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Synchronization in networks of nonlinear dynamical systems coupled via a directed graph

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Abstract
We study synchronization in an array of coupled identical nonlinear dynamical systems where the coupling topology is expressed as a directed graph and give synchronization criteria related to properties of a generalized Laplacian matrix of the directed graph. In particular, we extend recent results by showing that the array synchronizes for sufficiently large cooperative coupling if the underlying graph contains a spanning directed tree. This is an intuitive yet nontrivial result that can be paraphrased as follows: if there exists a dynamical system which influences directly or indirectly all other systems, then synchronization is possible for strong enough coupling. The converse is also true in general.

Mathematics Subject Classification: 05C50, 15A48, 34C15, 34D20

1. Introduction

For the last decade or so, synchronization in arrays of coupled chaotic systems has been widely studied [1–9]. Of interest to this paper is the case where the dynamical systems are coupled via a directed graph, i.e. the individual systems are considered as vertices and system v influences the dynamics of system w if and only if (v, w) is a directed edge of the underlying directed graph. Lyapunov's direct method has been used successfully to derive sufficient conditions for synchronization in such arrays [2, 4, 7, 8]. When the graph is undirected, such sufficient conditions depend on the second smallest eigenvalue of the Laplacian matrix, also known as algebraic connectivity [10]. In particular, since the algebraic connectivity is positive for connected graphs, it was shown in [2] that sufficiently strong cooperative coupling synchronizes the array when the underlying undirected graph is connected.

Recently, generalizations of algebraic connectivity to directed graphs have been proposed and used to derive synchronization in coupled arrays of dynamical systems [11, 12]. For instance, in [12] it was shown that sufficiently large cooperative coupling synchronizes the array if the underlying graph is strongly connected. The purpose of this paper is to

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extend this result to connected directed graphs which are not strongly connected. We derive a synchronization criterion which depends on algebraic properties of the underlying graph. Furthermore, we show that sufficiently large cooperative coupling synchronizes the array if the underlying graph contains a spanning directed tree. This is an intuitive result since this graph-theoretical condition implies the existence of a system (located at the root of the tree) which directly or indirectly influences all other systems. This result is obtained by using a mixture of results from linear algebra, graph theory and stability theory. When the underlying graph does not contain a spanning directed tree, we show that in general the array will not synchronize, especially if the individual systems are chaotic.

In section 2 we introduce results from stability theory which will be useful to this paper. In section 3 we present some definitions and results in graph theory. Finally, the main synchronization results are presented in section 4.

We denote the vector of all 1s by **1** and the jth unit vector by e_j. The matrix of all 1s is denoted by J. The identity matrix is denoted by I. The dimensions of these vectors and matrices will be clear from the context. A (not necessarily symmetric) real matrix A is positive definite (semidefinite) if $x^T A x > 0$ ($\geqslant 0$) for all nonzero x. We denote this as $A \succ 0$ ($A \succeq 0$). The Kronecker product of an n by m matrix A and a p by q matrix B is the np by mq matrix $A \otimes B$ defined as

$$A \otimes B = \begin{pmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \\ A_{n1}B & & A_{nm}B \end{pmatrix}.$$

2. Lyapunov's direct method

Definition 2.1. *Given an m by m matrix V, a function $f(y,t) : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is V-uniformly decreasing if $(y-z)^T V(f(y,t) - f(z,t)) \leqslant -\mu\|y-z\|^2$ for some $\mu > 0$ and all $y, z \in \mathbb{R}^m$ and $t \in \mathbb{R}$.*

By the mean value theorem, a differentiable function $f(y,t)$ is V-uniformly decreasing if and only if $V(\partial f(x,t)/\partial x) + \delta I \preceq 0$ for some $\delta > 0$ and all x, t [13]. Consider the following synchronization result which generalizes the results in [2, 8, 14] for the coupled network of identical dynamical systems with state equations:

$$\dot{x} = (f(x_1,t), \ldots, f(x_n,t))^T + (C(t) \otimes D(t))x + u(t), \tag{1}$$

where $x = (x_1, \ldots, x_n)^T$, $u = (u_1, \ldots, u_n)^T$ and $C(t)$ is a zero row sums matrix for all t. Each x_i is a state vector in $\mathbb{R}^m$. $C(t)$ is an n by n matrix and $D(t)$ is an m by m matrix for each t.

Theorem 2.1. *Let $Y(t)$ be an m by m time-varying matrix and V be an m by m symmetric positive definite matrix such that $f(x,t) + Y(t)x$ is V-uniformly decreasing. Then the network of coupled dynamical systems in equation (1) synchronizes in the sense that $\|x_i - x_j\| \to 0$ as $t \to \infty$ for all i, j if the following two conditions are satisfied:*

(i) $\lim_{t\to\infty} \|u_i - u_j\| = 0$ *for all i, j.*

(ii) There exists an n by n symmetric irreducible zero row sums matrix U with nonpositive off-diagonal elements such that

$$(U \otimes V)(C(t) \otimes D(t) - I \otimes Y(t)) \preceq 0$$

for all t.

Proof. Construct the Lyapunov function $g(x) = \frac{1}{2}x^T(U \otimes V)x$. As in the proof of theorem 3.2 in [4], the hypotheses imply that the derivative of g along trajectories of equation (1) is given by

$$\begin{aligned}
\dot{g} &= x^T(U \otimes V)\dot{x} \\
&= x^T(U \otimes V)\begin{pmatrix} f(x_1, t) + Y(t)x_1 + u_1(t) \\ \vdots \\ f(x_n, t) + Y(t)x_n + u_n(t) \end{pmatrix} + x^T(U \otimes V)(C(t) \otimes D(t) - I \otimes Y(t))x \\
&\leqslant \sum_{i<j} -U_{ij}(x_i - x_j)^T V(f(x_i, t) + Y(t)x_i - f(x_j, t) - Y(t)x_j + u_i(t) - u_j(t)) \\
&\leqslant \sum_{i<j} -U_{ij}(-\mu\|x_i - x_j\|^2 + (x_i - x_j)^T V(u_i(t) - u_j(t))).
\end{aligned}$$

Note that $-U_{ij} \geqslant 0$ for $i < j$. For each $-U_{ij} > 0$ and $\delta > 0$, and sufficiently large t, $(u_i(t) - u_j(t))$ is small enough such that if $\|x_i - x_j\| \geqslant \delta$, then $\dot{g} \leqslant -(\mu/2)\|x_i - x_j\|^2$. This implies that for large enough t, $\|x_i - x_j\| < \delta$. Therefore $\lim_{t\to\infty} \|x_i - x_j\| = 0$. Irreducibility of U implies that enough U_{ij} are nonzero to ensure $\|x_i - x_j\| \to 0$ for all i and j. □

3. Laplacian matrices of directed graphs

Definition 3.1. *For an irreducible square matrix B with nonpositive off-diagonal elements, the functions $\beta(B)$ and $\gamma(B)$ are defined as follows: decompose B uniquely as $B = L+D$, where L is a zero row sum matrix and D is a diagonal matrix. Let w be the unique positive vector such that $w^T L = 0$ and $\max_v w_v = 1$. The vector w exists by Perron–Frobenius theory [15]. Let $W = \mathrm{diag}(w)$. Then $\gamma(B) = \min_{x\neq 0, x\perp\mathbf{1}}(x^T WBx/(x^T(W - ww^T/\sum_v w_v)x))$ and $\beta(B) = \min_{x\neq 0}(x^T WBx/x^T Wx)$.*

Lemma 3.1. *Let B be an irreducible matrix with nonpositive off-diagonal elements and nonnegative row sums with decomposition $B = L + D$ as in definition 3.1. Then $\gamma(B) > 0$, $\beta(B) \geqslant 0$. Furthermore, $\beta(B) > 0$ if and only if $D \neq 0$.*

Proof. It is easy to see that $D \succeq 0$, $\gamma(B) \geqslant \min_{x\perp\mathbf{1}, x\neq 0}(x^T WBx/x^T x) = \frac{1}{2}\lambda_2(WB + B^T W)$ and $\beta(B) \geqslant \min_{x\neq 0}(x^T WBx/x^T x) = \frac{1}{2}\lambda_{\min}(WB + B^T W)$ where $\lambda_{\min}$ and λ_2 denote the smallest and the second smallest eigenvalues, respectively. Since WL has zero column sums, $WL + L^T W$ is a zero row sums matrix. As w is a positive vector, this in turns implies that $WB + B^T W$ is irreducible and has nonnegative row sums and thus $\gamma(B) \geqslant \frac{1}{2}\lambda_2(WB+B^T W) > 0$ [15]. By Gershgorin's circle criterion, $\beta(B) \geqslant \frac{1}{2}\lambda_{\min}(WB+B^T W) \geqslant 0$. Suppose that $D = 0$. Then WB is a zero row sums and zero column sums matrix so that $\mathbf{1}^T WB = WB\mathbf{1} = 0$ and thus $\beta(B) = 0$. If $D \neq 0$, there exists i such that $D_{ii} > 0$ and thus $(WB + B^T W)_{ii} > |\sum_{j\neq i}(WB + B^T W)_{ij}|$. By [16], $WB + B^T W$ is nonsingular and thus $\beta(B) > 0$. □

Lemma 3.2. *If B is an irreducible zero row sums matrix with nonpositive off-diagonal elements, then there exists an irreducible symmetric zero row sums matrix U with nonpositive off-diagonal elements such that $U(B - \alpha I) \succeq 0$ for all $\alpha \leqslant \gamma(B)$.*

Proof. Let w be a positive vector such that $w^T B = 0$ and $\max_v w_v = 1$ with $W = \mathrm{diag}(w)$. Define $U = W - (ww^T/\sum_v w_v)$. Then U is a symmetric positive semidefinite irreducible zero row sums matrix with nonpositive off-diagonal elements. Since $\mathbf{1}^T U = U\mathbf{1} = 0$, we have

$\mathbf{1}^T U(B-\alpha I) = U(B-\alpha I)\mathbf{1} = 0$. Therefore $U(B-\alpha I) \succeq 0$ if $\min_{y\perp\mathbf{1}} y^T U(B-\alpha I)y \geqslant 0$. Since $w^T B = 0$, $U(B-\alpha I) = WB - \alpha U$. Therefore for $y \perp \mathbf{1}$, $y^T U(B-\alpha I)y \geqslant \gamma(B)y^T U y - \alpha y^T U y \geqslant 0$. □

Definition 3.2. *Let A be a zero row sums matrix written in Frobenius normal form [17]:*

$$A = P \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1k} \\ & B_2 & \cdots & B_{2k} \\ & & \ddots & \vdots \\ & & & B_k \end{pmatrix} P^T, \tag{2}$$

where P is a permutation matrix and B_i are square irreducible matrices. Then $\eta(A)$ is defined as

$$\eta(A) = \min(\beta(B_1), \beta(B_2), \ldots, \beta(B_{k-1}), \gamma(B_k)).$$

Definition 3.3. *For a weighted directed graph, its adjacency matrix A is defined as $A_{ij} = \rho$ if and only if there is a directed edge with weight ρ from vertex i to vertex j. The outdegree of a vertex is the sum of the weights of all edges emanating from it. The Laplacian matrix is defined as $L = D - A$, where A is the adjacency matrix and D is the diagonal matrix of vertex outdegrees.*

Thus the Laplacian matrix is a zero row sums matrix. A graph is strongly connected if there exists a directed path between any ordered pair of distinct vertices. The decomposition of the Laplacian matrix into Frobenius normal form corresponds to decomposing the graph into maximally strongly connected subgraphs and is a standard problem in graph algorithms that can be solved in linear time [18]. A reversal of a graph is obtained by reversing the orientation of all the edges. If a graph has adjacency matrix A, then its reversal has adjacency matrix A^T. A graph is a directed tree if it is a tree as an undirected graph and there is a directed path from the root to every other vertex.

Lemma 3.3. *Let L be the Laplacian matrix of a graph with nonnegative weights. Then $\eta(L) > 0$ if and only if the reversal of the graph contains a spanning directed tree.*

Proof. Let V_i be the subset of vertices corresponding to B_i. If the reversal of the graph contains a spanning directed tree, then its root must be in V_k. Furthermore, there is a directed path from every other vertex to the root. If $D_i = 0$ for some $i < k$, then there are no paths from V_i to V_k, a contradiction. Therefore by lemma 3.1, $\eta(L) > 0$.

If the reversal of the graph does not have a spanning directed tree, then there exist a pair of vertices v and w such that for all vertices z, there is either no directed paths from v to z or no directed paths from w to z [19]. Let $R(v)$ and $R(w)$ be the set of vertices reachable from v and w respectively, which must necessarily be disjoint. Let $H(v)$ and $H(w)$ be the subgraphs of G corresponding to $R(v)$ and $R(w)$, respectively. Expressing the Laplacian matrix of $H(v)$ in Frobenius normal form, let $B(v)$ be the square irreducible matrix in the lower right corner. We define $B(w)$ similarly. Note that $B(w)$ and $B(v)$ are zero row sums singular matrices. By the construction, it is easy to see that $B(v) = B_i$ and $B(w) = B_j$ in the Frobenius normal form (equation (2)) of L for some i, j. By lemma 3.1, $\beta(B_i) = \beta(B_j) = 0$ and thus $\eta(L) = 0$ since either $i \neq k$ or $j \neq k$. □

At first glance, $\eta(A)$ appears not to be well-defined in definition 3.2 because even though the matrices B_i are uniquely defined (up to simultaneous row and column permutation), their ordering within the Frobenius normal form (equation (2)) is not [17]. However, it is easy to see that the lower right block B_k is uniquely defined if and only if the reversal of the graph contains a spanning directed tree. In this case $\eta(A)$ is well-defined. If B_k is not uniquely defined, $\eta(A) = 0$ by lemma 3.3 and $\eta(A) = 0$ for *any* admissible ordering of the B_i within the Frobenius normal form. Thus $\eta(A)$ is well-defined for any A.

4. Main results

Consider an array of dynamical systems coupled via a directed graph with state equation (equation (1)) where we require that C is a constant zero row sums matrix, i.e. the state equation is

$$\dot{x} = (f(x_1, t), \dots, f(x_n, t))^T + (C \otimes D(t))x + u(t). \tag{3}$$

The matrix C describes the coupling topology and the matrix D describes the coupling between two dynamical systems. Together the term $(C \otimes D(t))x$ denotes a time-varying coupling where the coupling topology does not change with time. The underlying graph is defined as follows: there is an edge from system i to system j if and only if there is a coupling term from system i to system j, i.e. $C_{ji} \neq 0$. In other words, the Laplacian matrix of the *reversal* of the underlying graph is C. The following theorem gives sufficient conditions for synchronization which is related to the quantity $\eta(C)$ associated with the underlying graph.

Theorem 4.1. *Consider an array of dynamical systems coupled via a directed graph with state equation (3). The array synchronizes in the sense that* $\forall i, j$, $\lim_{t\to\infty} \|x_i - x_j\| = 0$ *if the following conditions are satisfied:*

(i) C is a zero row sums matrix with nonpositive off-diagonal elements,
(ii) $\forall i, j$, $\lim_{t\to\infty} \|u_i - u_j\| = 0$,
(iii) $f(x, t) + D(t)x$ is V-uniformly decreasing for some symmetric positive definite V,
(iv) $VD(t) \preceq 0$ and is symmetric for all t,
(v) $\eta(C) \geqslant 1$.

Proof. Without loss of generality, we can assume that the permutation matrix P in the Frobenius normal form of C (equation (2)) is equal to the identity matrix. Let $\tilde{x}$ and $\tilde{u}$ be the part of the state vector x and input vector u corresponding to B_k, with $\tilde{x} = (x_s, x_{s+1}, \dots, x_n)^T$, $\tilde{u} = (u_s, u_{s+1}, \dots, u_n)^T$. The state equation for $\tilde{x}$ is then

$$\dot{\tilde{x}} = (f(x_s, t), \dots, f(x_n, t))^T + (B_k \otimes D(t))\tilde{x} + \tilde{u}. \tag{4}$$

By theorem 2.1 the array in equation (4) synchronizes if there exists an irreducible symmetric zero row sums matrix $\tilde{U}$ with nonpositive off-diagonal elements such that $(\tilde{U} \otimes V)((B_k - I) \otimes D(t)) \preceq 0$. Since $VD(t)$ is symmetric negative semidefinite, this is equivalent to $\tilde{U}(B_k - I) \succeq 0$. Since $\eta(C) \geqslant 1$ implies $\gamma(B_k) \geqslant 1$, such a matrix $\tilde{U}$ exists by lemma 3.2 and the array in equation (4) synchronizes and thus $\lim_{t\to\infty} \|x_i - x_j\| = 0$ for $s \leqslant i \leqslant j \leqslant n$. Since the systems in equation (4) are synchronized, we can collapse their dynamics to that of a single system. In particular, the state equation for x_s can be written as $\dot{x}_s = f(x_s, t) + u_s(t) + \phi_s(t)$ where $\phi_s(t) \to 0$ as $t \to \infty$.

Let us rewrite the Frobenius normal form of C as

$$C = \begin{pmatrix} F + G & H \\ & B_k \end{pmatrix},$$

where F is a square zero row sums matrix and G is diagonal. Note that $F + G$ is block upper-triangular with the diagonal blocks equal to $B_1, \dots, B_{k-1}$. Then the dynamics of $(x_1, \dots, x_s)$

can be written as:

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_s \end{pmatrix} = \begin{pmatrix} f(x_1,t) \\ \vdots \\ f(x_s,t) \end{pmatrix} + \left(\tilde{C} \otimes D(t)\right) \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} + \begin{pmatrix} \phi_1(t) + u_1(t) \\ \vdots \\ \phi_s(t) + u_s(t) \end{pmatrix}, \tag{5}$$

where

$$\tilde{C} = \begin{pmatrix} F+G & h \\ 0 & 0 \end{pmatrix}$$

and $\phi_i(t) \to 0$ as $t \to \infty$ and h is a vector of the row sums of H. Since C has zero row sums, this means that the elements of $-h$ are equal to the diagonal elements of G. The functions ϕ_i can be considered as residual errors that occurred when replacing $x_i, i > s$ in the state equation (equation (3)) with x_s. Construct the following matrices:

$$R = \begin{pmatrix} I & -\mathbf{1} \end{pmatrix}, \qquad Q = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

If C is an n by n matrix and B_k is an l by l matrix, then the dimensions of $\tilde{C}$, R, Q and F are $n-l+1$ by $n-l+1$, $n-l$ by $n-l+1$, $n-l+1$ by $n-l$ and $n-l$ by $n-l$, respectively. It is easy to verify that $R\tilde{C}Q = F+G$ and $R\tilde{C}QR = R\tilde{C}$. Decompose B_i as $B_i = L_i + D_i$ where L_i is a zero row sum matrix and D_i is a diagonal matrix. Let $W = \mathrm{diag}(w_1, \ldots, w_{k-1})$ where w_i are positive vectors such that $w_i^T L_i = 0$ and $\max_v w_i(v) = 1$. Note that $I \succeq W \succ 0$.

Let $\Delta = \mathrm{diag}(\alpha_1 I_1, \ldots, \alpha_{k-1} I_{k-1})$ where I_j are identity matrices of the same dimension as B_j and $\alpha_j > 0$. Let $Z = \Delta R\tilde{C}Q\Delta^{-1}$. If we choose α_j much larger than α_i for $j > i$, then Z is nearly block-diagonal with the blocks equal to $B_1, B_2, \ldots, B_{k-1}$. Now let $U = R^T \Delta W \Delta R$. It is easy to see that U is a symmetric irreducible zero row sums matrix and has nonpositive off-diagonal elements.

Then $U(\tilde{C} - I) = R^T \Delta W \Delta R(\tilde{C} - I) = R^T \Delta(WZ - W)\Delta R$. By choosing appropriate α_j Z can be made as close to block diagonal as possible. The condition $\eta(C) \geqslant 1$ implies that $\beta(B_i) \geqslant 1$ for $i < k$ and thus $WZ - W \succeq -\epsilon I$ for arbitrarily small ϵ and theorem 2.1 can again be applied (when ϵ is small enough) to show that equation (5) synchronizes. □

Assuming conditions (i)–(iv) are satisfied in theorem 4.1, the quantity $\eta(C)$, which depends on the underlying graph, provides a bound on the amount of coupling needed to synchronize the array.

Corollary 4.1. *Consider the array of coupled dynamical systems with state equations:*

$$\dot{x} = (f(x_1,t), \ldots, f(x_n,t))^T + \kappa(C \otimes D(t))x + u(t), \tag{6}$$

where κ is a scalar. Assume that conditions (i)–(iv) in theorem 4.1 are satisfied. If the underlying weighted directed graph, i.e. the graph whose reversal has Laplacian matrix C, contains a spanning directed tree, then the array in equation (6) synchronizes for sufficiently large $\kappa > 0$.

Proof. By theorem 4.1 the array synchronizes if $\kappa\eta(C) \geqslant 1$. Since $\eta(C) > 0$ by lemma 3.3, the result follows. □

Corollary 4.2. *Suppose that f has a bounded Jacobian matrix, i.e. $\|\partial f(x,t)/\partial x\| \leqslant M$ for all x, t. Suppose also that C is a zero row sums matrix with nonpositive off-diagonal elements, $D(t)$ is symmetric and for some $\epsilon > 0$, $D(t) \preceq -\epsilon I$ for all t. Suppose further that* $\lim_{t\to\infty} \|u_i - u_j\| = 0$ *for all i, j. If the underlying weighted directed graph contains a spanning directed tree, then the array in equation (6) synchronizes for sufficiently large $\kappa > 0$.*

Proof. If f has a bounded Jacobian matrix, we can choose $V = I$ and $f(x,t) + \psi D(t)x$ is V-uniformly decreasing for sufficiently large scalar ψ. The result then follows from corollary 4.1. $\square$

Corollaries 4.1 and 4.2 are intuitive results as the condition that the underlying graph contains a spanning directed tree implies that there exists a system that directly or indirectly couples into all other systems. So one can expect all other systems to synchronize to this system when the coupling is sufficiently large.

If the underlying graph does not contain a spanning directed tree, then the proof of lemma 3.3 shows that there exists two groups of systems which are not influenced by other systems. Therefore, in general, these two groups of systems will not synchronize with each other, especially when the systems are chaotic and exhibit sensitive dependence on initial conditions. In this case the Frobenius normal form can be written as

$$A = P \begin{pmatrix} B_1 & B_{12} & \cdots & & & B_{1k} \\ & B_2 & \cdots & & & B_{2k} \\ & & \ddots & & & \vdots \\ & & & B_r & 0 & 0 \\ & & & & \ddots & 0 \\ & & & & & B_k \end{pmatrix} P^T. \tag{7}$$

Let V_i denote the set of systems corresponding to B_i and assume conditions (i)–(iv) in theorem 4.1 are satisfied. In this case the systems within V_j will synchronize with each other if $\gamma(B_j) > 1$ for each $r \leqslant j \leqslant k$. Thus we have at least $k - r + 1$ separate clusters of synchronized systems. Similar arguments as above can be used to show that the systems belonging to $\bigcup_{i=1}^{r-1} V_i$ are synchronized with each other if for each $1 \leqslant j \leqslant r-1$, $\beta(B_j) \geqslant 1$ and for each $r \leqslant j \leqslant k$, B_{ij} are constant row sums matrices with the row sum of B_{ij} equal to the row sum of $B_{i'j}$ for $1 \leqslant i < i' < r$.

In [3] a Lyapunov exponents based approach is used to derive synchronization criteria. This method is based on numerical approximation of Lyapunov exponents and can only provide local results. The requirement that the underlying graph contains a spanning directed tree also exists in the Lyapunov exponents approach to synchronization. In this approach, the synchronization criteria depend on the nonzero eigenvalue of C with the smallest real part. For chaotic systems, this eigenvalue needs to have a positive real part for the array to synchronize. In [12] it was shown that this eigenvalue has a positive real part if and only if the underlying graph contains a spanning directed tree.

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