Questions

(1) Does the number $\pi^n$ have first three significant digits 3, 1, and 4 (in that order) for some $n \in \mathbb{N}$? If so, does

$$\lim_{N \to \infty} \frac{\# \{1 \leq n \leq N : \pi^n \text{ has first three significant digits 3, 1, and 4} \}}{N}$$

exist?

(2) Assume $T : \mathbb{T} \to \mathbb{T}$ is $\lambda$-preserving. Show that the following statements are equivalent:

(i) $T$ is ergodic;

(ii) if $f \circ T(z) = f(z)$ holds for some (measurable, bounded) function $f : \mathbb{T} \to \mathbb{C}$ and $\lambda$-almost every $z \in \mathbb{T}$ then $f$ is constant ($\lambda$-a.e.).

(3) (i) Is the sequence $(\log_{10} n)$ u. d. mod 1?

(ii) Let $\vartheta \in \mathbb{R}$ be irrational. Is the sequence $(n\vartheta + \log_{10} n)$ u. d. mod 1?

(4) Given a partition $D_1, \ldots, D_9$ of $\mathbb{N}$ into nine infinite sets, write $D_j = \{d_{j,1}, d_{j,2}, \ldots \} = \{d_{j,n} : n \in \mathbb{N}\}$ with $d_{j,1} < d_{j,2} < \ldots$, and let $\delta_{j,n} = d_{j,n+1} - d_{j,n}$ for each $j \in \{1, \ldots, 9\}$ and $n \in \mathbb{N}$. Also, let $p = (p_1, \ldots, p_9) \in \mathbb{R}^9$ be a (non-degenerate) probability vector, i.e., $0 < p_j < 1$ and $\sum_{j=1}^{9} p_j = 1$. Consider the following three statements about the partition $D_1, \ldots, D_9$:

(a) $\lim_{N \to \infty} \frac{\# \{1 \leq n \leq N : n \in D_j \}}{N} = p_j$ $\forall j \in \{1, \ldots, 9\}$;

(b) $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{j,n} = \frac{1}{p_j}$ $\forall j \in \{1, \ldots, 9\}$;

(c) $\lim_{N \to \infty} \frac{\# \{1 \leq n \leq N : \delta_{j,n} = m \}}{N} = p_j (1 - p_j)^{m-1} \forall j \in \{1, \ldots, 9\}$ and $m \in \mathbb{N}$.

Turn (a)$\heartsuit$(b)$\spadesuit$(c) into a true logical statement by replacing $\heartsuit$, $\spadesuit$ with either $\Leftarrow$, $\Rightarrow$, or $\Leftrightarrow$. Whenever your choice is $\Leftarrow$ (resp. $\Rightarrow$) rather than $\Leftrightarrow$, give an example for which $\Rightarrow$ (resp. $\Leftarrow$) is false.

(5) Given any sequence $(a_n)$ of positive real numbers, let

$D_j = \{n \in \mathbb{N} : a_n \text{ has leading (decimal) digit } j \} \quad \forall j \in \{1, \ldots, 9\}$

Note that $D_1, \ldots, D_9$ is a partition of $\mathbb{N}$. Choose $p \in \mathbb{R}^9$ appropriately and determine which of the statements (a), (b), and (c) of Question 4 are correct for this partition, where

(i) $a_n = 6^n$ for all $n \in \mathbb{N}$;

(ii) $a_n = 6^{n^2}$ for all $n \in \mathbb{N}$.
Recommended reading.

Given the broad and diverse nature of the subject, the literature on dynamical systems is huge. Below is but a very short selection of books that you may find helpful when starting out to explore things for yourself. As far as I know, the University of Alberta library has copies of all of them. I’ll be happy to provide further references in case you need some.


