Graph Theory and Global Stability Problem in Heterogeneous Epidemic Models

Michael Y. Li
Applied Mathematics Institute
University of Alberta
Edmonton, Alberta, Canada

Collaborators: Hongbin Guo and Zhisheng Shuai

Outline of Talk

Part I: Modeling Infectious Diseases in Heterogeneous Populations

- Simple epidemic models and their dynamics
- Basic reproduction number and the threshold theorem
- Multi-group models for heterogeneous populations

Part II: Global-Stability Problem in Multi-Group Models

- Global-stability problem and Lyapunov functions
- A Lyapunov function for multi-group models
- Why is global-stability difficult to prove?

Outline of Talk

Part I: Modeling Infectious Diseases in Heterogeneous Populations

- Simple epidemic models and their dynamics
- Basic reproduction number and the threshold theorem
- Multi-group models for heterogeneous populations

Part II: Global-Stability Problem in Multi-Group Models

- Global-stability problem and Lyapunov functions
- A Lyapunov function for multi-group models
- Why is global-stability difficult to prove?

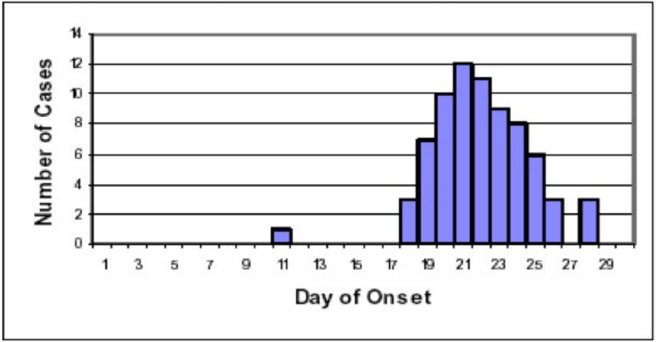
Part III: Matrix-Tree Theorem in Graph Theory

- Rooted directed trees and unicyclic graphs
- Kirchhoff's Matrix-Tree Theorem

Part IV: How do all of these come together?

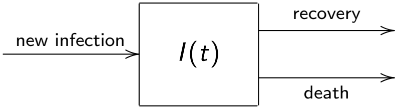
- Global-stability result for multi-group models.
- A general theorem

How to Model an Epidemic?

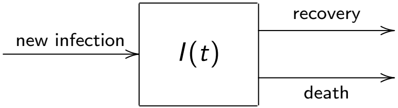


An Epidemic Curve

How to Model an Epidemic?

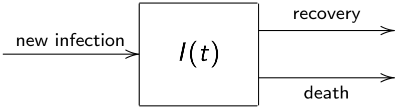


How to Model an Epidemic?



$$
\begin{aligned}
I'(t) &= \boxed{\text{Incidence Rate}} - \boxed{\text{Recovery Rate}} - \boxed{\text{Death Rate}} \\
&= f(I(t), S(t), N(t)) - \gamma\, I(t) - d\, I(t)
\end{aligned}
$$

How to Model an Epidemic?



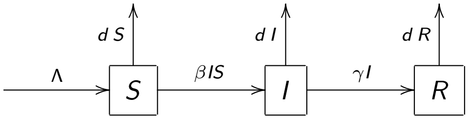
$$
\begin{aligned}
I'(t) &= \boxed{\text{Incidence Rate}} - \boxed{\text{Recovery Rate}} - \boxed{\text{Death Rate}} \\
&= f(I(t), S(t), N(t)) - \gamma I(t) - d\, I(t)
\end{aligned}
$$

$$f(I, S, N) = \beta I\, S : \quad \text{bilinear incidence}$$

$$f(I, S, N) = \lambda \frac{I\, S}{N} : \quad \text{proportionate incidence}$$

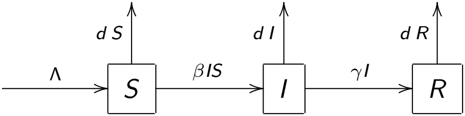
A Single-Group SIR Model

S : Susceptibles $\qquad$ I : Infectious $\qquad$ R : *Removed*



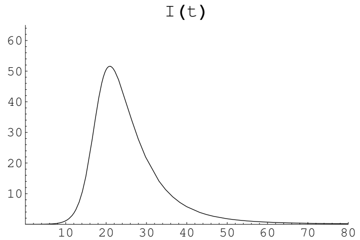
A Single-Group SIR Model

S : Susceptibles $\qquad I$: Infectious $\qquad R$: *Removed*



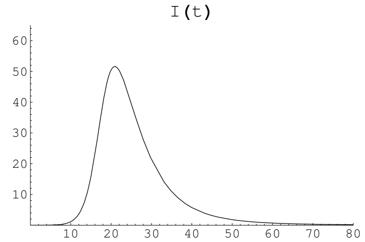
$$
\begin{aligned}
S' &= \Lambda - \beta\, I\, S - d\, S \\
I' &= \beta\, I\, S - (\gamma + d)\, I \\
R' &= \gamma\, I - d\, R
\end{aligned}
$$

A Single-Group SIR Model

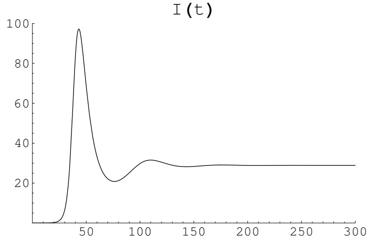


Numerical output I: epidemic case

A Single-Group SIR Model



Numerical output I: epidemic case



Numerical output II: endemic case

Threshold Theorem

The basic reproduction number is

$$R_0 = \frac{\beta \Lambda}{(\gamma + d)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d}$$

The average secondary infections produced by a single infective during its entire infectious period.

Threshold Theorem

The basic reproduction number is

$$R_0 = \frac{\beta \Lambda}{(\gamma + d)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d}$$

The average secondary infections produced by a single infective during its entire infectious period.

Theorem (Threshold Theorem)

(1) *If* $R_0 \leq 1$*, then the disease-free equilibrium* $P_0 = (\Lambda/d, 0)$ *is stable and attracts all solutions in* R^2_+.

(2) *If* $R_0 > 1$*, then* P_0 *is unstable, and a unique endemic (positive) equilibrium* P^* *is stable and attracts all positive solutions in* R^2_+.

Threshold Theorem

The basic reproduction number is

$$R_0 = \frac{\beta\,\Lambda}{(\gamma + d\,)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d}$$

The average secondary infections produced by a single infective during its entire infectious period.

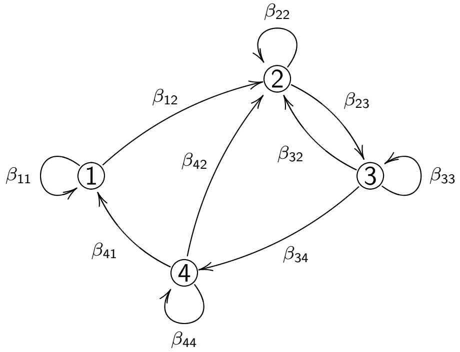
Theorem (Threshold Theorem)

(1) *If $R_0 \leq 1$, then the disease-free equilibrium $P_0 = (\Lambda/d\,, 0)$ is stable and attracts all solutions in R^2_+.*

(2) *If $R_0 > 1$, then P_0 is unstable, and a unique endemic (positive) equilibrium P^* is stable and attracts all positive solutions in R^2_+.*

Proof uses the Poincaré-Bendixson theory for 2d systems.

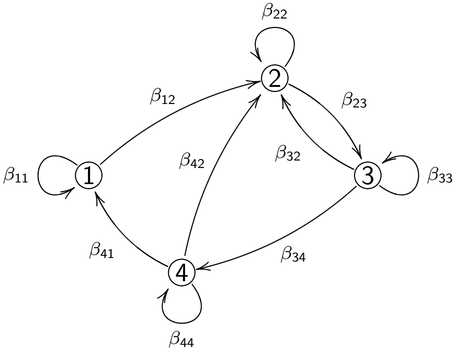
n-Group Models for Heterogeneous Populations



Each circled number represents a homogeneous group.

β_{jk} : transmission coefficient between I_j and S_k.

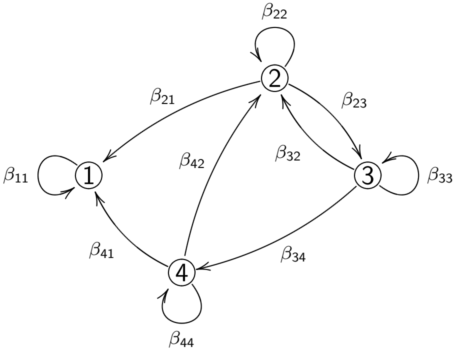
n-Group Models for Heterogeneous Populations



$$B = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 & 0 \\ 0 & \beta_{22} & \beta_{23} & 0 \\ 0 & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & 0 & \beta_{44} \end{bmatrix}$$

Transmission Matrix B is irreducible.

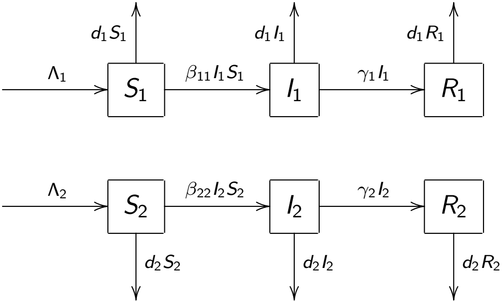
n-Group Models for Heterogeneous Populations



$$B = \begin{bmatrix} \beta_{11} & 0 & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & 0 \\ 0 & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & 0 & \beta_{44} \end{bmatrix}$$

Transmission Matrix B is reducible.

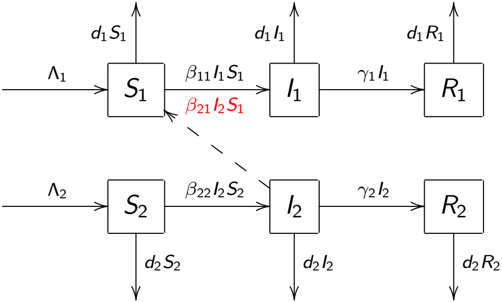
A Two-Group SIR Model



Incidence terms (bilinear):

- Group 1: $\beta_{11} I_1 S_1$
- Group 2: $\beta_{22} I_2 S_2$

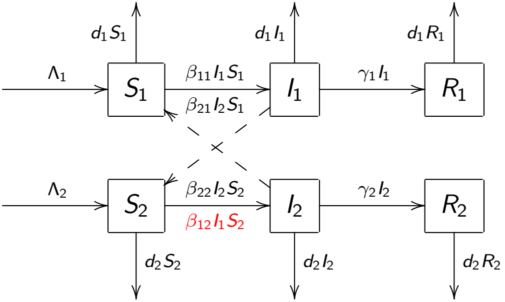
A Two-Group SIR Model



Incidence terms (bilinear):

- Group 1: $\beta_{11} I_1 S_1 + \beta_{21} I_2 S_1$
- Group 2: $\beta_{22} I_2 S_2$

A Two-Group SIR Model



Incidence terms (bilinear):

- Group 1: $\beta_{11} I_1 S_1 + \beta_{21} I_2 S_1$
- Group 2: $\beta_{22} I_2 S_2 + \beta_{12} I_1 S_2$

An n-Group SIR Model

$$\begin{cases} S_k' = \Lambda_k - d_k S_k - \sum_{j=1}^{n} \beta_{jk} I_j S_k, \\ I_k' = \sum_{j=1}^{n} \beta_{jk} I_j S_k - (d_k + \gamma_k) I_k, \end{cases} \qquad k = 1, \cdots, n.$$

Mathematical Questions:

- If $R_0 > 1$, is P^* unique?

An n-Group SIR Model

$$\begin{cases} S_k' = \Lambda_k - d_k S_k - \sum_{j=1}^{n} \beta_{jk} I_j S_k, \\ I_k' = \sum_{j=1}^{n} \beta_{jk} I_j S_k - (d_k + \gamma_k) I_k, \end{cases} \qquad k = 1, \cdots, n.$$

Mathematical Questions:

- If $R_0 > 1$, is P^* unique?
- When P^* is unique, is it globally stable?

Previous Results on GAS of P^*

For Models using bilinear incidence:

- Lajmanovich and Yorke (1976)
  - n-group SIS model, by Lyapunov function
  - later extended by Nold, Hirsch
- Hethcote (1975)
  - n-group SIR model with no vital dynamics
- Thieme (1983)
  - n-group SEIRS model, small latent and immune periods
- Beretta and Capasso (1986)
  - n-group SIR model, constant group sizes
- Lin and So (1993)
  - n-group SIRS model, constant group sizes
  - β_{ij} $(i \neq j)$ small

Non-uniqueness of P^* when $R_0 > 1$

- Lin (1992)
  n-group model for HIV

- Huang, Cooke, Castillo-Chavez (1992)
  n-group model for HIV with delay

These models use proportionate incidence.

Global-Stability and Lyapunov Functions

Consider a general system of ODE

$$x' = F(x), \quad x \in D \subset \mathbb{R}^d.$$

$\bar{x}$ is an equilibrium if $F(\bar{x}) = 0$.

An equilibrium $\bar{x}$ is globally stable in D if it is locally stable and all solutions in D converge to $\bar{x}$ as $t \to \infty$.

Global-Stability and Lyapunov Functions

Consider a general system of ODE

$$x' = F(x), \quad x \in D \subset \mathbb{R}^d.$$

$\bar{x}$ is an equilibrium if $F(\bar{x}) = 0$.

An equilibrium $\bar{x}$ is globally stable in D if it is locally stable and all solutions in D converge to $\bar{x}$ as $t \to \infty$.

Global-Stability and Lyapunov Functions

Theorem (Lyapunov)

Suppose $\exists$ *a Lipschitz function* $V(x)$ *such that*

(1) $V(x) \geq V(\bar{x})$ *and* $V(x) = V(\bar{x}) \iff x = \bar{x}$.

(2) $\overset{*}{V}(x) = \nabla V(x) \cdot F(x) \leq 0, \quad x \in D$, *and*

$\overset{*}{V}(x) = 0 \iff x = \bar{x}$.

Then $\bar{x}$ *is globally stable in* D.

Global-Stability and Lyapunov Functions

Theorem (Lyapunov)

Suppose $\exists$ a Lipschitz function $V(x)$ such that

(1) $V(x) \geq V(\bar{x})$ *and* $V(x) = V(\bar{x}) \iff x = \bar{x}$.

(2) $\overset{*}{V}(x) = \nabla V(x) \cdot F(x) \leq 0, \quad x \in D$, *and*
$\overset{*}{V}(x) = 0 \iff x = \bar{x}$.

Then $\bar{x}$ is globally stable in D.

$V(x(t))$ strictly decreasing along a solution $x(t)$

Constructing a Lyapunov Function for the n-Group Model

Consider a candidate

$$V = \sum_{k=1}^{n} v_k \Big[\underbrace{(S_k - S_k^* \ln S_k) + (I_k - I_k^* \ln I_k)}_{\text{A Lyapunov function for a single-group model}} \Big]$$

Constructing a Lyapunov Function for the n-Group Model

Consider a candidate

$$V = \sum_{k=1}^{n} v_k \Big[\underbrace{(S_k - S_k^* \ln S_k) + (I_k - I_k^* \ln I_k)}_{\text{A Lyapunov function for a single-group model}} \Big]$$

Choose appropriate v_k so that $\overset{*}{V}(x)$ is negative definite.

Derivative of V

$$V' = \sum_{k=1}^{n} v_k \left[\left(S_k' - \frac{S_k^*}{S_k} S_k'\right) + \left(I_k' - \frac{I_k^*}{I_k} I_k'\right) \right]$$

Derivative of V

$$V' =$$

$$\sum_{k=1}^{n} v_k \Big[(S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{I_k} I_k') \Big]$$

$$= \sum_{k=1}^{n} v_k \Big[d_k S_k^* \Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \Big) \Big]$$

$$+ \sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big]$$

$$+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

Derivative of V

$$
\begin{aligned}
V' = \\
&\sum_{k=1}^{n} v_k \left[\left(S_k' - \frac{S_k^*}{S_k} S_k'\right) + \left(I_k' - \frac{I_k^*}{I_k} I_k'\right) \right] \\
&= \sum_{k=1}^{n} v_k \left[d_k S_k^* \left(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right] \qquad \leq 0 \\
&+ \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right] \\
&+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right)
\end{aligned}
$$

Derivative of V

$$V' =$$

$$\sum_{k=1}^{n} v_k \Big[(S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{I_k} I_k') \Big]$$

$$= \sum_{k=1}^{n} v_k \Big[d_k S_k^* \Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \Big) \Big]$$

$$+ \sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \qquad \equiv 0$$

$$+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

Derivative of V

$$V' =$$

$$\sum_{k=1}^{n} v_k \Big[\big(S_k' - \frac{S_k^*}{S_k} S_k'\big) + \big(I_k' - \frac{I_k^*}{I_k} I_k'\big) \Big]$$

$$= \sum_{k=1}^{n} v_k \Big[d_k S_k^* \Big(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \Big) \Big]$$

$$+ \sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big]$$

$$+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

$$H_n := \sum_{j,k=1}^{n} v_k \, \bar{\beta}_{kj} \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$

Derivative of V

$$V' =$$

$$\sum_{k=1}^{n} v_k \left[\left(S_k' - \frac{S_k^*}{S_k} S_k'\right) + \left(I_k' - \frac{I_k^*}{I_k} I_k'\right) \right]$$

$$= \sum_{k=1}^{n} v_k \left[d_k S_k^* \left(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*}\right) \right]$$

$$+ \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right]$$

$$+ \sum_{j,k=1}^{n} v_k \, \beta_{kj} I_k^* S_j^* \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k}\right)$$

$$H_n := \sum_{j,k=1}^{n} v_k \, \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k}\right)$$

Choosing Constants v_k

Choose v_k so that

$$\sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \equiv 0$$

for all $I_1, \cdots, I_n > 0$.

Choosing Constants v_k

Choose v_k so that

$$\sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \equiv 0$$

for all $I_1, \cdots, I_n > 0$. This is equivalent to

$$\begin{bmatrix} \beta_{11} I_1^* S_1^* & \cdots & \beta_{n1} I_n^* S_1^* \\ \vdots & \ddots & \vdots \\ \beta_{1n} I_1^* S_n^* & \cdots & \beta_{nn} I_n^* S_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} \beta_{j1} I_j^* S_1^* v_1 \\ \vdots \\ \sum_{j=1}^{n} \beta_{jn} I_j^* S_n^* v_n \end{bmatrix}$$

Choosing Constants v_k

Choose v_k so that

$$\sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \equiv 0$$

for all $I_1, \cdots, I_n > 0$. This is equivalent to

$$\begin{bmatrix} \beta_{11} I_1^* S_1^* & \cdots & \beta_{n1} I_n^* S_1^* \\ \vdots & \ddots & \vdots \\ \beta_{1n} I_1^* S_n^* & \cdots & \beta_{nn} I_n^* S_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} \beta_{j1} I_j^* S_1^* v_1 \\ \vdots \\ \sum_{j=1}^{n} \beta_{jn} I_j^* S_n^* v_n \end{bmatrix}$$

since, at P^*,

$$(d_k + \gamma_k) = \sum_{j=1}^{n} \beta_{jk} I_j^* S_k^*.$$

Choosing Constants v_k

Choose v_k so that

$$\sum_{k=1}^{n} v_k \Big[\sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \Big] \equiv 0$$

for all $I_1, \cdots, I_n > 0$. This is equivalent to

$$\begin{bmatrix} \beta_{11} I_1^* S_1^* & \cdots & \beta_{n1} I_n^* S_1^* \\ \vdots & \ddots & \vdots \\ \beta_{1n} I_1^* S_n^* & \cdots & \beta_{nn} I_n^* S_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} \beta_{j1} I_j^* S_1^* v_1 \\ \vdots \\ \sum_{j=1}^{n} \beta_{jn} I_j^* S_n^* v_n \end{bmatrix}$$

since, at P^*,

$$(d_k + \gamma_k) = \sum_{j=1}^{n} \beta_{jk} I_j^* S_k^* .$$

Set $\bar{\beta}_{jk} = \beta_{jk} I_j^* S_k^*$. Then $(v_1, \cdots, v_k)$ are determined by the linear system

$$\begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Set $\bar{\beta}_{jk} = \beta_{jk} I_j^* S_k^*$. Then $(v_1, \cdots, v_k)$ are determined by the linear system

$$\begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The solution space is 1d and a basis is given by

$$v_k = C_{kk}, \quad \text{the } k\text{-th principal minor}, \quad k = 1, \cdots, n.$$

Set $\bar{\beta}_{jk} = \beta_{jk} I_j^* S_k^*$. Then $(v_1, \cdots, v_k)$ are determined by the linear system

$$\begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The solution space is 1d and a basis is given by

$$v_k = C_{kk}, \quad \text{the } k\text{-th principal minor}, \quad k = 1, \cdots, n.$$

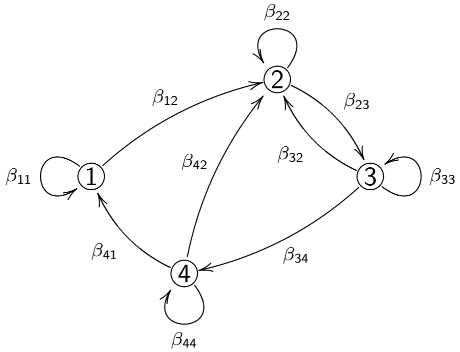
Need to show

$$V' \leq H_n = \sum_{j,k=1}^{n} v_k \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \right) \leq 0,$$

for all $S_1, I_1, \cdots, S_n, I_n \geq 0$.

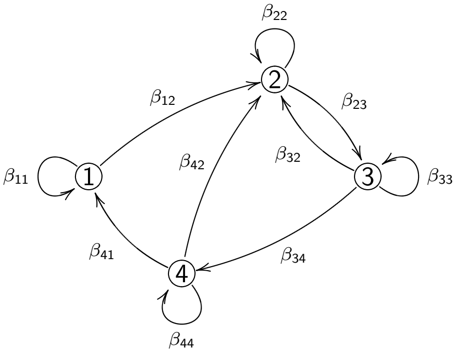
Directed Graphs and Rooted Spanning Trees

Let G be a directed graph with vertex set $V(G) = \{1, \cdots, n\}$ and weight matrix $B = (\beta_{ij})_{n \times n}$.



Directed Graphs and Rooted Spanning Trees

Let G be a directed graph with vertex set $V(G) = \{1, \cdots, n\}$ and weight matrix $B = (\beta_{ij})_{n \times n}$.

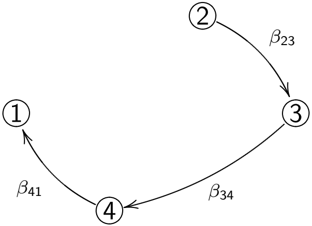


A spanning tree T of G is a sub-tree of G of $n-1$ edges.

A rooted spanning tree is oriented towards a vertex.

Directed Graphs and Rooted Spanning Trees

Let G be a directed graph with vertex set $V(G) = \{1, \cdots, n\}$ and weight matrix $B = (\beta_{ij})_{n \times n}$.



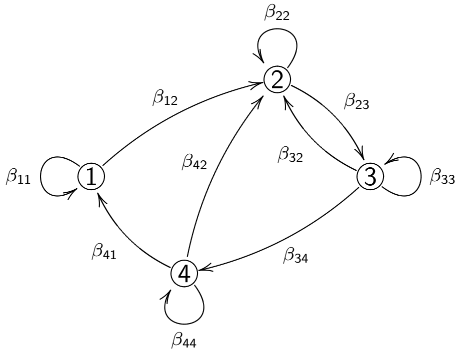
A spanning tree T of G is a sub-tree of G of $n-1$ edges.

A rooted spanning tree is oriented towards a vertex.

The weight of tree T is $w(T) = \prod \beta_{ij}$ over all edges (i, j).

Directed Graphs and Rooted Spanning Trees

Let G be a directed graph with vertex set $V(G) = \{1, \cdots, n\}$ and weight matrix $B = (\beta_{ij})_{n \times n}$.



A spanning tree T of G is a sub-tree of G of $n - 1$ edges.

A rooted spanning tree is oriented towards a vertex.

The weight of tree T is $w(T) = \prod \beta_{ij}$ over all edges (i, j).

The Matrix-Tree Theorem

Let $B = (\bar{\beta}_{ij})_{n \times n}$ be the weight matrix of graph G.

The Kirchhoff Matrix (combinatorial Laplacian) of B is

$$\overline{B} = \begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix}.$$

Note that all column sums of $\overline{B}$ are 0.

The Matrix-Tree Theorem

Let $B = (\bar{\beta}_{ij})_{n\times n}$ be the weight matrix of graph G.

The Kirchhoff Matrix (combinatorial Laplacian) of B is

$$\overline{B} = \begin{bmatrix} \sum_{l\neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l\neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l\neq n} \bar{\beta}_{nl} \end{bmatrix}.$$

Note that all column sums of $\overline{B}$ are 0.

Theorem (Matrix Tree Theorem, Kirchhoff 1847)

$$C_{kk} = \sum_{T\in\mathbb{T}_k} w(T).$$

$\mathbb{T}_k$: The set of spanning trees rooted at vertex k.

Solving System $\bar{B}\, v = 0: \;\; n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

$$\begin{tikzcd} 3 \arrow[d, "\bar{\beta}_{32}"] \\ 2 \arrow[d, "\bar{\beta}_{21}"] \\ 1 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[dr, "\bar{\beta}_{21}"'] & & 3 \arrow[dl, "\bar{\beta}_{31}"'] \\ & 1 & \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[d, "\bar{\beta}_{23}"] \\ 3 \arrow[d, "\bar{\beta}_{31}"] \\ 1 \end{tikzcd}$$

Solving System $\bar{B}\,v = 0$: $n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

$$3 \xrightarrow{\bar{\beta}_{32}} 2 \xrightarrow{\bar{\beta}_{21}} 1 \qquad 2 \xrightarrow{\bar{\beta}_{21}} 1 \xleftarrow{\bar{\beta}_{31}} 3 \qquad 2 \xrightarrow{\bar{\beta}_{23}} 3 \xrightarrow{\bar{\beta}_{31}} 1$$

Furthermore

$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Solving System $\bar{B}\, v = 0$: $n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

$$\begin{tikzcd} 3 \arrow[d, "\bar{\beta}_{32}"] \\ 2 \arrow[d, "\bar{\beta}_{21}"] \\ 1 \arrow[uu, bend left, "\bar{\beta}_{13}"] \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[dr, "\bar{\beta}_{21}"'] & & 3 \arrow[dl, "\bar{\beta}_{31}"'] \\ & 1 & \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[d, "\bar{\beta}_{23}"] \\ 3 \arrow[d, "\bar{\beta}_{31}"] \\ 1 \end{tikzcd}$$

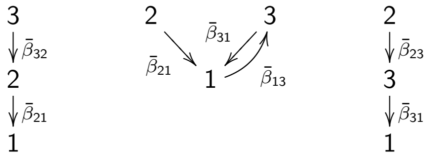
Furthermore

$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Solving System $\bar{B}\, v = 0 :\ \ n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:



Furthermore

$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Solving System $\bar{B}\, v = 0 : \;\; n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

$$\begin{tikzcd}
3 \arrow[d, "\bar{\beta}_{32}"] \\
2 \arrow[d, "\bar{\beta}_{21}"] \\
1
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[dr, "\bar{\beta}_{21}"'] & & 3 \arrow[dl, "\bar{\beta}_{31}"'] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[d, "\bar{\beta}_{23}"] \\
3 \arrow[d, "\bar{\beta}_{31}"] \\
1 \arrow[u, bend left, "\bar{\beta}_{13}"]
\end{tikzcd}$$

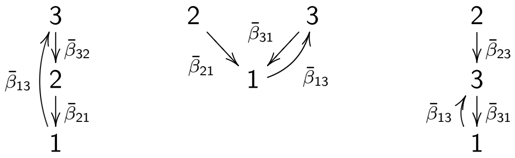
Furthermore

$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Solving System $\bar{B}\,v = 0: \;\; n = 3$

$$v_1 = C_{11} = \sum_{T \in \mathbb{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:



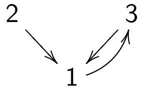
Furthermore

$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Each product is the weight of a unicyclic graph with a cycle of length $1 \le r \le 3$.

Unicyclic Graphs and Rooted Trees

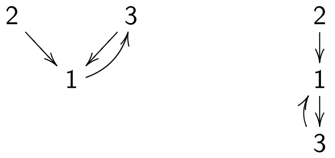
How does a unicyclic graph correspond to products in $v_k \beta_{kj}$?



$$v_1 \bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \color{red}{\bar{\beta}_{21}\bar{\beta}_{13}\bar{\beta}_{31}} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Unicyclic Graphs and Rooted Trees

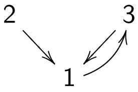
How does a unicyclic graph correspond to products in $v_k \beta_{kj}$?

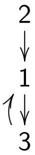


$$v_1 \bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{13}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Unicyclic Graphs and Rooted Trees

How does a unicyclic graph correspond to products in $v_k\beta_{kj}$?





$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{13}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$
$$v_3\bar{\beta}_{31} = \bar{\beta}_{12}\bar{\beta}_{23}\bar{\beta}_{31} + \color{red}{\bar{\beta}_{21}\bar{\beta}_{13}\bar{\beta}_{31}} + \bar{\beta}_{13}\bar{\beta}_{23}\bar{\beta}_{31}$$

Another Unicyclic Graph

$$
\begin{tikzcd}
3 \arrow[d] \\
2 \arrow[d] \\
1 \arrow[uu, bend left]
\end{tikzcd}
$$

$$v_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$

Another Unicyclic Graph

$$
\begin{tikzcd}
3 \arrow[d] \\
2 \arrow[d] \\
1 \arrow[uu, bend left]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 \arrow[d] \\
3 \arrow[d] \\
2 \arrow[uu, bend left]
\end{tikzcd}
$$

$$v_1\bar\beta_{13} = \bar\beta_{32}\bar\beta_{21}\bar\beta_{13} + \bar\beta_{21}\bar\beta_{31}\bar\beta_{13} + \bar\beta_{23}\bar\beta_{31}\bar\beta_{13}$$
$$v_2\bar\beta_{21} = \bar\beta_{13}\bar\beta_{32}\bar\beta_{21} + \bar\beta_{12}\bar\beta_{32}\bar\beta_{21} + \bar\beta_{31}\bar\beta_{12}\bar\beta_{21}$$

Another Unicyclic Graph

$$
\begin{tikzcd}
3 \arrow[d] \\
2 \arrow[d] \\
1 \arrow[uu, bend left]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 \arrow[d] \\
3 \arrow[d] \\
2 \arrow[uu, bend left]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
2 \arrow[d] \\
1 \arrow[d] \\
3 \arrow[uu, bend left]
\end{tikzcd}
$$

$$
\begin{aligned}
v_1\bar{\beta}_{13} &= \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13} \\
v_2\bar{\beta}_{21} &= \bar{\beta}_{13}\bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{12}\bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{31}\bar{\beta}_{12}\bar{\beta}_{21} \\
v_3\bar{\beta}_{32} &= \bar{\beta}_{12}\bar{\beta}_{23}\bar{\beta}_{32} + \bar{\beta}_{13}\bar{\beta}_{23}\bar{\beta}_{32} + \bar{\beta}_{21}\bar{\beta}_{13}\bar{\beta}_{32}
\end{aligned}
$$

Another Unicyclic Graph

$$\begin{tikzcd}[column sep=huge] 3 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] \\ 2 \arrow[d] & 3 \arrow[d] & 1 \arrow[d] \\ 1 \arrow[uu, bend left] & 2 \arrow[uu, bend left] & 3 \arrow[uu, bend left] \end{tikzcd}$$

$$\begin{aligned} v_1\bar{\beta}_{13} &= \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13} \\ v_2\bar{\beta}_{21} &= \bar{\beta}_{13}\bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{12}\bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{31}\bar{\beta}_{12}\bar{\beta}_{21} \\ v_3\bar{\beta}_{32} &= \bar{\beta}_{12}\bar{\beta}_{23}\bar{\beta}_{32} + \bar{\beta}_{13}\bar{\beta}_{23}\bar{\beta}_{32} + \bar{\beta}_{21}\bar{\beta}_{13}\bar{\beta}_{32} \end{aligned}$$

Another Unicyclic Graph

$$\begin{tikzcd} 3 \arrow[d] \\ 2 \arrow[d] \\ 1 \arrow[uu, bend left] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[d] \\ 3 \arrow[d] \\ 2 \arrow[uu, bend left] \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[d] \\ 1 \arrow[d] \\ 3 \arrow[uu, bend left] \end{tikzcd}$$

$$\begin{aligned}
v_1\bar\beta_{13} &= \bar\beta_{32}\bar\beta_{21}\bar\beta_{13} + \bar\beta_{21}\bar\beta_{31}\bar\beta_{13} + \bar\beta_{23}\bar\beta_{31}\bar\beta_{13} \\
v_2\bar\beta_{21} &= \bar\beta_{13}\bar\beta_{32}\bar\beta_{21} + \bar\beta_{12}\bar\beta_{32}\bar\beta_{21} + \bar\beta_{31}\bar\beta_{12}\bar\beta_{21} \\
v_3\bar\beta_{32} &= \bar\beta_{12}\bar\beta_{23}\bar\beta_{32} + \bar\beta_{13}\bar\beta_{23}\bar\beta_{32} + \bar\beta_{21}\bar\beta_{13}\bar\beta_{32}
\end{aligned}$$

$$V' \le H_n = \sum_{j,k=1}^{n} v_k\,\bar\beta_{kj}\left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*}\frac{I_j}{I_j^*}\frac{I_k^*}{I_k}\right)$$

H_n is Summed over all Unicyclic Graphs

$$H_n = \sum_{j,k=1}^{n} v_k \, \bar{\beta}_{kj} \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big)$$
$$= \sum_{Q} H_Q$$

H_n is Summed over all Unicyclic Graphs

$$
\begin{aligned}
H_n &= \sum_{j,k=1}^{n} v_k\, \bar{\beta}_{kj} \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*}\frac{I_j}{I_j^*}\frac{I_k^*}{I_k}\Big) \\
&= \sum_{Q} H_Q
\end{aligned}
$$

and

$$
\begin{aligned}
H_Q &= w(Q) \cdot \sum_{(p,q)\in E(C_Q)} \Big(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*}\frac{I_p}{I_p^*}\frac{I_q^*}{I_q}\Big) \\
&= w(Q) \cdot \Big[2r - \sum_{(p,q)\in E(C_Q)} \Big(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*}\frac{I_p}{I_p^*}\frac{I_q^*}{I_q}\Big)\Big]
\end{aligned}
$$

H_n is Summed over all Unicyclic Graphs

$$H_n = \sum_{j,k=1}^{n} v_k\, \bar{\beta}_{kj} \Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} \Big) \\ = \sum_Q H_Q$$

and

$$H_Q = w(Q) \cdot \sum_{(p,q)\in E(C_Q)} \Big(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \Big) \\ = w(Q) \cdot \Big[2r - \sum_{(p,q)\in E(C_Q)} \Big(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{I_p}{I_p^*} \frac{I_q^*}{I_q} \Big) \Big]$$

Finally, because C_Q is a cycle,

$$\prod_{(p,q)\in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = \prod_{(p,q)\in E(C_Q)} \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = 1.$$

H_n is Summed over all Unicyclic Graphs

$$H_n = \sum_{j,k=1}^{n} v_k\, \bar{\beta}_{kj}\left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*}\frac{I_j}{I_j^*}\frac{I_k^*}{I_k}\right)$$
$$= \sum_{Q} H_Q$$

and

$$H_Q = w(Q) \cdot \sum_{(p,q)\in E(C_Q)} \left(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*}\frac{I_p}{I_p^*}\frac{I_q^*}{I_q}\right)$$
$$= w(Q) \cdot \left[2r - \sum_{(p,q)\in E(C_Q)} \left(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*}\frac{I_p}{I_p^*}\frac{I_q^*}{I_q}\right)\right] \quad \leq 0$$

Finally, because C_Q is a cycle,

$$\prod_{(p,q)\in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = \prod_{(p,q)\in E(C_Q)} \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = 1.$$

H_n is Summed over all Unicyclic Graphs

$$V' \leq H_n = \sum_{j,k=1}^{n} v_k\, \bar{\beta}_{kj}\Big(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*}\frac{I_j}{I_j^*}\frac{I_k^*}{I_k}\Big)$$

$$= \sum_Q H_Q \qquad \leq 0$$

and

$$H_Q = w(Q) \cdot \sum_{(p,q)\in E(C_Q)} \Big(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*}\frac{I_p}{I_p^*}\frac{I_q^*}{I_q}\Big)$$

$$= w(Q) \cdot \Big[2r - \sum_{(p,q)\in E(C_Q)} \Big(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*}\frac{I_p}{I_p^*}\frac{I_q^*}{I_q}\Big)\Big]$$

Finally, because C_Q is a cycle,

$$\prod_{(p,q)\in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = \prod_{(p,q)\in E(C_Q)} \frac{I_p}{I_p^*} \cdot \frac{I_q^*}{I_q} = 1.$$

Main Result

For the *n*-group SIR model with bilinear incidence,

Theorem (Guo, Li, Shuai, 2007)

Assume that transmission matrix B is irreducible.

If $R_0 > 1$*, then* P^* *is unique and is globally stable in* R_+^{2n}.

The same graph-theoretical approach can be used to:

Build Lyapunov function V for a large-scale system

$$V = \sum_{k=1}^{n} c_k V_k$$

using the known Lyapunov function V_k for each component.

Construction of Lyapunov Functions for Complex Networks

Given a coupled system over a digraph $\mathcal{G}$:

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \quad i = 1, 2, \ldots, n. \tag{1}$$

Assume: Each vertex system $u_i' = f_i(t, u_i)$ has a global Lyapunov function V_i.

Question: How to construct a global Lyapunov function for the coupled system?

A simple idea: For $u = (u_1, \cdots, u_n)$, try

$$V(u) = \sum_{i=1}^{n} c_i V_i(u_i)$$

and find suitable constants c_i.

A New Result

Theorem [Z. Shuai and ML, 2009] Assume

(1) There exist $F_{ij}(t,u)$ such that, for $t > 0, u \in D$,

$$\dot{V}_i(u) \leq \sum_{j=1}^{n} a_{ij} F_{ij}(t,u). \tag{2}$$

Let $\mathcal{G}_A$ be the weighted graph with weight matrix $A = (a_{ij})$.

(2) $\{F_{ij}(t,u)\}$ satisfies the following cycle condition: along each directed cycle $\mathcal{C}$ of graph $\mathcal{G}_A$,

$$\sum_{(r,s)\in E(\mathcal{C})} F_{rs}(t,u) \leq 0, \quad t > 0,\ u \in D. \tag{3}$$

A New Result

Theorem [Z. Shuai and ML, 2009] Assume

(1) There exist $F_{ij}(t, u)$ such that, for $t > 0, u \in D$,

$$\dot{V}_i(u) \le \sum_{j=1}^{n} a_{ij} F_{ij}(t, u). \qquad (2)$$

Let $\mathcal{G}_A$ be the weighted graph with weight matrix $A = (a_{ij})$.

(2) $\{F_{ij}(t, u)\}$ satisfies the following cycle condition: along each directed cycle $\mathcal{C}$ of graph $\mathcal{G}_A$,

$$\sum_{(r,s)\in E(\mathcal{C})} F_{rs}(t, u) \le 0, \quad t > 0, \; u \in D. \qquad (3)$$

Then there exist constants $c_i \ge 0$ such that

$$V(u) = \sum_{i=1}^{n} c_i V_i(u)$$

satisfies

$$\dot{V}(u) \le 0, \quad u \in D_1 \times \cdots \times D_n.$$

Kirchhoff Matrix-Tree Theorem

Let $\mathcal{G}_A$ be a weighted digraph with weight matrix $A = (a_{ij})$. The Laplacian matrix of graph $\mathcal{G}_A$ is

$$L = \begin{bmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k\neq n} a_{nk} \end{bmatrix}.$$

Let c_i be the cofactor of the i-th diagonal element of L.

Matrix-Tree Theorem [Kirchhoff (1847)] Assume $n \geq 2$. Then

$$c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} w(\mathcal{T}), \qquad i = 1, 2, \ldots, n, \tag{4}$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of $\mathcal{G}_A$ rooted at vertex i, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$.

The Tree-Cycle Identity

Tree-Cycle Identity (Z. Shuai and ML, 2009) Let c_i be given by the Matrix-Tree Theorem. Then the following identity holds.

$$\sum_{i,j=1}^{n} c_i \, a_{ij} \, F_{ij}(x) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x), \qquad (5)$$

where $F_{ij}(x), 1 \leq i, j \leq n$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs $\mathcal{Q}$ of $\mathcal{G}_A$, $w(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $\mathcal{C}_{\mathcal{Q}}$ denotes the oriented cycle of $\mathcal{Q}$.

The Tree-Cycle Identity

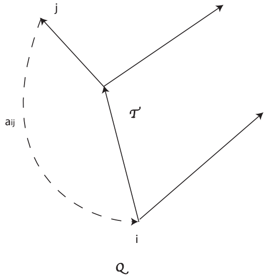
Tree-Cycle Identity (Z. Shuai and ML, 2009) Let c_i be given by the Matrix-Tree Theorem. Then the following identity holds.

$$\sum_{i,j=1}^{n} c_i \, a_{ij} \, F_{ij}(x) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_\mathcal{Q})} F_{rs}(x), \qquad (5)$$

where $F_{ij}(x), 1 \leq i,j \leq n$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs $\mathcal{Q}$ of $\mathcal{G}_A$, $w(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $\mathcal{C}_\mathcal{Q}$ denotes the oriented cycle of $\mathcal{Q}$.

Proof: Note $w(\mathcal{T}) \, a_{ij} = w(\mathcal{Q})$,

where $\mathcal{Q}$ is the unicyclic graph obtained by adding an arc (j,i) to $\mathcal{T}$.



Proof of Main Theorem

Let

$$V = \sum_{i=1}^{n} c_i V_i.$$

Choose c_i as in the Matrix-Tree Theorem

$$c_i = \sum_{\mathcal{T} \in \mathbb{T}_i} w(\mathcal{T}), \qquad i = 1, 2, \ldots, n.$$

Then

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n} c_i \dot{V}_i \leq \sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(u) \quad \text{(assumption (1))} \\ &= \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(u) \quad \text{(Tree-Cycle Identity)} \\ &\leq 0. \end{aligned}$$

Our Theorem offers a systematic way to construct global Lyapunov functions for complex networks, utilizing individual Lyapunov functions for the vertex systems.

Thank you!