

**Assume:** Independent vertex dynamics are simple or identical

**Investigate:** If, what, how complex dynamic behaviours emerge through network interactions.

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- Phase transition and bifurcation
- Stability and control

# Global Stability in Network Dynamics

Given a coupled system over a digraph  $G$ :

$$u'_i = f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j), \quad i = 1, 2, \dots, n. \quad (1)$$

**Assume:** Each vertex  $u'_i = f_i(t, u_i)$  is globally stable, as insured by a global Lyapunov function  $V_i$ .

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**Assume:** Each vertex  $u'_i = f_i(t, u_i)$  is globally stable, as insured by a global Lyapunov function  $V_i$ .

**Question:** Under what conditions on the underlying network and coupling is the coupled system globally stable?

Of significance in disease control, stability of eco-systems, power distribution grids etc.

# Main Result

**Theorem** [Z. Shuai and ML, 2009] Assume

(1) There exist  $F_{ij}(t, u_i, u_j)$  such that

$$\dot{V}_i(u) \leq \sum_{j=1}^n a_{ij} F_{ij}(t, u_i, u_j), \quad t > 0, u_i \in D_i, u_j \in D_j, j = 1, \dots, n. \quad (2)$$

(2) Along each directed cycle  $\mathcal{C}$  of  $G$ ,

$$\sum_{(r,s) \in E(\mathcal{C})} F_{rs}(t, u_r, u_s) \leq 0, \quad t > 0, u_r \in D_r, u_s \in D_s. \quad (3)$$

Then there exist constants  $c_i \geq 0$  such that  $V(u) = \sum_{i=1}^n c_i V_i(u)$  satisfies

$$\dot{V}(u) \leq 0, \quad u \in D_1 \times \dots \times D_n.$$

## Kirchhoff Matrix-Tree Theorem

Let  $(G, A)$  be a weighted digraph with weight matrix  $A = (a_{ij})$ .  
The **Laplacian matrix** of graph  $G$  is

$$L = \begin{bmatrix} \sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk} \end{bmatrix}.$$

Let  $c_i$  be the **cofactor** of the  $i$ -th diagonal element of  $L$ .

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**Theorem** [Kirchhoff (1847)] Assume  $n \geq 2$ . Then

$$c_i = \sum_{\mathcal{T} \in \mathbb{T}_i} w(\mathcal{T}), \quad i = 1, 2, \dots, n, \quad (4)$$

where  $\mathbb{T}_i$  is the set of all spanning trees  $\mathcal{T}$  of  $(G, A)$  rooted at vertex  $i$ ,  
and  $w(\mathcal{T})$  is the **weight** of  $\mathcal{T}$ .

## Reordering of a Double Sum

**Proposition** [Tree-Cycle-Identity, Z. Shuai and ML 2009] Let  $c_i$  be given by the Matrix-Tree Theorem. Then the following identity holds.

$$\sum_{i,j=1}^n c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{Q \in \mathcal{Q}} w(Q) \sum_{(r,s) \in E(\mathcal{C}_Q)} F_{rs}(x_r, x_s), \quad (5)$$

where  $F_{ij}(x_i, x_j)$ ,  $1 \leq i, j \leq n$ , are arbitrary functions,  $\mathcal{Q}$  is the set of all spanning unicyclic graphs  $Q$  of  $(\mathcal{G}, A)$ ,  $w(Q)$  is the weight of  $Q$ , and  $\mathcal{C}_Q$  denotes the oriented cycle of  $Q$ .

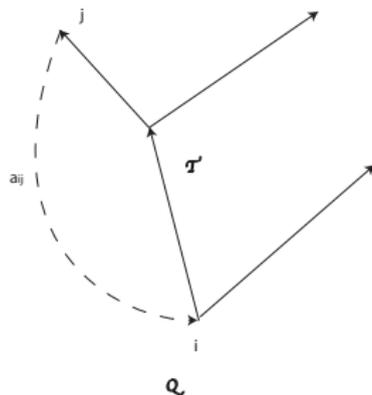
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Proof: Note  $w(\mathcal{T}) a_{ij} = w(Q)$ ,  
where  $Q$  is the unicyclic graph  
obtained by adding an arc  $(j, i)$  to  
 $\mathcal{T}$ .



# Proof of Main Theorem

$$\begin{aligned}\dot{V} &= \sum_{i=1}^n c_i \dot{V}_i \leq \sum_{i,j=1}^n c_i a_{ij} F_{ij}(t, u_i, u_j) \quad (\text{assumption (1)}) \\ &= \sum_{Q \in \mathcal{Q}} w(Q) \sum_{(r,s) \in E(\mathcal{C}_Q)} F_{rs}(t, u_r, u_s) \quad (\text{Proposition}) \\ &\leq 0.\end{aligned}$$

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Is the theorem any good?

## Application I: A Network of Coupled Oscillators

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^n \epsilon_{ij} (\dot{x}_i - \dot{x}_j) = 0, \quad (6)$$

or in systems

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^n \epsilon_{ij} (y_i - y_j). \end{aligned} \quad (7)$$

Each vertex dynamics is given by a damped nonlinear oscillator

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) = 0.$$

Assume that the damping  $\alpha_i \geq 0$  and the potential energy  $F_i(x_i) = \int^{x_i} f_i(s) ds$  has a strictly global minimum at  $x_i = x_i^*$ . Then  $x = x_i^*$  is globally stable (using the Lyapunov function)

$$V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}.$$

## Application I: A Network of Coupled Oscillators

**Theorem** Assume  $\alpha_k > 0$  for some  $k$  and digraph  $\mathcal{G}$  is strongly connected. Then  $E^*(x_1^*, 0, x_2^*, 0, \dots, x_n^*, 0)$  is globally asymptotically stable in  $\mathbb{R}^{2n}$ .

**Proof.**  $V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}$

$$\begin{aligned} \dot{V}_i &= -\alpha_i y_i^2 - \sum_{j=1}^n \epsilon_{ij} (y_i - y_j) y_i \\ &\leq \sum_{j=1}^n \epsilon_{ij} \left[ -\frac{1}{2} (y_i - y_j)^2 - \frac{1}{2} y_i^2 + \frac{1}{2} y_j^2 \right] \\ &\leq \sum_{j=1}^n \epsilon_{ij} F_{ij}(y_i, y_j) \end{aligned}$$

where

$$F_{ij}(y_i, y_j) = -\frac{1}{2} y_i^2 + \frac{1}{2} y_j^2.$$

## Application II: A Single Species Model with Dispersal

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij} x_i), \quad i = 1, 2, \dots, n. \quad (8)$$

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**Theorem** [Z. Shuai and ML (2009)] Assume

- (1) matrix  $(d_{ij})$  is irreducible;
- (2)  $f'_i(x_i) \leq 0, x_i > 0, i = 1, 2, \dots, n; \exists k, f'_k(x_k) \not\equiv 0$  in any open interval of  $\mathbb{R}^+$ ;
- (3) system (8) is uniformly persistent;
- (4) solutions of (8) are uniformly bounded.

Then system (8) has a globally asymptotically stable positive equilibrium  $E^*$ .

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**Theorem** [Z. Shuai and ML (2009)] Assume

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Then system (8) has a globally asymptotically stable positive equilibrium  $E^*$ .

**Note:** Lu and Tacheuchi (1993) proved the result under the assumption  $f_i'(x_i) < 0, x_i > 0$  for all  $i$ , using the theory of monotone dynamical systems.

## Application III: An $n$ -Patch Predator-Prey Model

$$\begin{aligned}x_i' &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij} x_i), & i = 1, 2, \dots, n. & \quad (9) \\y_i' &= y_i(-\gamma_i - \delta_i y_i + \epsilon_i x_i),\end{aligned}$$

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**Theorem** [Z. Shuai and ML (2009)] Assume that  $(d_{ij})$  is irreducible, and that  $\exists k$  such that  $b_k \delta_k > 0$ . Then the positive equilibrium  $E^*$ , whenever it exists, is unique and globally asymptotically stable in the positive cone  $\mathbb{R}_+^{2n}$ .

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$$V_i(x_i, y_i) = \epsilon_i(x_i - x_i^* \ln x_i) + e_i(y_i - y_i^* \ln y_i)$$

## Application IV: A Multi-group Delayed Epidemic Model

$$\begin{aligned} S'_i &= \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j), \\ I'_i &= \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j) - (d_i^I + \gamma_i) I_i, \end{aligned} \quad i = 1, 2, \dots, n. \quad (10)$$

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**Theorem** [Z. Shuai and ML (2009)] Assume that  $B = (\beta_{ij})$  is irreducible. If  $R_0 > 1$ , then the unique endemic equilibrium  $P^*$  for system (10) is globally asymptotically stable in  $\overset{\circ}{\Theta}$ .

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When  $n = 1$ , C. McCluskey proved the global stability with Lyapunov function

$$\begin{aligned}
 V_i &= (S_i - S_i^* + S_i^* \ln \frac{S_i}{S_i^*}) + (I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}) + \\
 &\quad \sum_{j=1}^n \beta_{ji} S_i^* \int_0^{\tau_j} \left( I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.
 \end{aligned}$$

A video on Youtube

<https://www.youtube.com/watch?v=QmWD76jwjbQ>

GRASP Lab, University of Pennsylvania

Each robotic agent has position vector  $r_i = (x_i, y_i) \in \mathbb{R}^2$  and velocity vector  $v_i = \dot{r}_i = (\dot{x}_i, \dot{y}_i)$ .

The system's evolution is governed by Newton's equation

$$\begin{aligned}\dot{r}_i &= v_i, \\ \dot{v}_i &= u_i,\end{aligned} \quad i = 1, \dots, n. \quad (11)$$

Here

- $u_i$ ,  $i = 1, \dots, n$ , define the **control protocol**
- **Formation control** is achieved through communications among agents
- **Network** represents the communication graph (topology)
- A complete communication graph is too costly.

# Hierarchical Potential Clustering (HPC) Protocol

Proposed by J. Maidens and ML:

- 1) Divide the agents into clusters
- 2) Assign a leader to each cluster
- 3) Implement an artificial potential scheme (with a complete graph) within each cluster
- 4) Implement a velocity consensus scheme among the cluster leaders.

## HPC Protocol: control within a cluster $i$

For  $j \neq 1$ , (i.e.  $r_{ij}$  is not a leader in cluster  $i$ )

$$u_{ij} = -\nabla_{r_{ij}} P_{ij} - \sum_k \frac{(\theta_{ij} - \theta_{ik}) \|v_{ij}\|}{\|r_{ij} - r_{ik}\|} \hat{n}(ij),$$

where

$$P_{ij} = \sum_{k=1}^{n_i} P_{ij}^{ik}$$

controls distance of agents in the cluster and

$$\theta_{ij} = \tan^{-1} \left( \frac{\dot{y}_{ij}}{\dot{x}_{ij}} \right)$$

is the heading of agent  $(i, j)$ .

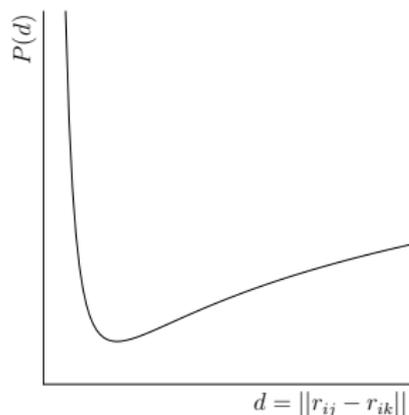


Figure: Potential function  $P_{ij}^{ik}$ .

## HPC Protocol: control among leaders

For  $j = 1$ , (i.e.,  $r_{i1}$  is the leader in cluster  $i$ ), we add additional force to control there heading

$$u_{i1} = -\nabla_{r_{i1}} P_{i1} - \sum_k \frac{(\theta_{i1} - \theta_{ik}) \|v_{i1}\|}{\|r_{i1} - r_{ik}\|} \hat{n}(i1) \\ + \sum_{h \in N_i} b_{ih} (v_{h1} - v_{i1})$$

where matrix  $B = (b_{ij})$  is any nonnegative **irreducible** matrix. The correspond communication graph  $\mathcal{G}_B$  among leaders is **strongly connected**.

# Formation Stabilization Problem

**Definition** A control protocol is said to solve the **formation stabilization problem** if solutions of (11) converge asymptotically to a state such that

- (a) the relative positions of each agent  $(i, j)$  within a cluster are such that a local minimum of the total vertex potential  $P_{ij}$  is achieved,
- (b) the headings of any two agents  $(i, j)$  and  $(h, k)$  satisfy  $\theta_{ij} = \theta_{hk}$ .

# Main Result

**Theorem** (J. Maidens and ML, 2013)

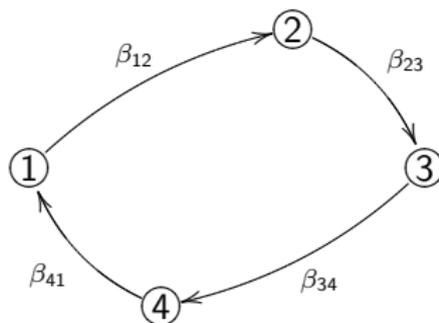
Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph  $\mathcal{G}_B$  is **strongly connected**.

# Main Result

**Theorem** (J. Maidens and ML, 2013)

Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph  $\mathcal{G}_B$  is **strongly connected**.

An example graph that is strongly connected:

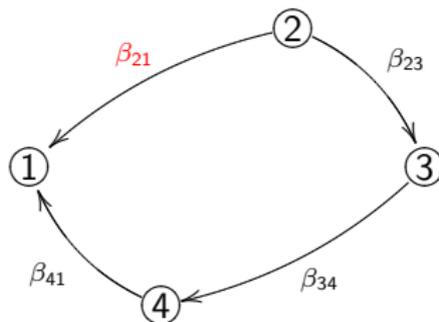


# Main Result

**Theorem** (J. Maidens and ML, 2013)

Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph  $\mathcal{G}_B$  is **strongly connected**.

An example graph that is **not** strongly connected:



# Simulations

- Clustering without control protocol
  - Video 1
- Clustering without leader control
  - Video 2
  - Video 3
- Clustering with leader control
  - Video 4
  - Video 5

# Synchronization

Synchronization of metronomes: a video

<https://www.youtube.com/watch?v=Aaxw4zbULMs>

# Coupled Oscillators Revisited

Consider a system of coupled oscillators:

$$\ddot{x}_i + f_i(x_i) + \sum_{j=1}^n \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0,$$

Assume that  $f_i(x_i)$  and  $F_i(x_i) = \int_i^x f_i(t)dt$  satisfy

(C<sub>1</sub>)  $f_i(x_i)x_i > 0$ ,  $x_i \neq 0$ ,  $i = 1, 2, \dots, n$ ,

(C<sub>2</sub>)  $F_i(x_i) \rightarrow \infty$  as  $|x_i| \rightarrow \infty$ ,  $i = 1, 2, \dots, n$ .

Both (C<sub>1</sub>) and (C<sub>2</sub>) are satisfied for  $f_i(x_i) = x_i^3$ .

# Global Synchronization

**Definition:** System (29) is said to achieve global synchronization if, for every solution  $x(t)$  of system (29) and all  $1 \leq i, j \leq n$ ,

$$\dot{x}_i(t) - \dot{x}_j(t) = 0.$$

**Question:** Under what conditions of matrix  $A = (a_{ij})$  does the system (29) achieves global synchronization?

# A Theorem

## Theorem (P. Du and ML 2015)

*In system (29), suppose that the direct graph  $\mathcal{G}_A$  is strongly connected, and assumptions  $(C_1)$  and  $(C_2)$  are satisfied. Then system (29) achieves global synchronization.*

Considering the equivalent system

$$\begin{aligned}\dot{x}_i &= y_i \\ \dot{y}_i &= -f_i(x_i) + \sum_{j=1}^n \epsilon_{ij}(y_j - y_i)\end{aligned}$$

Using Lyapunov functions:

$$V_i = \frac{1}{2}y_i^2 + F_i(x_i),$$

and

$$V = \sum_{i=1}^n c_i V_i.$$