Wavelets/Framelets for Computer Graphics

The following is based on book manuscript: B. Han, Framelets and Wavelets: Algorithms, Analysis and Applications.

In this project, we only deal with computer generated curves (not surfaces). This is an easier project than the project on wavelets/framelets for signal/image processing.

To introduce a subdivision curve, we need some definitions and notation. By $l(\mathbb{Z})$ we denote the linear space of all sequences $v = \{v(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ of complex numbers on $\mathbb{Z}$. One -dimensional discrete input data or signal is often treated as an element in $l(\mathbb{Z})$. Similarly, by $l_0(\mathbb{Z})$ we denote the linear space of all sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ on $\mathbb{Z}$ such that $\{k \in \mathbb{Z} \,:\, u(k) \neq 0\}$ is a finite set. An element in $l_0(\mathbb{Z})$ is often regarded as a finite-impulse-response (FIR) filter (also called a finitely supported mask in the literature of wavelet analysis). In this book we often use u for a general filter and v for a general signal or data. It is often convenient to use the formal Fourier series (or symbol) $\widehat{v}$ of a sequence $v = \{v(k)\}_{k\in\mathbb{Z}}$, which is defined as follows:

$$\widehat{v}(\xi) := \sum_{k\in\mathbb{Z}} v(k)e^{-ik\xi}, \qquad \xi \in \mathbb{R}, \tag{1}$$

where i in this book always denotes the imaginary unit. For $v \in l_0(\mathbb{Z})$, $\widehat{v}$ is a 2π-periodic trigonometric polynomial.

Let M be a positive integer greater than one. For a filter $a \in l_0(\mathbb{Z})$ and $v \in l(\mathbb{Z})$, the *subdivision operator* $\mathscr{S}_{\mathsf{M},a} : l(\mathbb{Z}) \to l(\mathbb{Z})$ is defined to be

$$[\mathscr{S}_{\mathsf{M},a}v](n) := 2\sum_{k\in\mathbb{Z}} v(k)a(n-\mathsf{M}k), \qquad n \in \mathbb{Z}. \tag{2}$$

Given an initial control polygonal shape $\{v(k)\}_{k\in\mathbb{Z}}$. We can generate a smooth curve through subdivision schemes. For $j \in \mathbb{N}$, define

$$v_j := \mathscr{S}^j_{\mathsf{M},a}v.$$

That is, we apply the subdivision operator j times (see the other project about how to efficiently implement a subdivision operator). Now we define “a function” f_j on the lattice $2^{-j}\mathbb{Z}$ as:

$$f_j(2^{-j}k) := v_j(k), \qquad k \in \mathbb{Z}.$$

Then we connect these discrete points one-by-one to create a function f_j. When $j \to \infty$, then $f_j \to f$, where f is the smooth subdivision curves. In practice, we only apply the subdivision scheme no more than 10 times.

To efficiently compute values $\mathscr{S}_{a,\mathsf{M}}v$ on the refined reference mesh $\mathsf{M}^{-1}\mathbb{Z}$ from v on the coarse mesh $\mathbb{Z}$, we often rewrite the subdivision operator using coset masks and convolution: For $\beta, \gamma \in \mathbb{Z}$,

$$[\mathscr{S}_{a,\mathsf{M}}v](\gamma+\mathsf{M}\beta) = |\mathsf{M}|\sum_{k\in\mathbb{Z}} v(k)a(\gamma+\mathsf{M}\beta-Mk) = |\mathsf{M}|[v*a^{[\gamma]}](\beta), \tag{3}$$

where the *coset mask* $a^{[\gamma]}$ of the mask a is defined to be

$$a^{[\gamma]}(k) := a(\gamma+\mathsf{M}k), \qquad k,\gamma \in \mathbb{Z}. \tag{4}$$

If $\sum_{k\in\mathbb{Z}} a(k)=1$, then $|\mathsf{M}|\sum_{k\in\mathbb{Z}} a^{[\gamma]}(k)=1$ for all $\gamma\in\mathbb{Z}$. Hence, a subdivision scheme is a local averaging rule. Moreover,

$$[\mathscr{S}_{a,\mathsf{M}}v](\gamma+\mathsf{M}\beta)=|\mathsf{M}|[v*a^{[\gamma]}](\beta)=\langle v(\beta+\cdot),|\mathsf{M}|\overline{a^{[\gamma]}(-\cdot)}\rangle, \tag{5}$$

which is attached to the point $\beta+\mathsf{M}^{-1}\gamma-\mathsf{M}^{-1}c_a$. Consequently, the filter

$$\{|\mathsf{M}|\overline{a^{[\gamma]}(-k)}\}_{k\in\mathbb{Z}}=\{|\mathsf{M}|\overline{a(\gamma-\mathsf{M}k)}\}_{k\in\mathbb{Z}},\qquad \gamma\in\{0,\dots,\mathsf{M}-1\}$$

is called the $\mathsf{M}^{-1}\gamma$*-stencil* of the mask a for computing the values $[\mathscr{S}_{a,\mathsf{M}}v](\gamma+\mathsf{M}\cdot)$ on the cosets in $\mathsf{M}^{-1}\gamma+\mathbb{Z}$ of the refined mesh $\mathsf{M}^{-1}\mathbb{Z}$. It is more convenient to use stencils for subdivision schemes in computer graphics than a filter/mask a.

To deal with curves in two or three dimensions, we simply apply the subdivision scheme componentwise (that is, entrywise). Quite often we also need a to have symmetry:

$$a(c-k)=a(k),\qquad k\in\mathbb{Z}$$

for some integer c. That is, we see that a has $\{1,-1\}$-symmetry. For a subdivision scheme, we often use subdivision triplets: $(a,\mathsf{M},\{-1,1\})$: a is the mask, M is the dilation factor, and $\{-1,1\}$ is the symmetry group. For dimension one and a dilation factor M, the reference coarse mesh $\mathbb{Z}$ is refined into a finer mesh $\frac{1}{\mathsf{M}}\mathbb{Z}$ by inserting new vertices at $\frac{\gamma}{\mathsf{M}}+\mathbb{Z}$ with $\gamma=1,\dots,|\mathsf{M}|-1$.

In the following, we provide a few examples of subdivision triplets.

Example 1 $(a,2,\{-1,1\})$ is a primal subdivision triplet with

$$a=\frac{1}{2}\{w_3,w_2,w_1,\underline{w_0},w_1,w_2,w_3\}_{[-3,3]},$$

where

$$w_0=\tfrac{3+t}{4},\quad w_1=\tfrac{8+t}{16},\quad w_2=\tfrac{1-t}{8},\quad w_3=-\tfrac{t}{16}\qquad \text{with}\quad t\in\mathbb{R}. \tag{6}$$

For $t=-\frac{1}{2}$, then $a=a_6^B(\cdot-3)$ and $\operatorname{sr}(a,2)=6$, $\operatorname{lpm}(a)=2$ and $\operatorname{sm}_p(a,2)=5+1/p$ for all $1\leqslant p\leqslant\infty$. $\operatorname{sr}(a,2)=4$ if $t\neq-1/2$. Since $\widehat{a}(\xi)=e^{i3\xi}(1+e^{-i\xi})^4\widehat{b}(\xi)$ with $\widehat{b}(\xi):=-\frac{t}{32}+\frac{1+t}{16}e^{-i\xi}-\frac{t}{32}e^{-i2\xi}$, by item (5) of Corollary **??**, we have $\operatorname{sm}_\infty(a,2)=3-\log_2(1+t)$ provided $t>-1/2$. We only have $\operatorname{sm}_\infty(a,2)\geqslant 3-\log_2|t|$ for $t\leqslant-1/2$. When $t=0$, $a=a_4^B(\cdot-2)$ is the centered B-spline filter of order 4 with $\operatorname{sr}(a,2)=4$ and $\operatorname{lpm}(a)=2$. When $t=1$, a is an interpolatory 2-wavelet filter with $\operatorname{sr}(a,2)=4$ and $\operatorname{lpm}(a)=4$. See Figure 1 for its subdivision stencils.

Example 2 $(a,2,\{-1,1\})$ is a dual subdivision triplet with

$$a=\frac{1}{2}\{w_2,w_1,\underline{w_0},w_0,w_1,w_2\}_{[-2,3]},$$

where

$$w_0=\tfrac{12+3t}{16},\quad w_1=\tfrac{8-3t}{32},\quad w_2=-\tfrac{3t}{32}\qquad \text{with}\quad t\in\mathbb{R}. \tag{7}$$

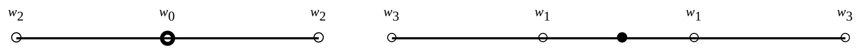

Figure 1: The 0-stencil (left) and the $\frac{1}{2}$-stencil (right) of the primal subdivision scheme in Example 1, where $w_0,\dots,w_3$ are given in (6). It is an interpolatory 2-wavelet filter if $w_2 = \frac{1-t}{8} = 0$. Since $\mathsf{M}=2$, each line segment (with endpoints $\circ$) in the coarse mesh $\mathbb{Z}$ is equally split into two line segments with one new vertex ($\bullet$) in the middle.

For $t = -\frac{2}{3}$, $a = a_5^B(\cdot - 2)$ and $\operatorname{sr}(a,2) = 5, \operatorname{lpm}(a) = 2$ and $\operatorname{sm}_p(a,2) = 4 + 1/p$ for all $1 \leqslant p \leqslant \infty$. Since $\widehat{a}(\xi) = e^{i2\xi}(1+e^{-i\xi})^3\widehat{b}(\xi)$ with $\widehat{b}(\xi) := -\frac{3t}{8} + \frac{4+3t}{32}e^{-i\xi} - \frac{3t}{8}e^{-i2\xi}$, by item (5) of Corollary **??**, we have $\operatorname{sr}(a,2) = 3$ and $\operatorname{sm}_\infty(a,2) = 4 - \log_2(4+3t)$ provided $t > -2/3$. We only have $\operatorname{sm}_\infty(a,2) \geqslant 1 - \log_2(3|t|)$ for $t \leqslant -2/3$. When $t = 0$, $a = a_3^B(\cdot - 1)$ is the shifted B-spline filter of order 3 with $\operatorname{sr}(a,2) = 3$ and $\operatorname{lpm}(a) = 2$. When $t = 1$, $\operatorname{sr}(a,2) = 3$ and $\operatorname{lpm}(a) = 4$. See Figure 2 for its subdivision stencils.

Figure 2: The 0-stencil (left) and the $\frac{1}{2}$-stencil (right) of the dual subdivision scheme in Example 2, where w_0, w_1, w_2 are given in (7). The $\frac{1}{2}$-stencil is the same as the 0-stencil. The value $[\mathscr{S}_{a,2}v](k)$ for $k \in \mathbb{Z}$ is attached to the center $\frac{k-1}{2}$ of the line segment $[k-1,k]$ instead of the vertex $\frac{k}{2}$. Since $\mathsf{M}=2$, each line segment is equally split into two.

Example 3 $(a, 3, \{-1, 1\})$ is a primal subdivision triplet with

$$a = \frac{1}{3}\{w_5, w_4, w_3, w_2, w_1, \underline{w_0}, w_1, w_2, w_3, w_4, w_5\}_{[-5,5]},$$

where

$$\begin{aligned} w_0 &= \tfrac{7-2t_1-8t_2}{9}, \quad w_1 = \tfrac{6-2t_1-5t_2}{9}, \quad w_2 = \tfrac{3+t_1+t_2}{9}, \\ w_3 &= \tfrac{1+t_1+4t_2}{9}, \quad w_4 = \tfrac{t_1+3t_2}{9}, \quad w_5 = \tfrac{t_2}{9} \end{aligned} \qquad \text{with} \quad t_1, t_2 \in \mathbb{R}. \tag{8}$$

For $t_1 = 2/9$ and $t_2 = 1/9$, $\operatorname{sr}(a,3) = 5$ and $\operatorname{sm}_p(a,3) = 4 + 1/p$ for all $1 \leqslant p \leqslant \infty$ whose 3-refinable function is the B-spline of order 5. Since $\widehat{a}(\xi) = (e^{i\xi} + 1 + e^{-i\xi})^3\widehat{b}(\xi)$ with

$$\widehat{b}(\xi) := \tfrac{t_2}{27}e^{i2\xi} + \tfrac{t_1}{27}e^{i\xi} + \tfrac{1-2t_1-2t_2}{27} + \tfrac{t_1}{27}e^{-i\xi} + \tfrac{t_2}{27}e^{-i2\xi},$$

by a similar result to item (5) of Corollary **??**, we have

$$\operatorname{sm}_\infty(a,2) \geqslant 2 - \log_3 \max(|1-2t_1-2t_2|, |2t_1|, |2t_2|).$$

For $t_1 = 7/9$ and $t_2 = -4/9$, a is an interpolatory 3-wavelet filter with $\operatorname{sr}(a,3) = 4 = \operatorname{lpm}(a)$ and $\operatorname{sm}_\infty(a,3) \geqslant \log_3 14 - 4 \approx 1.5978$. For $t_1 = 5/11$ and $t_2 = -4/11$, a is an interpolatory 3-wavelet filter with $\operatorname{sr}(a,3) = 3 = \operatorname{lpm}(a)$ and $\operatorname{sm}_\infty(a,3) \geqslant 2 + \log_3(11/10) \approx 2.0867$ (Using joint spectral radius, we in fact have $\operatorname{sm}_2(a,3) = \log_3 11 \approx 2.18266$). See Figure 3 for its subdivision stencils.

We now provide some subdivision curves in Figure 4 using the above subdivision triplets.

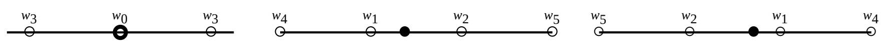

Figure 3: The 0-stencil (left), the $\frac{1}{3}$-stencil (middle), and $\frac{2}{3}$-stencil of the subdivision scheme in Example 3, where $w_0, \dots, w_5$ are given in (8). Due to symmetry, $\frac{2}{3}$-stencil is the same as the $\frac{1}{3}$-stencil. It is an interpolatory 3-wavelet filter if $w_3 = \frac{1+t_1+4t_2}{9} = 0$. Since $\mathsf{M} = 3$, each line segment (with endpoints $\circ$) is equally split into three line segments with two new inserted vertices ($\bullet$) at $\frac{1}{3} + \mathbb{Z}$ and $\frac{2}{3} + \mathbb{Z}$.

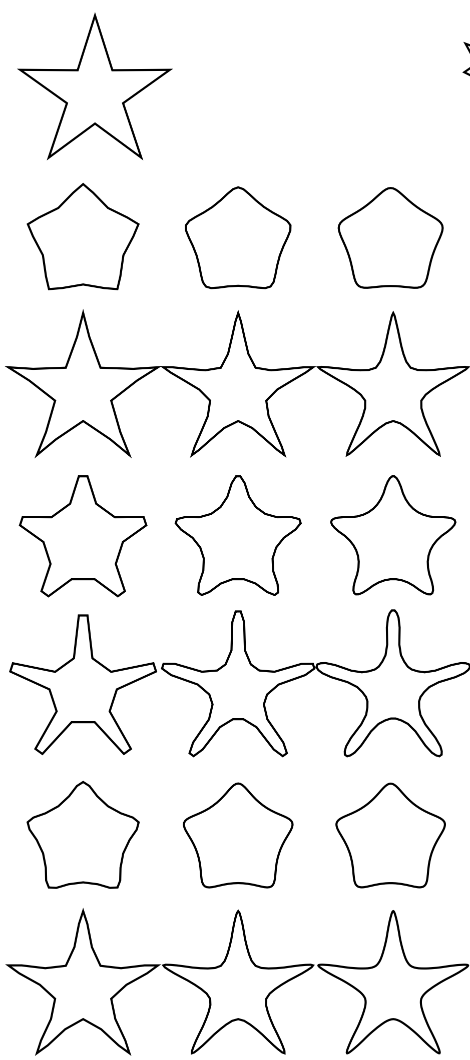

Figure 4: Subdivision curves at levels $1,2,3$ with the initial control polygons at the first row. The subdivision triplet $(a,2,\{-1,1\})$ in Example 1 is used with $t=-\frac{1}{2}$ ($a_4^B(\cdot-2)$) for the 2nd row and with $t=1$ (interpolatory) for the 3rd row. $(a,2,\{-1,1\})$ in Example 2 is used with $t=0$ ($a_3^B(\cdot-1)$, the corner cutting scheme) for the 4th row and with $t=1$ and $\operatorname{lpm}(a)=4$ for the 5th row. $(a,3,\{-1,1\})$ is used with $t_1=\frac{2}{9}, t_2=\frac{1}{9}$ for the 6th row and with $t_1=\frac{5}{11}, t_2=-\frac{4}{11}$ (interpolatory, $\operatorname{sm}_\infty(a,3)=\log_3 11$) for the 7th row.