A Research Story: Compound Equations and Dynamics. 
Part 3

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Curves and Surfaces
A smooth curve $\gamma$ in $\mathbb{R}^n$ is a $C^1$ function $s \to x(s)$, $s \in I \subset \mathbb{R}$, $x(s) \in \mathbb{R}^n$.
A measure of the length of $\gamma$ is

$$l(\gamma) = \int_{\gamma} dl \overset{\text{def}}{=} \int_I \left\| \frac{dx}{ds}(s) \right\| ds$$

where $\left\| \cdot \right\|$ is a norm on $\mathbb{R}^n$. For example, the euclidean norm

$$\|x\| = \sqrt{(x_1)^2 + \cdots + (x_n)^2}$$

gives the usual measure of length

$$l(\gamma) = \int_I \sqrt{\frac{dx_1}{ds}^2 + \cdots + \frac{dx_n}{ds}^2} \, ds$$
A smooth 2-surface $\sigma$ in $\mathbb{R}^n$ is a $C^1$ function $(s_1, s_2) \to x(s_1, s_2)$, $(s_1, s_2) \in U \subset \mathbb{R}^2$, $x(s_1, s_2) \in \mathbb{R}^n$.

A measure of the area of $\sigma$ is

$$a_2(\sigma) = \int_{\sigma} da \overset{\text{def}}{=} \int_U \| x_{s_1} \wedge x_{s_2} \| \, ds_1 \, ds_2$$

where $x_{s_i} = \frac{\partial}{\partial s_i} x(s_1, s_2)$ and $\| \cdot \|$ is a norm on $\mathbb{R}^{(n_2)}$. If $\| \cdot \|$ is the Euclidean norm we have

$$a_2(\sigma) = \int_U \sqrt{\sum_{1 \leq i < j \leq n} \frac{\partial (x_i, x_j)}{\partial (s_1, s_2)}^2} \, ds_1 \, ds_2$$

where

$$\frac{\partial (x_i, x_j)}{\partial (s_1, s_2)} = \det \begin{bmatrix} \frac{\partial x_i}{\partial s_1} & \frac{\partial x_i}{\partial s_2} \\ \frac{\partial x_i}{\partial s_1} & \frac{\partial x_i}{\partial s_2} \end{bmatrix}.$$
A smooth \( k \)-surface \( \sigma \) in \( \mathbb{R}^n \) is a \( C^1 \) function
\[
(s_1, \ldots, s_k) \mapsto x(s_1, \ldots, s_k),
\]
\( s_1, \ldots, s_k \in U \subset \mathbb{R}^k \), \( x(s_1, \ldots, s_k) \in \mathbb{R}^n \).

A measure of the \( k \)-area of \( \sigma \) is

\[
a_k(\sigma) = \int_\sigma d\mathbf{a}_k \overset{\text{def}}{=} \int_U \| x_{s_1} \wedge \cdots \wedge x_{s_k} \| \, ds_1 \cdots ds_k
\]

where \( x_{s_i} = \frac{\partial}{\partial s_i} x(s_1, \ldots, s_k) \) and \( \| \cdot \| \) is a norm on \( \mathbb{R}^{(n)}_k \). If \( \| \cdot \| \) is the Euclidean norm we have

\[
a_k(\sigma) = \int_U \sqrt{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\partial (x_{i_1}, \ldots, x_{i_k})}{\partial (s_1, \ldots, s_k)}^2} \, ds_1 ds_2
\]

where

\[
\frac{\partial (x_{i_1}, \ldots, x_{i_k})}{\partial (s_1, \ldots, s_k)} = \det \begin{bmatrix}
\frac{\partial x_{i_1}}{\partial s_1} & \frac{\partial x_{i_1}}{\partial s_1} & \cdots & \frac{\partial x_{i_1}}{\partial s_k} \\
\frac{\partial x_{i_1}}{\partial s_1} & \frac{\partial x_{i_1}}{\partial s_1} & \cdots & \frac{\partial x_{i_1}}{\partial s_k} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial x_{i_k}}{\partial s_1} & \frac{\partial x_{i_k}}{\partial s_1} & \cdots & \frac{\partial x_{i_k}}{\partial s_k}
\end{bmatrix}.
\]
Nonlinear Differential Equations

\[ f \in C^1 (\mathbb{R}^n \to \mathbb{R}^n) \]

\[ \dot{x} = f(x) \quad \text{(N)} \]

Solution: \( x(t) = \phi(t) = \phi(t, x_0) \), is uniquely determined by \( x(0) = x_0 \) and, for simplicity, we will only consider equations for which solutions exist for all \( t > 0 \).

If \( \phi(t, x_0) = x_0 \) for all \( t \), then \( x_0 \) is called an equilibrium.

If \( \phi(t + \omega) = \phi(t) \), \( \omega > 0 \), the solution is periodic of period \( \omega \).

An orbit (positive semi-orbit) is a set \( \{ \phi(t) : 0 \leq t < \infty \} \).

The orbit of an equilibrium is a single point.

The orbit of a periodic solution is a simple closed curve (Jordan curve).
Linearization about a solution $\phi(t)$:

$$\dot{y} = \frac{\partial f}{\partial x}(\phi(t)) y$$

Solution is:

$$y(t) = \frac{\partial \phi}{\partial x_0}(t, x_0) y(0), \quad x_0 = \phi(0)$$

$$Y(t) = \frac{\partial \phi}{\partial x_0}(t, x_0) \text{ fundamental matrix, } Y(0) = I$$

"Proof": $y = \phi(t, x_0)$ solves $\dot{y} = f(y)$

$$\Rightarrow \frac{\partial \phi}{\partial t}(t, x_0) = f(\phi(t, x_0))$$

Differentiate with respect to $x_0$

$$\Rightarrow \frac{\partial^2 \phi}{\partial t \partial x_0}(t, x_0) = \frac{\partial^2 \phi}{\partial x_0 \partial t}(t, x_0) = \frac{\partial f}{\partial x}(\phi(t, x_0)) \frac{\partial \phi}{\partial x_0}(t, x_0)$$

$$\Rightarrow \dot{Y} = \frac{\partial f}{\partial x}(\phi(t)) Y$$
The $k$–th compound equation of $(L)$ is:

$$\dot{z} = \frac{\partial f^{[k]}}{\partial x} (\phi(t)) z$$  \hspace{1cm} (L_k)$$

Solution: $z(t) = \frac{\partial \phi^{(k)}}{\partial x_0} (t, x_0) z(0), \ x_0 = \phi(0)$

The case $k = n$ of $(L_k)$ is the Liouville equation:

$$\dot{z} = \text{div} f (\phi(t)) z$$  \hspace{1cm} (L_n)$$

Solution: $z(t) = \det \frac{\partial \phi}{\partial x_0} (t, x_0) z(0), \ x_0 = \phi(0)$
Suppose that $D \subset \mathbb{R}^n$ has finite $n$-dimensional measure $a_n(D)$, then the measure of $\phi(t, D)$ is

$$a_n(\phi(t, D)) = \int_{x \in \phi(t, D)} dx = \int_{x_0 \in D} \left| \det \frac{\partial \phi}{\partial x_0}(t, x_0) \right| dx_0$$

$(L_n) \Rightarrow \det \frac{\partial \phi}{\partial x_0}(t, x_0) = \exp \left[ \int_0^t \text{div } f(\phi(s, x_0)) \, ds \right]$. So, for example, if $\text{div } f < 0$ in $\mathbb{R}^n$, then the measure of the set $\phi(t, D)$ decreases with time.

When $n = 2$ this observation implies that no simply connected region where $\text{div } f < 0$ can contain a non-trivial periodic orbit of $(L)$. This is known as Bendixson’s Condition. Most textbooks prove this as a very nice application of Green’s Theorem.
Stability of the linearized equations \((L)\) and its compounds \((L_k)\) have many implications for the dynamics of \((N)\).

If \(\gamma_0: x = x_0(s), \ 0 \leq s \leq 1\) is a curve in \(\mathbb{R}^n\), then \(\gamma_t: x = \phi(t, x_0(s)), \ 0 \leq s \leq 1\) is also a curve in \(\mathbb{R}^n\) for each \(t \geq 0\).

\[
\begin{align*}
  l\gamma_0 & = \int_0^1 \left\| \frac{d}{ds} x_0(s) \right\| \ ds \\
  l\gamma_t & = \int_0^1 \left\| \frac{d}{ds} \phi(t, x_0(s)) \right\| \ ds = \int_0^1 \left\| \frac{\partial \phi}{\partial x_0}(t, x_0(s)) \frac{d}{ds} (x_0(s)) \right\| \ ds \\
  & \leq \int_0^1 \left\| \frac{\partial \phi}{\partial x_0}(t, x_0(s)) \right\| \left\| \frac{d}{ds} (x_0(s)) \right\| \ ds
\end{align*}
\]

We can conclude for example that, if \(\left\| \frac{\partial \phi}{\partial x_0}(t, x_0) \right\| \rightarrow 0\) uniformly with respect to \(x_0 \in \mathbb{R}^n\), then

- there is at most one equilibrium of \((N)\) and,
- any equilibrium attracts all other orbits.
If $\sigma_0 : (s_1, s_2) \rightarrow x(s_1, s_2)$ is a 2-surface in $\mathbb{R}^n$ then so also is $\sigma_t : (s_1, s_2) \rightarrow \phi(t, x(s_1, s_2))$.

We can use similar ideas to get higher dimensional Bendixson Conditions to rule out the existence of periodic orbits. These are conditions on $(L_2)$ that typically imply that some measure of surface area decreases in the dynamics. Another related type of condition would imply that $a_2 \sigma_t \rightarrow 0$.

The central idea is to observe that a periodic orbit $\gamma$ is invariant in the dynamics, $\phi(t, \gamma) = \gamma$. So, if $\Sigma_0$ is any surface which has $\gamma$ as its boundary, then $\Sigma_t = \phi(t, \Sigma_0)$ is also a surface with $\gamma$ as boundary. But if, among all surfaces with boundary $\gamma$, $\Sigma_0$ is a surface with minimum area and $(N)$ diminishes area we would contradict the minimality of $\Sigma_0$. So no such invariant closed curve can exist.
The following are Bendixson conditions for various measures of 2-surface area. Each reduces to the classical result when $n = 2$:

$$
\lambda_1 + \lambda_2 < 0 \quad \text{(RA Smith)}
$$

$$
\max_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r,s} \left( \left| \frac{\partial f_r}{\partial x_q} \right| + \left| \frac{\partial f_s}{\partial x_q} \right| \right) \right\} < 0
$$

$$
\max_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r,s} \left( \left| \frac{\partial f_q}{\partial x_r} \right| + \left| \frac{\partial f_q}{\partial x_s} \right| \right) \right\} < 0
$$

$$
\lambda_{n-1} + \lambda_n > 0
$$

$$
\min_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r,s} \left( \left| \frac{\partial f_r}{\partial x_q} \right| + \left| \frac{\partial f_s}{\partial x_q} \right| \right) \right\} > 0
$$

$$
\min_{r \neq s} \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r,s} \left( \left| \frac{\partial f_q}{\partial x_r} \right| + \left| \frac{\partial f_q}{\partial x_s} \right| \right) \right\} > 0
$$

$$
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ are the eigenvalues of } \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{\partial f^*}{\partial x} \right)
$$
General Compounds


\( \mathcal{X} \subset \mathcal{Y} \): General compound \( A^{[k]} \in \mathcal{L} \left( \wedge^k \mathcal{X} \rightarrow \wedge^k \mathcal{Y} \right) \). \( 0 \leq m \leq k \)

\[
A^{[k,m]} \left( v^1 \wedge \cdots \wedge v^k \right) \overset{\text{def}}{=} \sum_{(\varepsilon_1, \ldots, \varepsilon_k)} A^{\varepsilon_1} v^1 \wedge A^{\varepsilon_2} v^2 \wedge \cdots \wedge A^{\varepsilon_k} v^k
\]

\( \varepsilon_i \in \{0, 1\}, \ \varepsilon_1 + \cdots + \varepsilon_k = m, \ A^0 = I \)

\[
A^{[k,0]} = I^{(k)}, \quad A^{[k,1]} = A^{[k]}, \quad A^{[k,k]} = A^{(k)}
\]

\[
D_h^m (I + hA)^{(k)} \bigg|_{t=0} = m! A^{[k,m]}
\]
\[ D_h^m (I + hA)^{(k)} \bigg|_{t=0} = m! A^{[k,m]} \]

\[(I + hA)^{(k)} = \sum_{m=0}^{k} h^m A^{[k,m]} \]
\[= h A^{[k,1]} + h^2 A^{[k,2]} + \cdots + h^k A^{[k,k]} \]

If \( \lambda_1, \cdots, \lambda_n \) are the eigenvalues of \( A \) with eigenvectors \( v^1, \cdots, v^n \), then the eigenvalues of \((I + hA)^{(k)}\) are

\[ h (\lambda_{i_1} + \cdots + \lambda_{i_k}) + h^2 (\lambda_{i_1} \lambda_{i_2} + \cdots + \lambda_{i_{k-1}} \lambda_{i_k}) + \cdots + h^k (\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}) \]

with eigenvectors \( v^{i_1} \wedge v^{i_2} \wedge \cdots \wedge v^{i_k} \).


