A Research Story: Compound Equations and Dynamics.
Part I

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Compound Matrices

$m \times n$ matrix:

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \ 1 \leq i \leq m, \ 1 \leq j \leq n$$

$p \times p$ minor:

$$a_{r_1...r_p}^{s_1...s_p} = \det \begin{bmatrix} a_{r_i}^{s_i} \end{bmatrix}, \ 1 \leq i, j \leq p,$$

minor of $A$ determined by the rows $r_1, \ldots, r_p$ and the columns $s_1, \ldots, s_p$

examples:

$$a_{12}^{12} = \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix} = 0$$

$$a_{12}^{12} = \begin{vmatrix} a_1^1 & a_2^1 \\ a_2^1 & a_2^2 \end{vmatrix}, \ a_{12}^{13} = \begin{vmatrix} a_1^1 & a_2^2 \\ a_3^1 & a_3^2 \end{vmatrix}$$
\[
a_{122}^{123} = \begin{vmatrix}
a_1^1 & a_1^2 & a_1^3 \\
a_2^1 & a_2^2 & a_2^3 \\
a_3^1 & a_3^2 & a_3^3 \\
\end{vmatrix} = 0
\]

\[
a_{123}^{123} = \begin{vmatrix}
a_1^1 & a_1^2 & a_1^3 \\
a_2^1 & a_2^2 & a_2^3 \\
a_3^1 & a_3^2 & a_3^3 \\
\end{vmatrix}, \ a_{123}^{124} = \begin{vmatrix}
a_1^1 & a_1^2 & a_1^4 \\
a_2^1 & a_2^2 & a_2^4 \\
a_3^1 & a_3^2 & a_3^4 \\
\end{vmatrix}
\]
$m = n > p$, cofactor matrix:

$$A^{s_1 \ldots s_p}_{r_1 \ldots r_p} \text{ is the cofactor of } a^{s_1 \ldots s_p}_{r_1 \ldots r_p},$$

i.e. it is the signed minor determined by the rows complementary to rows $r_1, \ldots, r_p$ and by the columns complementary to columns $s_1, \ldots, s_p$ multiplied by $(-1)^{r_1+s_1+\ldots+r_p+s_p}$.

If $p = n$, define

$$A^{12\ldots n}_{12\ldots n} = 1.$$
$3 \times 3$ matrix $A = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix}$:

$A_1^1 = a_{23}^{23} = \begin{vmatrix} a_2^2 & a_2^3 \\ a_3^2 & a_3^3 \end{vmatrix}$, $A_2^3 = -a_{13}^{12} = -\begin{vmatrix} a_1^1 & a_1^2 \\ a_3^1 & a_3^2 \end{vmatrix}$

$A_{12}^{12} = a_3^3$, $A_{13}^{12} = -a_2^3$ and $A_{123}^{123} = 1$.

Note that in this case we have

$\det A = a_i^1 A_i^1 + a_i^2 A_i^2 + a_i^3 A_i^3$, $i = 1, 2, 3$

and

$0 = a^1_i A_i^1 + a^2_i A_i^2 + a^3_i A_i^3$, if $i \neq r$.

$n \times n$ matrix $A$: Laplace expansion

by rows $\det A = \sum_{j=1}^{n} a_i^j A_i^j$, $i = 1, 2, \ldots, n$

by columns $\det A = \sum_{i=1}^{n} a_i^j A_i^j$, $j = 1, 2, \ldots, n$
If $A$ is a $n \times n$ matrix and $1 \leq k \leq n$, then the *Laplace* expansions by minors are:

\[
\det A = \sum_{1 \leq s_1 < \ldots < s_k \leq n} a_{r_1 \ldots r_k} ^{s_1 \ldots s_k} A_{r_1 \ldots r_k} ^{s_1 \ldots s_k}, \text{ if } 1 \leq r_1 < \ldots < r_k \leq n
\]

\[
\det A = \sum_{1 \leq r_1 < \ldots < r_k \leq n} a_{r_1 \ldots r_k} ^{s_1 \ldots s_k} A_{r_1 \ldots r_k} ^{s_1 \ldots s_k}, \text{ if } 1 \leq s_1 < \ldots < s_k \leq n
\]

Note: $0 = \sum_{(s)} a_{r_1 \ldots r_k} ^{s_1 \ldots s_k} A_{t_1 \ldots t_k} ^{s_1 \ldots s_k}$, if $(r) \neq (t)$, and $0 = \sum_{(r)} a_{r_1 \ldots r_k} ^{s_1 \ldots s_k} A_{r_1 \ldots r_k} ^{t_1 \ldots t_k}$, if $(s) \neq (t)$.
$n \times n$ matrix $A$: Cofactor matrix of $A$:

$$cofA = \left[ A^i_j \right], \ i, j = 1, ..., n$$

Adjugate (or classical adjoint) matrix of $A$:

$$adjA = (cofA)^T.$$ 

Properties:

$$A (adjA) = (adjA) A = (\det A) I$$

$$A^{-1} = \frac{1}{\det A} adjA$$

$$\det (cofA) = \det (adjA) = (\det A)^{n-1}$$
Multiplicative Compounds

$n \times m$ matrix $A$, $1 \leq k \leq \min\{n, m\}$

$k$-th multiplicative compound is the \( \binom{n}{k} \times \binom{m}{k} \) matrix

\[
A^{(k)} = \begin{bmatrix}
a_{s_1 \ldots s_k}^{r_1 \ldots r_k}
\end{bmatrix} = \begin{bmatrix}
a_{(s)}^{(r)}
\end{bmatrix}
\]

The entry in the $r$-th row and the $s$-th column of $A^{(k)}$ is $a_{s_1 \ldots s_k}^{r_1 \ldots r_k} = a_{(s)}^{(r)}$, where $(r) = (r_1, ..., r_k)$ is the $r$-th member of the lexicographic ordering of the integers $1 \leq r_1 < r_2 < ... < r_k \leq m$ and $(s) = (s_1, ... s_k)$ is the $s$-th member in the lexicographic (dictionary) ordering of all $k$-tuples of the integers $1 \leq s_1 < s_2 < ... < s_k \leq n$:

\[
\begin{align*}
1 & \leq r_1 < r_2 < r_3 \leq 5 \\
(1) & = (123), (2) = (124), (3) = (125), (4) = (134), (5) = (135), \\
(6) & = (145), (7) = (234), (8) = (235), (9) = (245), (10) = (345).
\end{align*}
\]
Example:

\[
A = \begin{bmatrix}
    a_1^1 & a_1^2 \\
    a_2^1 & a_2^2 \\
    \vdots & \vdots \\
    a_m^1 & a_m^2
\end{bmatrix}_{m \times 2}, \quad A^{(2)} = \begin{bmatrix}
    a_{12}^1 \\
    a_{12}^2 \\
    \vdots \\
    a_{m-1,m}^{12}
\end{bmatrix}_{(m \choose 2) \times 1}
\]

Binet-Cauchy Theorem:

\[AB = C \Rightarrow A^{(k)}B^{(k)} = C^{(k)}\]

Sylvester’s Theorem:

\[\det A^{(k)} = (\det A)^{\frac{n-1}{k-1}}\]
Linear Differential Equations

\[ \dot{x} = A(t) x \quad (L) \]

\( t \in [0, \infty), \ x \in \mathbb{R}^n, \ t \rightarrow A(t)_{n \times n} \) continuous.

A solution \( x(t) \) of \((L)\) is uniquely determined by its value \( x(t_0) \) at any point \( t_0 \in [0, \infty) \).

\( X(t)_{n \times m} \) is a solution matrix of \((L)\) if \( \dot{X}(t) = A(t) X(t) \)

\( X(t) \) is a fundamental matrix of \((L)\) if it is \( n \times n \), non-singular and \( \dot{X}(t) = A(t) X(t) \)
The columns of a fundamental matrix span the solution space of \((L)\):

\(x(t)\) is a solution of \((L)\) \iff there exists \(c \in \mathbb{R}^n\) such that

\[ x(t) = X(t)c. \]

Equivalently, the columns of \(X(t)\) are solutions of \((L)\) which span the solution space of \((L)\).

In particular, each column of \(X(t)\) is a solution of \((L)\).

Suppose that \(X(t)\) is a fundamental matrix of \((L)\), then a \(n \times n\) matrix \(Y(t)\) is a fundamental matrix of \((L)\) if and only if there is a constant non-singular matrix \(C\) such that \(Y(t) = X(t)C\).
Any continuously differentiable $n \times n$ matrix $X(t)$ is a fundamental matrix for some linear differential equation $(L) \iff X(t)$ is non-singular:

$$A(t) = \dot{X}(t)X^{-1}(t)$$
$$\dot{X}(t) = A(t)X(t)$$
Compound Differential Equations

Recall, from Sylvester’s Theorem, \( \det X(t)^{(k)} = (\det X(t))^{(n-1)}_{k-1} \) so that \( \det X(t) \neq 0 \Rightarrow \det X^{(k)}(t) \neq 0 \). So \( Y(t) = X^{(k)}(t) = [x_{r_1\cdots r_k}^{s_1\cdots s_k}(t)] \) is a fundamental matrix for a \( (n \choose k) \)-dimensional equation. The coefficient matrix in this equation is denoted \( A^{[k]} \)

\[
\dot{y} = A^{[k]}(t)y \quad (k)
\]

the \( k \)-th compound equation of \( (L) \). Note that \( A^{[1]} = A \), \( A^{[n]} = \text{tr } A \)

\[
\dot{y} = A(t)y \quad (1)
\]

\[
\dot{y} = \text{tr } A(t)y \quad (n)
\]

In the case \( k = n \), \( X^{(n)}(t) = \det X(t) \), and \( (n) \) is the famous Abel-Jacobi scalar equation which gives

\[
\det X(t) = \det X(t_0) \exp \left( \int_{t_0}^{t} \text{tr } A(s) \, ds \right)
\]
If $X(t)$ is a $n \times m$ solution matrix of $(L)$, then $Y(t) = X^{(k)}(t)$ is a \( \binom{n}{k} \times \binom{m}{k} \) solution matrix of $(k)$

Example:

\[
X = \begin{bmatrix}
 x_1^1 & x_2^1 \\
 x_1^2 & x_2^2 \\
 \vdots & \vdots \\
 x_m^1 & x_m^2
\end{bmatrix}_{m \times 2}, \quad x^{(2)} = \begin{bmatrix}
 x_{12}^{12} \\
 x_{12}^{13} \\
 \vdots \\
 x_{m-1,m}^{12}
\end{bmatrix}_{(m-1) \times 1}
\]
Additive Compounds

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix}, \ 1 \leq i, j \leq m = n \]

\[ C = A^{[k]}, 1 \leq k \leq m = n \] is called the \( k \)-th additive compound of \( A \)

\[ c_r^s = \begin{cases} 
  a_{r_1}^{r_{1}} + \cdots + a_{r_k}^{r_{k}}, & \text{if } (r) = (s) \\
  (-1)^{i+j} a_{r_i}^{s_j}, & \text{if exactly one entry } r_i \text{ in } (r) \\
  0, & \text{does not occur in } (s) \text{ and } s_j \\
 & \text{does not occur in } (r) \\
 & \text{if } (r) \text{ differs from } (s) \text{ in two or more entries} 
\end{cases} \]

Additivitiy:

\[ (A + B)^{[k]} = A^{[k]} + B^{[k]} \]
Examples:

\( n = 2 : \)

\[
A^{[1]} = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix} = A
\]

\[
A^{[2]} = a_{11} + a_{22} = \text{tr}A
\]

\( n = 3 : \)

\[
A^{[1]} = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix} = A
\]

\[
A^{[2]} = \begin{bmatrix}
    a_{11} + a_{22} & a_{23} & -a_{13} \\
    a_{32} & a_{11} + a_{33} & a_{12} \\
    -a_{31} & a_{21} & a_{22} + a_{33}
\end{bmatrix}
\]

\[
A^{[3]} = a_{11} + a_{22} + a_{33} = \text{tr}A
\]
\( n = 4 \):

\[
A^{[2]} = \begin{bmatrix}
    a_{11} + a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\
    a_{32} & a_{11} + a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\
    a_{42} & a_{43} & a_{11} + a_{44} & 0 & a_{12} & a_{13} \\
    -a_{31} & a_{21} & 0 & a_{22} + a_{33} & a_{34} & -a_{24} \\
    -a_{41} & 0 & a_{21} & a_{43} & a_{22} + a_{44} & a_{23} \\
    0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33} + a_{44}
\end{bmatrix}
\]

\[
A^{[3]} = \begin{bmatrix}
    a_{11} + a_{22} + a_{33} & a_{34} & -a_{24} & a_{14} \\
    a_{43} & a_{11} + a_{22} + a_{44} & a_{23} & -a_{13} \\
    -a_{42} & a_{32} & a_{11} + a_{33} + a_{44} & a_{12} \\
    a_{41} & -a_{31} & a_{21} & a_{22} + a_{33} + a_{44}
\end{bmatrix}
\]

\[A^{[4]} = a_{11} + a_{22} + a_{33} + a_{44} = trA\]
Geometrical Interpretation

Solutions $x^1(t)$, $x^2(t)$ of $(L)$ with $n = 3$ may be interpreted as oriented line segments in $\mathbb{R}^3$ whose projections on a basis $e^1, e^2, e^3$ are

$$
\begin{bmatrix}
  x_1^1(t) \\
  x_2^1(t) \\
  x_3^1(t)
\end{bmatrix}
$$

and whose evolution in time is governed by $(L)$. If

$$
X(t) = \begin{bmatrix}
  x_1^1(t) & x_1^2(t) \\
  x_2^1(t) & x_2^2(t) \\
  x_3^1(t) & x_3^2(t)
\end{bmatrix},
$$

then $X^{(2)}(t) = \begin{bmatrix}
  x_{12}^{12} \\
  x_{12}^{13} \\
  x_{12}^{23}
\end{bmatrix}$ satisfies (2) and may be considered as an oriented 2-dimensional parallelogram in $\mathbb{R}^3$ whose projection onto the $(e^i, e^j)$ coordinate plane, $i < j$, is a parallelogram with area $x_{ij}^{12}$. 
If \( x^1(t), \ldots, x^k(t) \) are considered as an ordered set of oriented line segments in \( \mathbb{R}^n \) changing with time, then \( y(t) = x_{r_1r_2\cdots r_k}(t) \) may be interpreted as the projection of the corresponding \( k \)-dimensional oriented parallelopiped in \( \mathbb{R}^n \) onto the \( k \)-dimensional coordinate subspace spanned by \( e_{r_1}, \ldots, e_{r_k} \).
\[(\exp A)^{(k)} = \exp \left( A^{[k]} \right)\]

\[\frac{d}{dt} \left( I + tA \right)^{(k)} \bigg|_{t=0} = A^{[k]}\]

The last expression is sometimes taken as the definition of \(A^{[k]}\)