Note Title: Geometric Analysis 17/07/2018

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Geometric Analysis = Geometry + Analysis (Calculus)

* Curves, surfaces, manifolds
* Curvature
* Applications: General relativity
  Topology
  Applied Mathematics
  Information Theory
  Statistics

A project: Can you write a computer program to do this?
http://a.carapetis.com/csf/

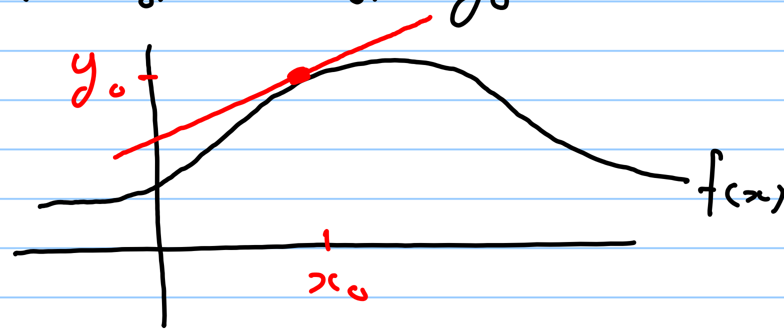
Today's topic: What is this webpage doing?
Answer: "Curve shortening flow"
To explain, first we have to study curves and their curvature.

Curvature

Graph of the function $x \mapsto f(x)$

Tangent line ("linear approximation") at (x_0, y_0)

$$y = f'(x_0)(x - x_0) + y_0$$



The osculating circle — Find a circle that

(i) passes through (x_0, y_0),

(ii) has the same tangent line as $f(x)$ does at (x_0, y_0), and

(iii) has the same second derivative (concavity/convexity) as $f(x)$ does at (x_0, y_0)

Notice: These are 3 conditions for the 3 free parameters a, b, c in the equation of a circle:

$$(x-a)^2 + (y-b)^2 = c^2 \quad --- (*)$$

Solution: Using (i), then $c^2 = (x_0 - a)^2 + (y_0 - b)^2 \quad --- (i)$

Differentiate $(*)$: $\quad x - a + (y-b)y' = 0 \quad --- (**)$

But when $(x,y) = (x_0, y_0)$ then $y' = f'(x_0)$.

$$\Rightarrow (x_0 - a)^2 = (f'(x_0))^2 (y_0 - b)^2 \quad --- (ii)$$

Combine (i), (ii) $\Rightarrow$ $c^2 = \left[1 + (f'(x_0))^2\right](y_0 - b)^2 \quad --- (ii')$

Finally, differentiate (**) to get

$$1+(y')^2+(y-b)y''=0$$

At (x_0, y_0) we have $y' = f'(x_0)$, $y'' = f''(x_0)$.

$$\Rightarrow \quad 1+(f'(x_0))^2+(y_0-b)f''(x_0) = 0$$

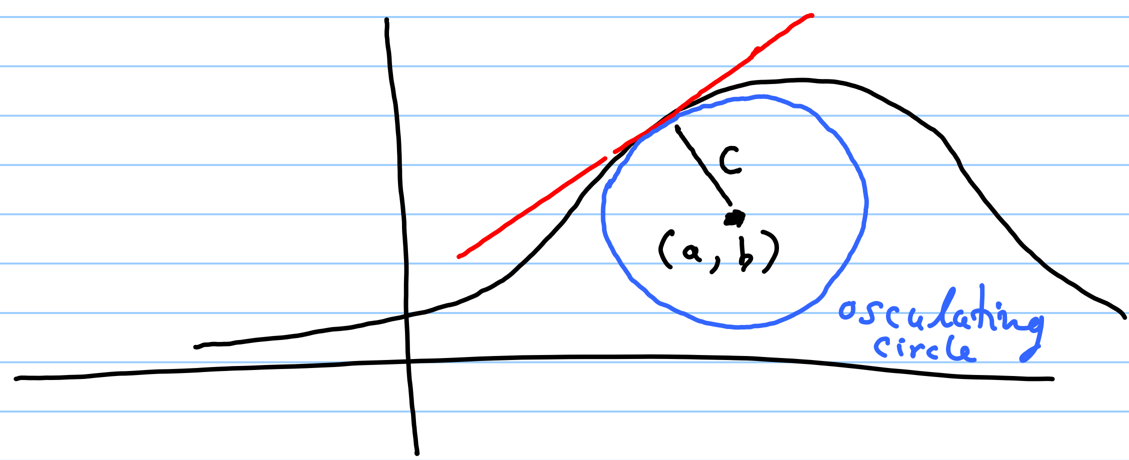
$$\Rightarrow \quad (y_0-b)^2\,(f''(x_0))^2 = \left[1+(f'(x_0))^2\right]^2 \quad \text{--- (iii)}$$

Using (ii'), we get:

$$c^2 = \left[1+(f'(x_0))^2\right]^3 \Big/ (f''(x_0))^2 \quad \text{--- (***)}$$

provided $f''(x_0) \neq 0$ (this happens, for example, at smooth points of inflection).

* Now can find a, b in terms of $f'(x_0)$, $f''(x_0)$.

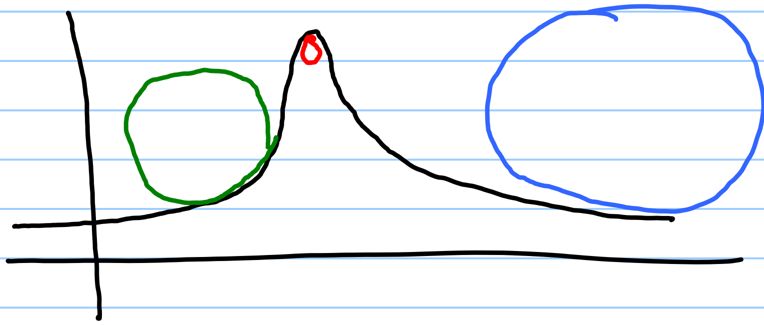


* The osculation circle is a "better" approximation to $x \mapsto f(x)$ at (x_0, y_0).

$$f(x) = \underbrace{\underbrace{f(x_0)+f'(x_0)(x-x_0)}_{\text{tangent line approximation}}+\tfrac{1}{2}f''(x_0)(x-x_0)^2}_{\text{quadratic approximation}}+\cdots$$

Small circle $\Rightarrow$ large curvature

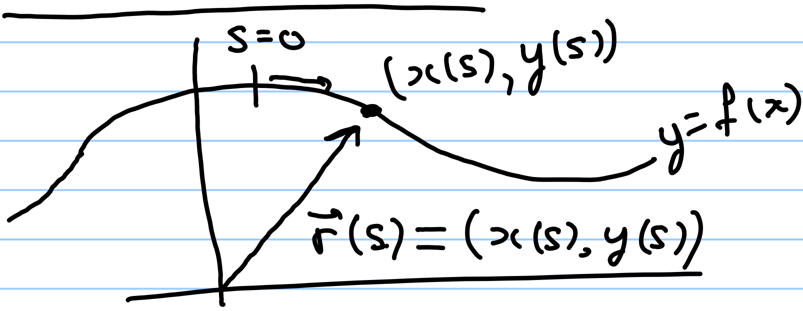
Large circle $\Rightarrow$ small curvature



Define the curvature κ to be the inverse of the radius of the osculating circle:

$$\kappa := \frac{1}{c} = \frac{f''(x_0)}{\left[1+(f'(x_0))^2\right]^{3/2}}$$

Parametrized Curves



$$\frac{dy}{ds} = \frac{df}{dx}\frac{dx}{ds} \quad \text{by chain rule.}$$

$$\Rightarrow \frac{df}{dx} = \frac{dy}{ds} \Big/ \frac{dx}{ds} \quad \left(\text{assume } \frac{dx}{ds} \neq 0\right) \text{ --- [1]}$$

Now $$\frac{d^2y}{ds^2} = \frac{d}{ds}\left(\frac{df}{dx}\frac{dx}{ds}\right) = \left(\frac{d}{ds}\frac{df}{dx}\right)\frac{dx}{ds} + \frac{df}{dx}\frac{d^2x}{ds^2}$$

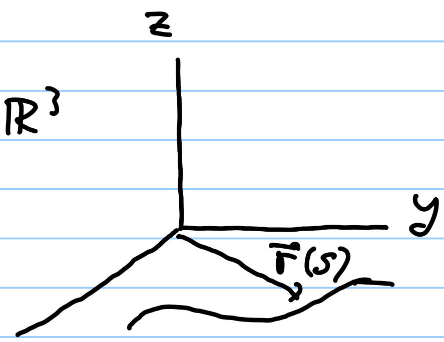
$$= \frac{d^2f}{dx^2}\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy/ds}{dx/ds}\right)\frac{d^2x}{ds^2}$$

$$\Rightarrow \frac{d^2f}{dx^2} = \frac{\frac{d^2y}{ds^2} - \frac{dy/d}{dx/ds}\frac{d^2x}{ds^2}}{(dx/ds)^2} = \frac{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}}{(dx/ds)^3} \quad \text{--[2]}$$

$$\Rightarrow K = \frac{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}}{\left(\frac{dx}{ds}\right)^3\left[1+\left(\frac{dy/ds}{dx/ds}\right)^2\right]^{3/2}}$$

$$= \frac{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}}{\left[\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right]^{3/2}}$$

Think of xy-plane as lying in $\mathbb{R}^3$



$\vec{r}(s) = (x(s), y(s), 0)$

$\vec{r}\,'(s) = (x'(s), y'(s), 0)$

$\vec{r}\,''(s) = (x''(s), y''(s), 0)$

$$\Rightarrow K = \pm \frac{|\vec{r}\,'(s) \times \vec{r}\,''(s)|}{|\vec{r}\,'(s)|^3}$$

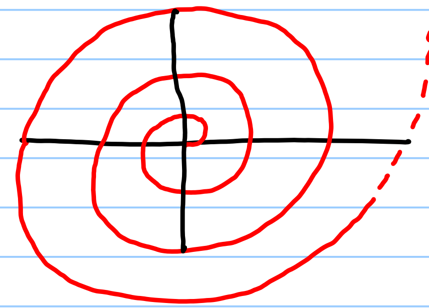
Examples:

1. Circle centred at origin, radius a.

$$\begin{matrix} x(\theta) = a\cos\theta \\ y(\theta) = a\sin\theta \end{matrix} \Rightarrow \begin{matrix} x'(\theta) = -a\sin\theta \\ y'(\theta) = a\cos\theta \end{matrix} \Rightarrow \begin{matrix} x''(\theta) = -a\cos\theta \\ y''(\theta) = -a\sin\theta \end{matrix}$$

$$\Rightarrow K = \frac{a^2\sin^2\theta + a^2\cos^2\theta}{(a^2\sin^2\theta + a^2\cos^2\theta)^{3/2}} = \frac{1}{a} \checkmark.$$

2. Spiral curve $x(\theta) = \theta\cos\theta$
$y(\theta) = \theta\sin\theta$
$\theta \in [0, \infty)$



$$\Rightarrow x'(\theta) = -\theta\sin\theta + \cos\theta$$
$$y'(\theta) = \theta\cos\theta + \sin\theta$$

$$\Rightarrow x''(\theta) = -\theta\cos\theta - 2\sin\theta$$
$$y''(\theta) = -\theta\sin\theta + 2\cos\theta$$

$$\Rightarrow x'(\theta)y''(\theta) - y'(\theta)x''(\theta) = (-\theta\sin\theta + \cos\theta)(-\theta\sin\theta + 2\cos\theta) - (\theta\cos\theta + \sin\theta)(-\theta\cos\theta - 2\sin\theta)$$

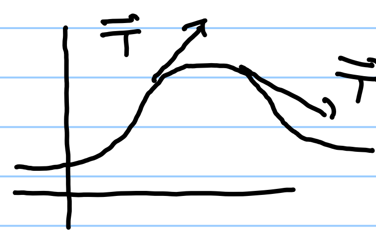
$$= \theta^2\sin^2\theta + 2\cos^2\theta + \theta^2\cos^2\theta + 2\sin^2\theta$$

$$= 2 + \theta^2$$

$$\left[\left(x'(\theta)\right)^2 + \left(y'(\theta)\right)^2\right]^{3/2} = \left[\theta^2\sin^2\theta + \cos^2\theta + \theta^2\cos^2\theta + \sin^2\theta\right]^{3/2}$$
$$= (1+\theta^2)^{3/2}$$

$$\Rightarrow K = \frac{2+\theta^2}{(1+\theta^2)^{3/2}} \sim \frac{1}{\theta} \text{ for large } \theta. \checkmark$$

Unit tangent vector $\vec{T}(s) = \dfrac{\vec{r}'(s)}{|\vec{r}'(s)|}$ so that $|\vec{T}(s)| = 1$.



Now $\dfrac{d}{ds}|\vec{r}'(s)|$

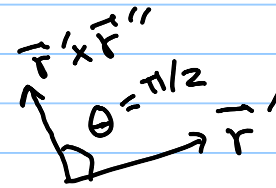
$$= \frac{1}{2|\vec{r}'(s)|}\frac{d}{ds}(\vec{r}'\cdot\vec{r}')$$

$$= \frac{\vec{r}'}{|\vec{r}'|}\cdot\frac{d}{ds}(\vec{r}') = \frac{\vec{r}'\cdot\vec{r}''}{|\vec{r}'|}$$

$$\Rightarrow \frac{d}{ds}\vec{T} = \frac{d}{ds}\left(\frac{\vec{r}'}{|\vec{r}'|}\right) = \frac{\vec{r}''|\vec{r}'| - \vec{r}'\frac{d}{ds}(|\vec{r}'|)}{|\vec{r}'|^2}$$

$$= \frac{\vec{r}\,''}{|\vec{r}\,'|} - \frac{(\vec{r}\,'\cdot\vec{r}\,'')\vec{r}\,'}{|\vec{r}\,'|^3}$$

$$= \frac{(\vec{r}\,'\cdot\vec{r}\,')\vec{r}\,'' - (\vec{r}\,'\cdot\vec{r}\,'')\vec{r}\,'}{|\vec{r}\,'|^3}$$



$$= \frac{\vec{r}\,'\times(\vec{r}\,''\times\vec{r}\,')}{|\vec{r}\,'|^3}$$

$$\Rightarrow \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{r}\,'\times(\vec{r}\,''\times\vec{r}\,')|}{|\vec{r}\,'|^3} = \frac{|\vec{r}\,'|\,|\vec{r}\,''\times\vec{r}\,'|}{|\vec{r}\,'|^3}\underbrace{\sin\theta}_{=1}$$

$$= \frac{|\vec{r}\,''\times\vec{r}\,'|}{|\vec{r}\,'|^2} = \pm\kappa\,|\vec{r}\,'|$$

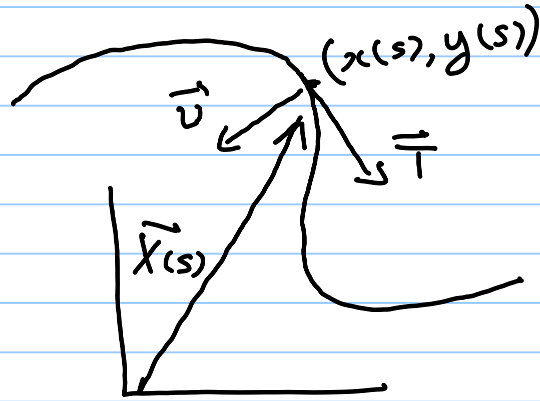
$$\Rightarrow \kappa = \pm\left|\frac{d\vec{T}}{ds}\right| \Big/ \left|\frac{d\vec{r}}{ds}\right|$$

$\Rightarrow$ κ measures
(i) the (inverse of the) radius of the oscalating circle.
(ii) the rate of change of the unit tangent vector along the curve

<u>Remark:</u> Often we choose the parameter s so that $\left|\frac{d\vec{r}}{ds}\right| = 1$. Then s is called an <u>arc length parameter</u>.

Curves parametrized by an arclength parameter are called <u>unit speed curves</u>.

A curvature flow:



$\vec{\nu}$ = unit normal vector

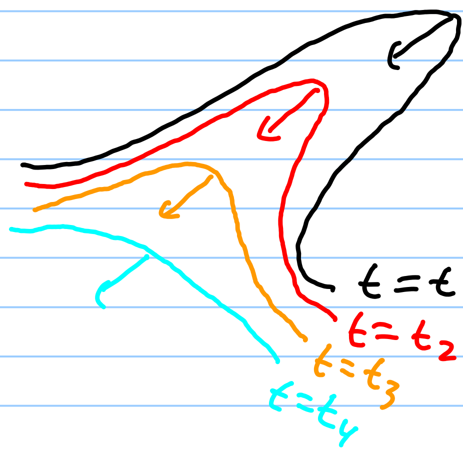
(i) $|\vec{\nu}| = 1$

(ii) $\vec{\nu} \perp \vec{T}$

(iii) $\vec{\nu}$ points toward centre of osculating circle.

The curve-shortening flow is $\vec{X}(s,t)$ where

$$\frac{\partial \vec{X}}{\partial t} = |\kappa| \, \vec{\nu}$$



$t_1 < t_2 < t_3 < t_4$

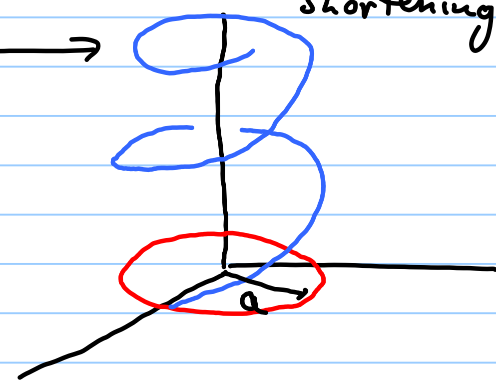
See the demonstration at a.carapetis.com/csf/

Space Curves: Curves in $\mathbb{R}^3$

Parametric form: $t \mapsto \vec{r}(t) = (x(t), y(t), z(t))$

t = parameter (not the t of the curve shortening flow)

e.g. Circular helix: $\longrightarrow$



$x(t) = a \cos t$

$y(t) = a \sin t$

$z(t) = t$

$t \in [0, \infty)$

arclength of a space curve: $s(t) = \int_0^t |\vec{r}\,'|\,dt$

For helix (above), $\vec{r}\,' = (-a \sin t, a \cos t, 1)$

$$|\vec{r}\,'| = \sqrt{a^2+1}$$

$$s = \sqrt{a^2+1}\; t.$$

Arclength parametrization: a parameter u is an arclength parameter if $|\vec{r}\,'(u)| = 1$.

e.g. In above example, use $t = s/\sqrt{a^2+1}$ to write

$$\vec{r}(s) = \left(a \cos \frac{s}{\sqrt{a^2+1}},\; a \sin \frac{s}{\sqrt{a^2+1}},\; \frac{s}{\sqrt{a^2+1}}\right).$$

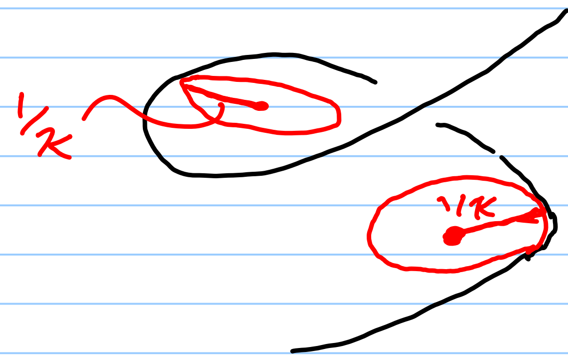
Then $\vec{r}\,'(s) = \left(\frac{-a}{\sqrt{a^2+1}} \sin \frac{s}{\sqrt{a^2+1}},\; \frac{a}{\sqrt{a^2+1}} \cos \frac{s}{\sqrt{a^2+1}},\; \frac{1}{\sqrt{a^2+1}}\right)$

and so $|\vec{r}\,'(s)| = \frac{a^2+1}{a^2+1} = 1. \Rightarrow$ s is an arclength parameter and $\vec{r}(s)$ is <u>unit speed.</u>

Curvature of a space curve: $\vec{T} = \vec{r}\,'(s) =$ unit tangent vector.

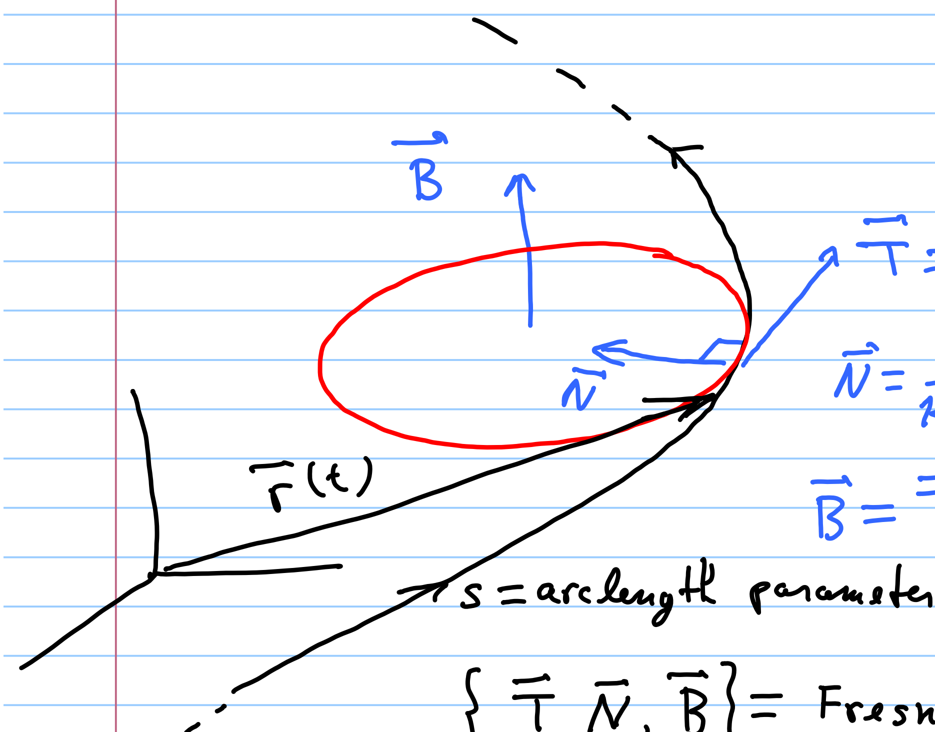
Then the curvature is

$$\kappa = \left| d\vec{T}/ds \right|$$



Helix with osculating circles with radii $1/\kappa$

<u>Torsion</u>: This measures how much the osculating circles "tip" or "tilt" as they move along the curve.



$\vec{T} = \vec{r}\,'/|\vec{r}\,'|$

$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} \Rightarrow \begin{cases} |\vec{N}| = 1 \\ \vec{N} \perp \vec{T} \end{cases}$

$\vec{B} = \vec{T} \times \vec{N} = \begin{cases} |\vec{B}| = 1 \\ \vec{B} \perp \vec{T} \\ \vec{B} \perp \vec{N} \end{cases}$

$\{\vec{T}, \vec{N}, \vec{B}\} =$ Fresnet frame

$=$ orthonormal basis along curve

$$\frac{d}{ds}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix},$$

τ is called the torsion.

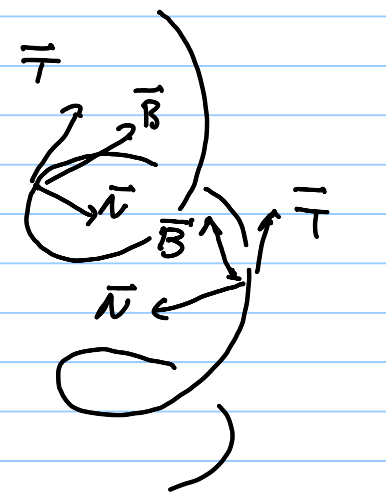
Exercise: For the circular helix

$$\vec{r}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right)$$

$a \neq 0$, $b \neq 0$ are constants,

find $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$

and $\kappa(s)$, $\tau(s)$.



Solution:

$$\vec{r}' = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\frac{s}{\sqrt{a^2+b^2}},\ a\cos\frac{s}{\sqrt{a^2+b^2}},\ b\right)$$

$$|\vec{r}'| = 1 \quad \text{so} \quad \vec{T} = \vec{r}' = \frac{1}{\sqrt{a^2+b^2}}\left(\text{—''—}\right)$$

$$\frac{d\vec{T}}{ds} = -\frac{1}{a^2+b^2}\left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ 0\right)$$

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{a}{a^2+b^2}$$

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = -\left(\cos\frac{s}{\sqrt{a^2+b^2}},\ \sin\frac{s}{\sqrt{a^2+b^2}}\right)$$

so the vector pointing to centre of osculating circle is always horizontal

$$\vec{B} = \vec{T}\times\vec{N} = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\frac{s}{\sqrt{a^2+b^2}},\ -b\cos\frac{s}{\sqrt{a^2+b^2}},\ a\right).$$

Then $$\frac{d\vec{B}}{ds} = \frac{b}{(a^2+b^2)}\left(\cos\frac{s}{\sqrt{a^2+b^2}},\ \sin\frac{s}{\sqrt{a^2+b^2}},\ 0\right) = \frac{-b}{(a^2+b^2)}\vec{N}$$

But $\frac{d\vec{B}}{ds} = -\tau\vec{N}$ from the matrix equation on the last page.

$$\Rightarrow \quad \tau = b/(a^2+b^2)$$

Fundamental Theorem of Space Curves:

Let $\vec{r}_1(s)$, $s \in I$, and $\vec{r}_2(s)$, $s \in I$ be two space curves parametrized by arclength s taking values in the same interval $I \subseteq \mathbb{R}$.

If the respective curvatures and torsions agree

$$\left.\begin{array}{l} \kappa_1(s) = \kappa_2(s) \\ \tau_1(s) = \tau_2(s) \end{array}\right\} \text{ for all } s \in I$$

then $\vec{r}_1$ and r_2 are <u>congruent</u> (i.e., they are the same curve up to a rotation and translation.