

Dynamical Systems on Networks: From Epidemics to Flight of Drones

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Outline of Talks

Part I: Global Stability Problems in Heterogeneous Epidemic Models

I.1: Modeling Infectious Diseases in Heterogeneous Populations

- ▶ Simple epidemic models and their dynamics
- ▶ Basic reproduction number and the threshold theorem
- ▶ Multi-group models for heterogeneous populations

I.2: Global-Stability Problem in Multi-Group Models

- ▶ Global-stability problem and Lyapunov functions
- ▶ A Lyapunov function for multi-group models
- ▶ Why is global-stability difficult to prove?

I.3: Matrix-Tree Theorem in Graph Theory

- ▶ Rooted directed trees and unicyclic graphs
- ▶ Kirchhoff's Matrix-Tree Theorem

I.4: How do all of these come together?

- ▶ Global-stability result for multi-group models.

Outline of Talks

Part II: Dynamical Systems on Networks

II.1: Dynamical systems on networks

- ▶ A network as a directed graph
- ▶ Dynamical systems on networks
- ▶ Examples

I.2: Global-stability problem

- ▶ Global-stability problem and Lyapunov functions
- ▶ A general theorem
- ▶ Applications

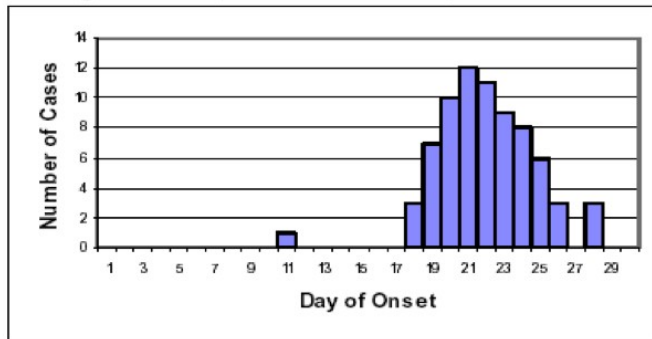
I.3: Flight formation control for drones

- ▶ Network of autonomous robotic agents
- ▶ Flight formation problems and HPC control protocol
- ▶ Simulations

I.4: Synchronization problems

- ▶ Synchronization of metronomes, a video
- ▶ Global synchronization of coupled oscillators.

How to Model an Epidemic?



An Epidemic Curve

Dynamical Systems on Networks: From Epidemics to Flight of Drones

Simple Epidemic Models



How to Model an Epidemic?

Simple Epidemic Models

n-Group Models

An *n*-Group SIR Model

Previous Results

Global-Stability and Lyapunov Functions

Derivative of V

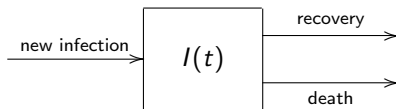
Choosing Constants v_k

The Matrix-Tree Theorem

The Double Sum H_n

Main Results

Summary



$$\begin{aligned} I'(t) &= \boxed{\text{Incidence Rate}} - \boxed{\text{Recovery Rate}} - \boxed{\text{Death Rate}} \\ &= f(I(t), S(t), N(t)) - \gamma I(t) - d I(t) \end{aligned}$$

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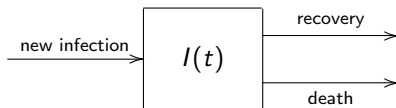
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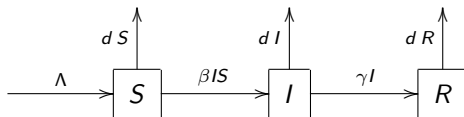
$$\begin{aligned} I'(t) &= \boxed{\text{Incidence Rate}} - \boxed{\text{Recovery Rate}} - \boxed{\text{Death Rate}} \\ &= f(I(t), S(t), N(t)) - \gamma I(t) - d I(t) \end{aligned}$$

$$f(I, S, N) = \beta I S : \quad \text{bilinear incidence}$$

$$f(I, S, N) = \lambda \frac{I S}{N} : \quad \text{proportionate incidence}$$

A Single-Group SIR Model

S : Susceptibles I : Infectious R : *Removed*



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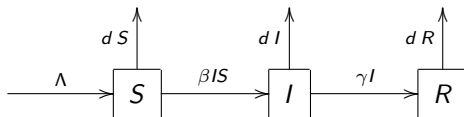
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S : Susceptibles I : Infectious R : *Removed*

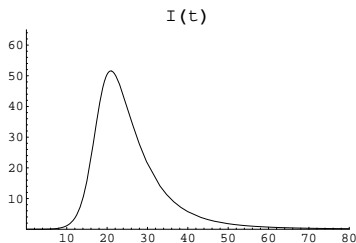


$$S' = \Lambda - \beta IS - dS$$

$$I' = \beta IS - (\gamma + d)I$$

$$R' = \gamma I - dR$$

A Single-Group SIR Model



Numerical output I: epidemic case

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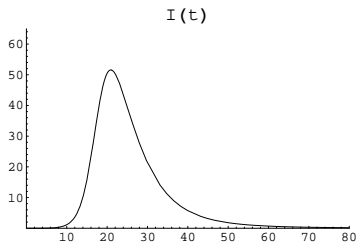
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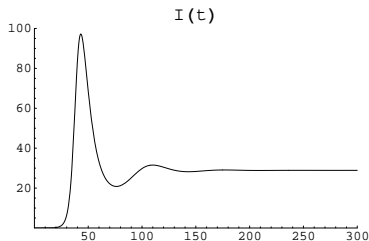
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Numerical output I: epidemic case



Numerical output II: endemic case

Threshold Theorem

The **basic reproduction number** is

$$R_0 = \frac{\beta \Lambda}{(\gamma + d)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d}$$

The average secondary infections produced by a single infective during its entire infectious period.

Threshold Theorem

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The average secondary infections produced by a single infective during its entire infectious period.

Theorem (Threshold Theorem)

- (1) If $R_0 \leq 1$, then the disease-free equilibrium $P_0 = (\Lambda/d, 0)$ is stable and attracts all solutions in R_+^2 .
- (2) If $R_0 > 1$, then P_0 is unstable, and a unique endemic (positive) equilibrium P^* is stable and attracts all positive solutions in R_+^2 .

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Proof uses the Poincaré-Bendixson theory for 2d systems.

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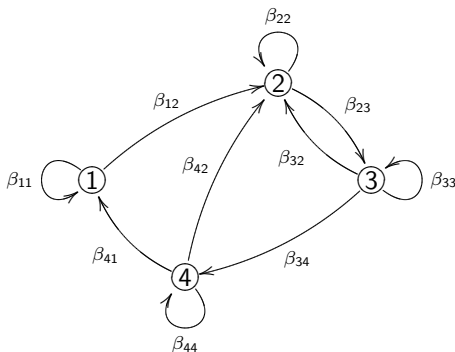
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n -Group Models for Heterogeneous Populations



Each circled number represents a homogeneous group.

β_{jk} : transmission coefficient between I_j and S_k .

n -Group Models for Heterogeneous Populations

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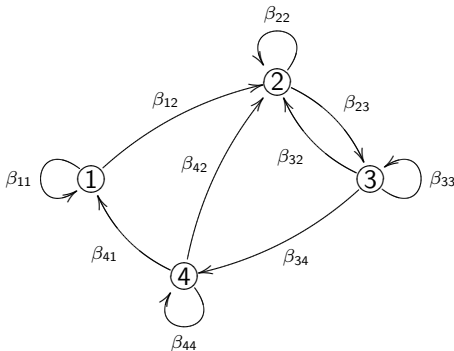
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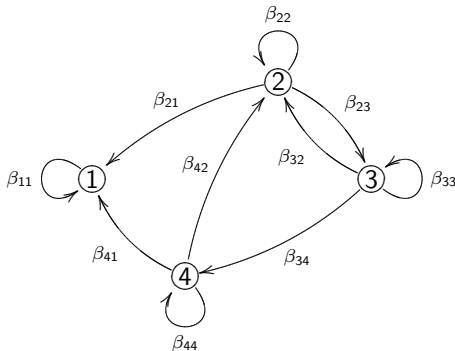
$$B = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 & 0 \\ 0 & \beta_{22} & \beta_{23} & 0 \\ 0 & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & 0 & \beta_{44} \end{bmatrix}$$

Transmission Matrix B is **irreducible**.

n -Group Models for Heterogeneous Populations

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$$B = \begin{bmatrix} \beta_{11} & \mathbf{0} & 0 & 0 \\ \mathbf{\beta_{21}} & \beta_{22} & \beta_{23} & 0 \\ 0 & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & 0 & \beta_{44} \end{bmatrix}$$

Transmission Matrix B is **reducible**.

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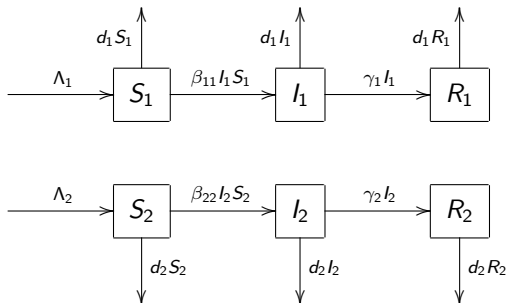
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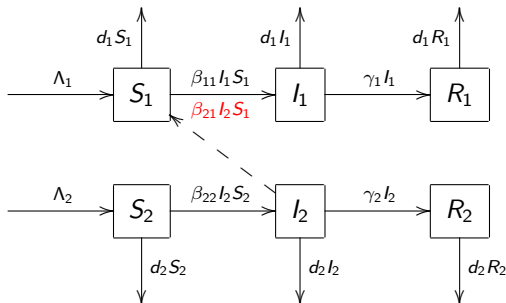
A Two-Group SIR Model



Incidence terms (bilinear):

- ▶ Group 1: $\beta_{11} I_1 S_1$
- ▶ Group 2: $\beta_{22} I_2 S_2$

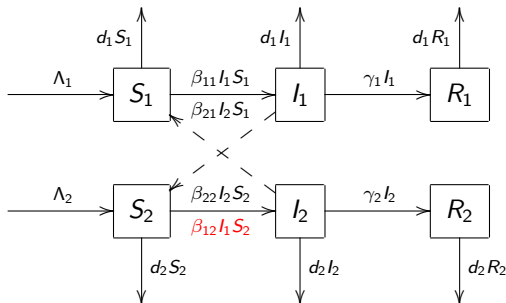
A Two-Group SIR Model



Incidence terms (bilinear):

- ▶ Group 1: $\beta_{11} I_1 S_1 + \beta_{21} I_2 S_1$
- ▶ Group 2: $\beta_{22} I_2 S_2$

A Two-Group SIR Model



Incidence terms (bilinear):

- ▶ Group 1: $\beta_{11} I_1 S_1 + \beta_{21} I_2 S_1$
- ▶ Group 2: $\beta_{22} I_2 S_2 + \beta_{12} I_1 S_2$

An n -Group SIR Model

$$\begin{cases} S'_k = \Lambda_k - d_k S_k - \sum_{j=1}^n \beta_{jk} I_j S_k, \\ I'_k = \sum_{j=1}^n \beta_{jk} I_j S_k - (d_k + \gamma_k) I_k, \end{cases} \quad k = 1, \dots, n.$$

Mathematical Questions:

- If $R_0 > 1$, is P^* unique?

An n -Group SIR Model

$$\begin{cases} S'_k = \Lambda_k - d_k S_k - \sum_{j=1}^n \beta_{jk} I_j S_k, \\ I'_k = \sum_{j=1}^n \beta_{jk} I_j S_k - (d_k + \gamma_k) I_k, \end{cases} \quad k = 1, \dots, n.$$

Mathematical Questions:

- ▶ If $R_0 > 1$, is P^* unique?
- ▶ When P^* is unique, is it globally stable?

Previous Results on GAS of P^*

For Models using **bilinear incidence**:

- Lajmanovich and Yorke (1976)
 - ▶ n -group SIS model, by Lyapunov function
 - ▶ later extended by Nold, Hirsch
- Hethcote (1975)
 - ▶ n -group SIR model with no vital dynamics
- Thieme (1983)
 - ▶ n -group SEIRS model, small latent and immune periods
- Beretta and Capasso (1986)
 - ▶ n -group SIR model, constant group sizes
- Lin and So (1993)
 - ▶ n -group SIRS model, constant group sizes
 - ▶ β_{ij} ($i \neq j$) small

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Non-uniqueness of P^* when $R_0 > 1$

- ▶ Lin (1992)
 n -group model for HIV
- ▶ Huang, Cooke, Castillo-Chavez (1992)
 n -group model for HIV with delay

These models use **proportionate incidence**.

Global-Stability and Lyapunov Functions

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Summary

Consider a general system of ODE

$$x' = F(x), \quad x \in D \subset \mathbb{R}^d.$$

\bar{x} is an **equilibrium** if $F(\bar{x}) = 0$.

An equilibrium \bar{x} is globally stable in D if it is locally stable and all solutions in D converge to \bar{x} as $t \rightarrow \infty$.

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\bar{x} is an equilibrium if $F(\bar{x}) = 0$.

An equilibrium \bar{x} is **globally stable in D** if it is locally stable and all solutions in D converge to \bar{x} as $t \rightarrow \infty$.

Global-Stability and Lyapunov Functions

Theorem (Lyapunov)

Suppose \exists a Lipschitz function $V(x)$ such that

$$(1) \quad V(x) \geq V(\bar{x}) \text{ and } V(x) = V(\bar{x}) \iff x = \bar{x}.$$

$$(2) \quad \overset{*}{V}(x) = \nabla V(x) \cdot F(x) \leq 0, \quad x \in D, \text{ and}$$

$$\overset{*}{V}(x) = 0 \iff x = \bar{x}.$$

Then \bar{x} is globally stable in D .

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$$\overset{*}{V}(x) = 0 \iff x = \bar{x}.$$

Then \bar{x} is globally stable in D .

$V(x(t))$ strictly decreasing along a solution $x(t)$

Constructing a Lyapunov Function for the n -Group Model

Consider a candidate

$$V = \sum_{k=1}^n v_k \left[\underbrace{(S_k - S_k^* \ln S_k) + (I_k - I_k^* \ln I_k)}_{\text{A Lyapunov function for a single-group model}} \right]$$

Constructing a Lyapunov Function for the n -Group Model

Consider a candidate

$$V = \sum_{k=1}^n v_k \left[\underbrace{(S_k - S_k^* \ln S_k) + (I_k - I_k^* \ln I_k)}_{\text{A Lyapunov function for a single-group model}} \right]$$

Choose appropriate v_k so that $V(x)$ is negative definite.

Derivative of V

$$V' = \sum_{k=1}^n v_k \left[(S'_k - \frac{S_k^*}{S_k} S'_k) + (I'_k - \frac{I_k^*}{I_k} I'_k) \right]$$

Derivative of V

$$\begin{aligned} V' &= \\ &\sum_{k=1}^n v_k \left[(S'_k - \frac{S_k^*}{S_k} S'_k) + (I'_k - \frac{I_k^*}{I_k} I'_k) \right] \\ &= \sum_{k=1}^n v_k \left[d_k S_k^* \left(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right] \\ &\quad + \sum_{k=1}^n v_k \left[\sum_{j=1}^n \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right] \\ &\quad + \sum_{j,k=1}^n v_k \beta_{kj} I_k^* S_j^* \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \end{aligned}$$

Derivative of V

$$\begin{aligned} V' &= \\ &\sum_{k=1}^n v_k \left[(S'_k - \frac{S_k^*}{S_k} S'_k) + (I'_k - \frac{I_k^*}{I_k} I'_k) \right] \\ &= \sum_{k=1}^n v_k \left[d_k S_k^* \left(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right] \leq 0 \\ &+ \sum_{k=1}^n v_k \left[\sum_{j=1}^n \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right] \\ &+ \sum_{j,k=1}^n v_k \beta_{kj} I_k^* S_j^* \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \end{aligned}$$

Derivative of V

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Derivative of V

$$V' =$$

$$\begin{aligned} & \sum_{k=1}^n v_k \left[(S'_k - \frac{S_k^*}{S_k} S'_k) + (I'_k - \frac{I_k^*}{I_k} I'_k) \right] \\ &= \sum_{k=1}^n v_k \left[d_k S_k^* \left(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right] \\ &+ \sum_{k=1}^n v_k \left[\sum_{j=1}^n \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right] \\ &+ \sum_{j,k=1}^n v_k \beta_{kj} I_k^* S_j^* \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \\ &H_n := \sum_{j,k=1}^n v_k \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \end{aligned}$$

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Choosing Constants v_k

Choose v_k so that

$$\sum_{k=1}^n v_k \left[\sum_{j=1}^n \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right] \equiv 0$$

for all $I_1, \dots, I_n > 0$.

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for all $I_1, \dots, I_n > 0$. This is equivalent to

$$\begin{bmatrix} \beta_{11} I_1^* S_1^* & \cdots & \beta_{n1} I_n^* S_1^* \\ \vdots & \ddots & \vdots \\ \beta_{1n} I_1^* S_n^* & \cdots & \beta_{nn} I_n^* S_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n \beta_{j1} I_j^* S_1^* v_1 \\ \vdots \\ \sum_{j=1}^n \beta_{jn} I_j^* S_n^* v_n \end{bmatrix}$$

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since, at P^* ,

$$(d_k + \gamma_k) = \sum_{j=1}^n \beta_{jk} I_j^* S_k^*.$$

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Theorem

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Set $\bar{\beta}_{jk} = \beta_{jk} I_j^* S_k^*$. Then (v_1, \dots, v_k) are determined by the linear system

$$\begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

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The solution space is 1d and a basis is given by

$v_k = C_{kk}$, the k -th principal minor, $k = 1, \dots, n$.

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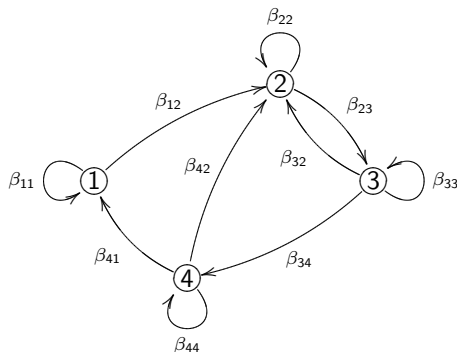
Need to show

$$V' \leq H_n = \sum_{j,k=1}^n v_k \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \leq 0,$$

for all $S_1, I_1, \dots, S_n, I_n \geq 0$.

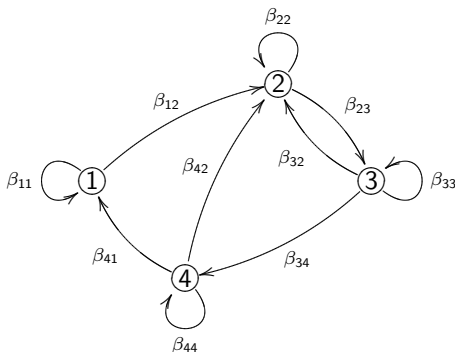
Directed Graphs and Rooted Spanning Trees

Let G be a directed graph with vertex set $V(G) = \{1, \dots, n\}$ and weight matrix $B = (\beta_{ij})_{n \times n}$.



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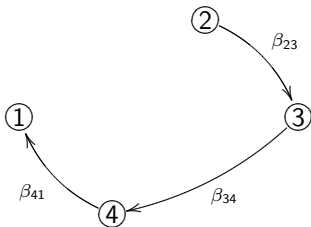


A **spanning tree** T of G is a sub-tree of G of $n - 1$ edges.

A **rooted** spanning tree is oriented towards a vertex.

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A rooted spanning tree is oriented towards a vertex.

The **weight** of tree T is $w(T) = \prod \beta_{ij}$ over all edges (i, j) .

The Matrix-Tree Theorem

Let $B = (\bar{\beta}_{ij})_{n \times n}$ be the weight matrix of graph G .

The **Kirchhoff Matrix** (combinatorial Laplacian) of B is

$$\bar{B} = \begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix}.$$

Note that **all column sums of \bar{B} are 0**.

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Theorem (Matrix Tree Theorem, Kirchhoff 1847)

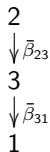
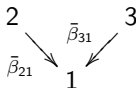
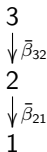
$$C_{kk} = \sum_{T \in \mathbb{T}_k} w(T).$$

\mathbb{T}_k : The set of spanning trees rooted at vertex k .

Solving System $\bar{B} v = 0 : n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:



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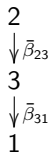
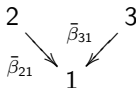
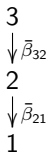
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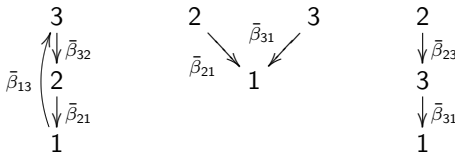
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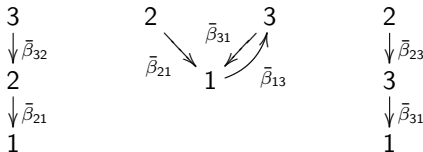
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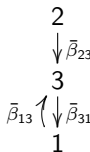
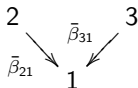
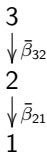
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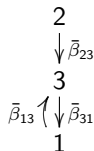
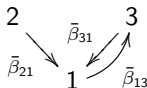
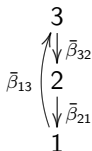
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Each product is the weight of a **unicyclic graph** with a cycle of length $1 \leq r \leq 3$.

Unicyclic Graphs and Rooted Trees

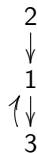
How does a unicyclic graph correspond to products in $v_k \beta_{kj}$?



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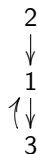
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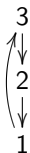
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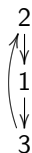
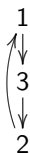
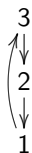
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Another Unicyclic Graph

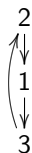
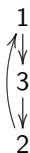
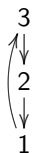


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Another Unicyclic Graph

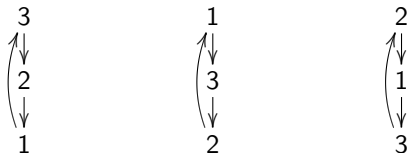


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$$V' \leq H_n = \sum_{j,k=1}^n v_k \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right)$$

H_n is Summed over all Unicyclic Graphs

$$\begin{aligned} H_n &= \sum_{j,k=1}^n v_k \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \\ &= \sum_Q H_Q \end{aligned}$$

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$$H_Q = w(Q) \cdot \sum_{(p,q) \in E(C_Q)} \left(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \right)$$

$$= w(Q) \cdot \left[2r - \sum_{(p,q) \in E(C_Q)} \left(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \right) \right]$$

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Finally, because C_Q is a cycle,

$$\prod_{(p,q) \in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = \prod_{(p,q) \in E(C_Q)} \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = 1.$$

H_n is Summed over all Unicyclic Graphs

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$$\begin{aligned} V' &\leq H_n = \sum_{j,k=1}^n v_k \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \\ &= \sum_Q H_Q \leq 0 \end{aligned}$$

and

$$\begin{aligned} H_Q &= w(Q) \cdot \sum_{(p,q) \in E(C_Q)} \left(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \right) \\ &= w(Q) \cdot \left[2r - \sum_{(p,q) \in E(C_Q)} \left(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \right) \right] \end{aligned}$$

Finally, because C_Q is a cycle,

$$\prod_{(p,q) \in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = \prod_{(p,q) \in E(C_Q)} \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = 1.$$

Main Result

For the *n*-group SIR model with bilinear incidence,

Theorem (Guo, Li, Shuai, 2007)

Assume that transmission matrix B is irreducible.

If $R_0 > 1$, then P^ is unique and is globally stable in R_+^{2n} .*

