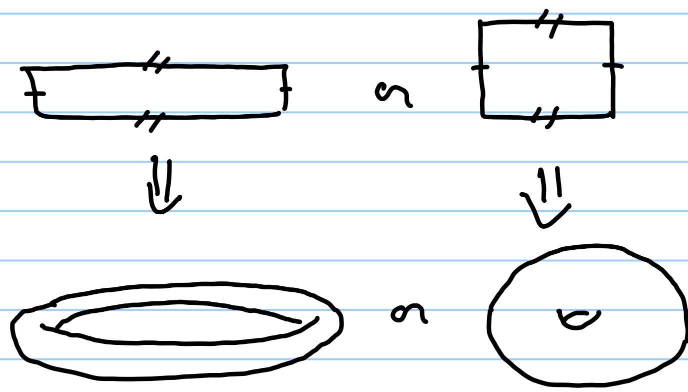
What is the "best" torus?

How to make a torus in $\mathbb{R}^3$

1. Get a rubber sheet and cut it to make a rectangle (or just a parallelogram)

2. Glue the opposite sides (you will have to stretch the rubber sheet).



The 2-dimensional surface is called a <u>torus</u> (plural: tori).

The 3-dimensional interior is called a <u>solid torus</u>.

For example, doughnuts (and bagels) are solid tori.

Which is (mathematically) best, the torus on the left or on the right (above)?

T. J. Willmore asked this question in 1965.
F. Codá Marques and A. Neves fully answered it in 2012.

Willmore Conjecture [now a theorem of Marques – Neves]:

Let Σ be a torus in $\mathbb{R}^3$ with mean curvature H. Then

(i) $W = \int_\Sigma H^2 \, dA \geq 2\pi^2$ and

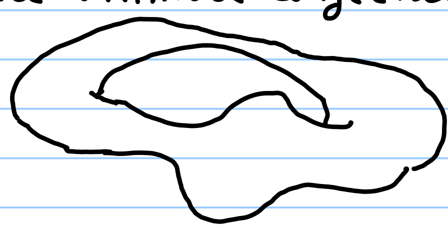
(ii) there is a torus (the Willmore torus) Σ such that

$$W = \int_\Sigma H^2 \, dA = 2\pi^2$$

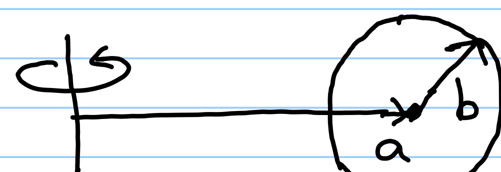
W is called the Willmore Energy.

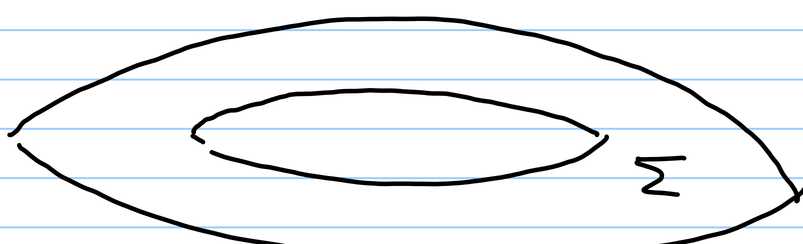
Note that here dA means the area element on the surface Σ (familiar from multivariable Calculus).

The Willmore conjecture considers all tori in $\mathbb{R}^3$:



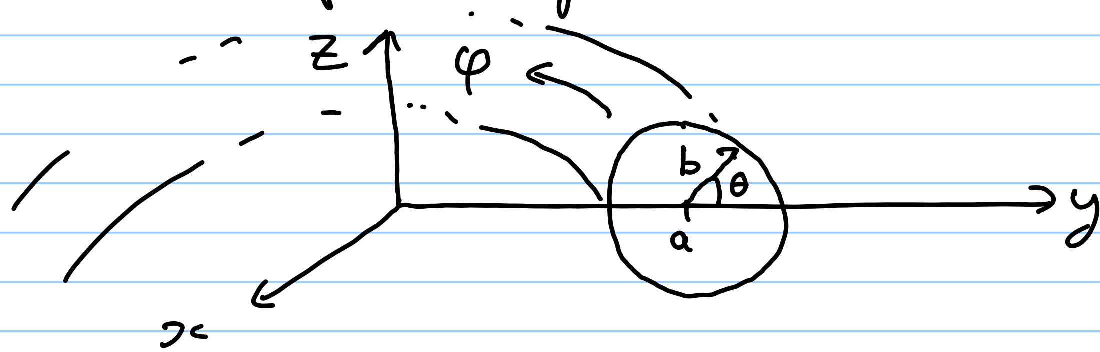
But we will just consider standard tori: draw a circle of radius a and then consider the tube of radius $b < a$ about that circle.





What ratio $\frac{a}{b} = k$ minimizes W?

Parametric equations of a standard torus



$$x(\theta,\varphi) = (a + b\cos\theta)\cos\phi$$

$$y(\theta,\varphi) = (a + b\cos\theta)\sin\phi$$

$$z(\theta,\varphi) = b\sin\theta$$

Area element of a standard torus

$$dA = b(a+b\cos\theta)\,d\theta\,d\varphi \quad \text{--- (1)}$$

The principal curvatures are

$$\left.\begin{aligned} &\text{(i)} \quad \kappa_1 = \frac{1}{b} \\ &\text{(ii)} \quad \kappa_2 = \frac{\cos\theta}{a+b\cos\theta} \end{aligned}\right\} \quad (2)$$

(Can you derive these?)

Then

$$W = \int_0^{2\pi} d\varphi \int_0^{2\pi} \frac{1}{4}\left[\frac{1}{b^2} + \frac{\cos^2\theta}{(a+b\cos\theta)^2}\right] b(a+b\cos\theta)\,d\theta$$

$$= \frac{\pi}{2}\int_0^{2\pi}\left[1 + \frac{\cos^2\theta}{(k+\cos\theta)^2}\right](k+\cos\theta)\,d\theta, \quad k = \frac{a}{b}.$$

$$= \frac{\pi}{2}\int_0^{2\pi}\left[k + \cos\theta + \frac{\cos^2\theta}{(k+\cos\theta)}\right]d\theta$$

Now $\dfrac{\cos^2\theta}{k+\cos\theta} = \dfrac{\cos^2\theta - k^2}{k+\cos\theta} + \dfrac{k^2}{k+\cos\theta}$

$$= (\cos\theta - k) + \frac{k^2}{k+\cos\theta}$$

$$\Rightarrow W = \frac{\pi}{2}\int_0^{2\pi}\left[2\cos\theta + \frac{k^2}{k+\cos\theta}\right]d\theta$$

$$= \frac{\pi k^2}{2}\int_0^{2\pi}\frac{d\theta}{k+\cos\theta} \quad \text{since } \int_0^{2\pi}\cos\theta\, d\theta = 0$$

$$= \frac{\pi^2 k^2}{\sqrt{k^2-1}}$$

(Compute this integral yourself. Hint: Start from the substitution $t = \tan(\theta/2)$.)

This is a minimum when $k = \sqrt{2}$ (You should check this).

$\Rightarrow W \geq 2\pi^2$ and $W = 2\pi^2$ if and only if $k = \frac{a}{b} = \sqrt{2}$ for the special case of standard tori.

$\Rightarrow$ For standard tori at least, the "best" torus (in the sense of Willmore) is given by

$$x(\theta,\varphi) = b\left(\sqrt{2} + \cos\theta\right)\cos\varphi$$
$$y(\theta,\varphi) = b\left(\sqrt{2} + \cos\theta\right)\sin\varphi$$
$$z(\theta,\varphi) = b\sin\theta$$

for any $b > 0$.

There are a large number of open questions about minimal surfaces, CMC surfaces, Willmore surfaces.

1. Prove that any 3-dimensional closed manifold contains infinitely many immersed minimal surfaces.

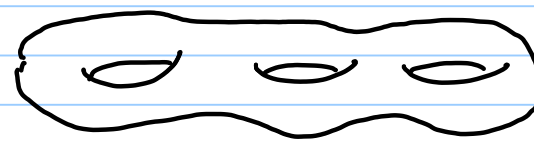
   * "Generic case" proved by Irie, Marques, Neves (2017)

2. Prove that any manifold diffeomorphic to the 3-sphere admits at least 4 distinct minimal spheres

   * Birkhoff (early 20^{th} century) found 1.
   * Haslhofer and Ketover (2017) found 2.

3. What is the correct Willmore conjecture for "higher genus surfaces", and are there higher genus Willmore surfaces?

Higher genus surface, also called a multi-torus



g = genus = number of holes

In this drawing, $g=3$

By "manifold" above, we mean a smooth set of points such that each point has a neighbourhood that looks like a neighbourhood of a point in $\mathbb{R}^n$.

Two-dimensional surfaces are 2-manifolds.

The 3-sphere $x^2+y^2+z^2+w^2$ in $\mathbb{R}^4$ is a 3-manifold.

Riemannian Manifolds

For surfaces in $\mathbb{R}^3$, we defined the Gauss Curvature

$$K_G = \kappa_1 \kappa_2$$
$$= \text{product of the principal curvatures}$$

* Gauss Curvature of round $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=a^2\}$ in $\mathbb{R}^3$:

$$\kappa_1 = \kappa_2 = 1/a \quad \text{so} \quad K_G = 1/a^2$$

$$\int_{S^2} K_G \, dS = \frac{1}{a^2} \cdot (\text{Surface Area}) = \frac{1}{a^2} \cdot 4\pi a^2 = 4\pi.$$

* Gauss Curvature of a Standard Embedded torus Σ in $\mathbb{R}^3$:

$$\kappa_1 = \frac{1}{b}, \quad \kappa_2 = \frac{\cos\theta}{a+b\cos\theta} \quad \text{so} \quad K_G = \frac{\cos\theta}{b(a+b\cos\theta)}.$$

$$\int_\Sigma K_G \, d\Sigma = \int_0^{2\pi} \left(\int_0^{2\pi} \frac{\cos\theta}{b(a+b\cos\theta)} (a+b\cos\theta)\, d\varphi \right) b \, d\theta$$

$$= 2\pi \int_0^{2\pi} \cos\theta \, d\theta = 0$$

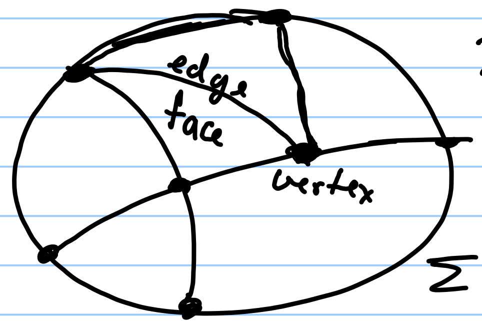
These are examples of the <u>Gauss-Bonnet theorem</u>:

If Σ is a compact smooth 2-dimensional surface without a boundary (i.e., a closed 2-manifold) then
$$\int_\Sigma K_G \, d\Sigma = 2\pi \chi(\Sigma)$$

where $\chi(\Sigma) = 2(1-g)$, g = genus = number of holes.

χ is called the <u>Euler characteristic</u>. This definition of χ is for <u>orientable</u> manifolds Σ.

For a general definition, cover Σ by "triangles" that meet along their edges (which don't have to be straight lines) and vertices.



$$\chi(\Sigma) = f - e + v$$

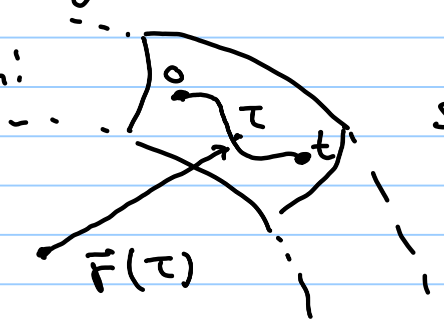
f = number of faces (2-dimensional)
e = number of edges (1-dimensional)
v = number of vertices (0-dimensional)

$\therefore \int_\Sigma K_G \, d\Sigma$ has nothing to do with the details of the shape (curvature) of Σ. It just measures the <u>topology</u> of Σ.

Gauss Curvature is "intrinsic"

How do you measure the arc length of a curve on a 2-sphere?

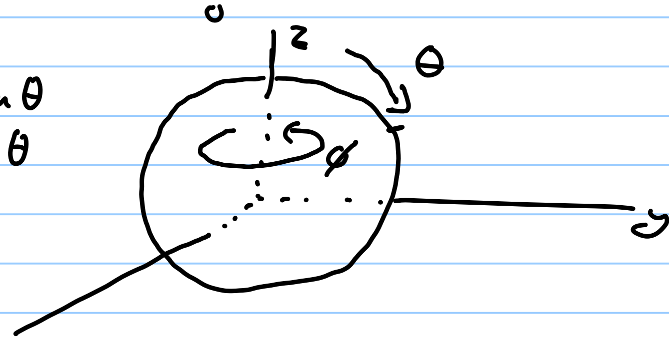
Answer:



$$s(t) = \int_0^t |\vec{r}\,'(\tau)| \, d\tau$$

$$= \int_0^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2} \, d\tau$$

Sphere:
$$x = a \cos\phi \sin\theta$$
$$y = a \sin\phi \sin\theta$$
$$z = a \cos\theta$$
$$\theta \in [0, \pi]$$
$$\phi \in [0, 2\pi]$$



Now $\frac{dx}{d\tau} = -a\sin\phi\sin\theta\frac{d\phi}{d\tau} + a\cos\phi\cos\theta\frac{d\theta}{d\tau}$

$$\frac{dy}{d\tau} = a\cos\phi\sin\theta\frac{d\phi}{d\tau} + a\sin\phi\cos\theta\frac{d\theta}{d\tau}$$

$$\frac{dz}{d\tau} = -a\sin\theta\frac{d\theta}{d\tau}$$

<u>Exercise</u>: Check that

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 = a^2\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]$$

$$= \begin{bmatrix}\frac{d\theta}{d\tau} & \frac{d\phi}{d\tau}\end{bmatrix}\underbrace{\begin{bmatrix}a^2 & 0\\ 0 & a^2\sin^2\theta\end{bmatrix}}_{\text{We call this matrix } g.}\begin{bmatrix}d\theta/d\tau\\ d\phi/d\tau\end{bmatrix}$$

$$\Rightarrow s(t) = \int_0^t \left(\left[\vec{r}\,'(\tau)\right]^T g(\tau)\left[\vec{r}\,'(\tau)\right]\right)^{1/2} d\tau$$

g is called a Riemannian <u>metric</u> <u>tensor</u> (or simply a <u>metric</u>, but the term <u>metric</u> is also used in analysis to describe a different but related object — you can define a <u>metric</u> which is not a Riemannian metric tensor).

- One can compute K_G from a formula, given g.
- But g uses only coordinates θ, ϕ on S^2, not the coordinates (x, y, z) of $\mathbb{R}^3$.

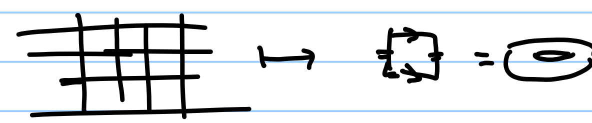
$\Rightarrow$ K_G is "intrinsic." The Gauss curvature of a 2-manifold does not depend on how that 2-manifold is embedded in $\mathbb{R}^3$, or even if it is embedded at all! This is Gauss's <u>Theorema Egregium</u> (latin for "Remarkable Theorem").

* True even though our original definition $K_G = \kappa_1 \kappa_2$ required $\mathbb{R}^3$ (since the κ_i were inverses of radii of osculating circles).

<u>Idea behind Riemannian Geometry</u>:

- Write down a metric tensor g. Note that g can depend on position (it can be different at different points p in M). So g is a map $g: p \mapsto \langle \, , \rangle$ giving an inner product $\langle \, , \rangle$ at each $p \in M$. We use this inner product to define lengths of tangent vectors (and angles between vectors).

<u>e.g.</u> "Flat tori": In the plane $\mathbb{R}^2$, identify points (x,y) with $(x+a, y+b)$ for fixed constants $a>0$ and $b>0$.



This gives a torus with the same flat geometry as the plane. $\Rightarrow K_G = 0$.