Notes to IUSEP Lectures in Mathematical Finance:

A Tour from the Binomial Model to the Black-Scholes Formula

July 24 & 26, 2018

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1. Single-period binomial model

A single-period model for a financial market:

- Consider the following very simplistic model with one stock and a bank account over one period:

prices	initial value	terminal value
stock	S_0	$\begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{with probability } 1-p \end{cases}$
bank account	1	1 + r

  for numbers $S_0 > 0$, $0 \leq d < u$, $p \in (0,1)$ and r.

- Assumptions: agent can invest in or short sell (= negative investment) the stock, and they can invest in and borrow from the bank account at the same interest rate r.

- We also assume

$$d < 1 + r < u \quad \text{(no-arbitrage condition).}$$

This assumption is reasonable. Example: if we had $d = 2$, $u = 3$ and $r = 0$, the agent could borrow a positive dollar amount x from the bank account and invest in stock to make a risk-free profit:

	initial value	terminal value
bank account	$-x$	$-x(1+r) = -x$
stock	x	$\begin{cases} ux = 3x \text{ with probability } p \\ dx = 2x \text{ with probability } 1-p \end{cases}$
total	0	$\begin{cases} 3x - x = 2x \text{ with probab. } p \\ 2x - x = x \text{ with probab. } 1-p \end{cases}$

Pricing financial derivatives:

- Consider now additionally a financial derivative with given payoff as follows:

initial value	terminal value
$x = ?$	$\begin{cases} f_u & \text{if stock price} = uS_0 \text{ (probab. } p) \\ f_d & \text{if stock price} = dS_0 \text{ (probab. } 1-p) \end{cases}$

for fixed numbers f_u and f_d.

- Example: (European) call option with strike K. *A call option gives the buyer the right, but not the obligation to buy the stock at maturity for the strike price K.* In our model the option payoff is

$$\begin{cases} \max\{uS_0 - K, 0\} & \text{with probability } p \\ \max\{dS_0 - K, 0\} & \text{with probability } 1-p \end{cases}$$

  because
  · two possibilities uS_0 or dS_0 for the stock price,
  · the buyer will use (= exercise) the option only if the stock price is higher than K.

- How can we find an initial value x for the financial derivative?

  We can use a replication argument because we will see that we can obtain here the same payoff by investing in the stock and bank account.

  Consider a portfolio consisting of Δ units of the stock and Ψ units of the bank account.

initial value	terminal value
$\Delta S_0 + \Psi$	$\begin{cases} \Delta uS_0 + \Psi(1+r) & \text{if stock} = uS_0 \\ \Delta dS_0 + \Psi(1+r) & \text{if stock} = dS_0 \end{cases}$

- We find Δ and Ψ of a replicating portfolio by setting its terminal value equal to that of the financial derivative, which implies

$$f_u = \Delta u S_0 + \Psi(1+r),$$
$$f_d = \Delta d S_0 + \Psi(1+r).$$

Solving this system of two linear equations for the unknowns gives

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}, \quad \Psi = \frac{u f_d - d f_u}{(1+r)(u-d)}$$

so that the initial value of the portfolio equals

$$\begin{aligned}\Delta S_0 + \Psi &= \frac{f_u - f_d}{u-d} + \frac{u f_d - d f_u}{(1+r)(u-d)} \\ &= \frac{1}{1+r}\Big(\frac{1+r-d}{u-d} f_u + \frac{u-1-r}{u-d} f_d\Big).\end{aligned}$$

To avoid arbitrage (= risk-free gains), this quantity must be equal to the initial value x of the financial derivative:

$$x = \frac{1}{1+r}\Big(\frac{1+r-d}{u-d} f_u + \frac{u-1-r}{u-d} f_d\Big).$$

Risk-neutral probabilities:

- Define

$$q_u = \frac{1+r-d}{u-d}, \qquad q_d = \frac{u-1-r}{u-d}$$

so that we can write

$$x = \frac{1}{1+r}(q_u f_u + q_d f_d).$$

- Note that

  · $q_u + q_d = 1$,

  · $d < 1 + r < u \Longrightarrow q_u > 0, q_d > 0$.

Therefore, we can consider q_u, q_d as the probabilities of a probability measure Q and we have

$$x = \frac{1}{1+r}E^Q[f] = \frac{1}{1+r}(q_u f_u + q_d f_d),$$

where f is the random variable of the option payoff and E^Q denotes the expectation under the measure Q. In other words,

option value = expectation of the discounted payoff under a measure Q

(discounted because payoff is divided by $1 + r$).

A crucial observation is that the probability measure Q used in the pricing formula does not equal the historical (from the model construction) probability measure because in general

$$q_u = \frac{1+r-d}{u-d} \neq p, \quad q_d = \frac{u-1-r}{u-d} \neq 1-p.$$

- If we calculate the expectation of the discounted stock price under Q, we obtain

$$\begin{aligned} & E^Q\left[\frac{\text{terminal value of stock}}{1+r}\right] \\ & = q_u \frac{uS_0}{1+r} + q_d \frac{dS_0}{1+r} \\ & = \frac{1+r-d}{u-d} \cdot \frac{uS_0}{1+r} + \frac{u-1-r}{u-d} \cdot \frac{dS_0}{1+r} \\ & = S_0, \end{aligned}$$

which shows that the expectation of the discounted terminal stock price under Q equals its initial value. Therefore, q_u and q_d are called risk-neutral probabilities and Q a risk-neutral probability measure.

- Remark. A financial market (like that we considered here) where every payoff can be replicated is called complete. It can be proved that a risk-neutral probability measure exists if there is no arbitrage in the market model and it is unique if the market is complete.

- The pricing formula we derived was based on a replication argument: we replicated the payoff of the derivative by investing in the stock and bank account. As a byproduct, we also saw the right number of stocks we need for the replication, which is $\Delta = \frac{f_u - f_d}{S_0(u-d)}$.

  This means that as a buyer of the option, we can "neutralize" the option by investing $-\Delta$ in the stock. Conversely, as a writer (= seller) of the option, we can buy Δ units of the stock to hedge against our risk. Consequently, this is called a replicating strategy or hedging strategy.

2. Two-period binomial model

- We can extend the model of Section 1 by adding a second period. We then have a tree of the form

$$
\begin{array}{ccccccc}
\text{time } 0 & & \text{time } 1 & & \text{time } 2 & & \text{probability} \\
 & & & & u^2 S_0 & & p^2 \\
\text{stock:} & & & \overset{\nearrow}{p} & & & \\
 & & u S_0 & & & & \\
 & \overset{\nearrow}{p} & & \overset{\searrow}{1-p} & & & \\
S_0 & & & & u d S_0 & & 2p(1-p) \\
 & \overset{\searrow}{1-p} & & \overset{\nearrow}{p} & & & \\
 & & d S_0 & & & & \\
 & & & \overset{\searrow}{1-p} & & & \\
 & & & & d^2 S_0 & & (1-p)^2
\end{array}
$$

bank account:

$$
\begin{array}{ccccc}
1 & \rightarrow & 1+r & \rightarrow & (1+r)^2
\end{array}
$$

- Trading is now also possible at the intermediate time 1. We still assume $d < 1 + r < u$.

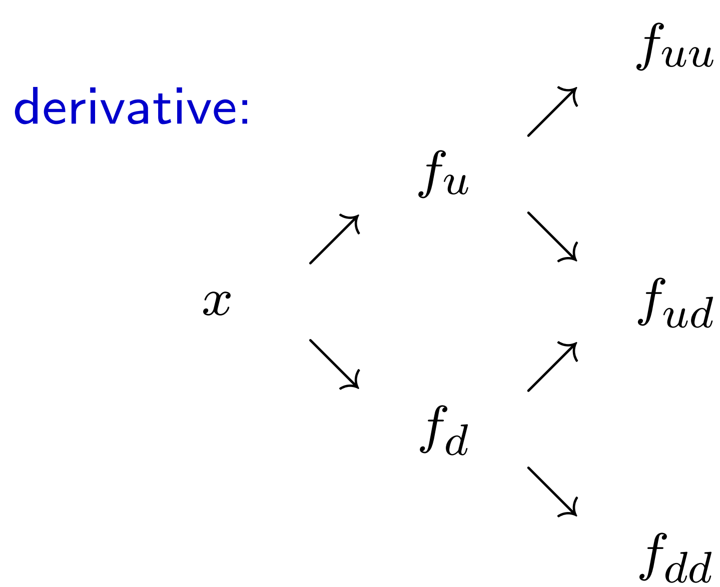
- Let us consider a European call option with maturity 2. It has the following payoff at time 2:

$$\begin{cases} f_{uu} = \max\{u^2 S_0 - K, 0\} & \text{if stock } = u^2 S_0 \\ f_{ud} = \max\{udS_0 - K, 0\} & \text{if stock } = udS_0 \\ f_{dd} = \max\{d^2 S_0 - K, 0\} & \text{if stock } = d^2 S_0 \end{cases}$$

- By a replication argument similarly to that in Section 1 applied to both trading periods, the price of the derivative equals

$$\frac{1}{(1+r)^2}\Big(q^2 f_{uu} + 2q(1-q) f_{ud} + (1-q)^2 f_{dd}\Big),$$

where $q = \frac{1+r-d}{u-d}$. Indeed, we have



Applying the reasoning of Section 1 to each branch of the tree gives

$$f_u = \frac{1}{1+r}\Big(qf_{uu} + (1-q)f_{ud}\Big),$$
$$f_d = \frac{1}{1+r}\Big(qf_{ud} + (1-q)f_{dd}\Big),$$
$$x = \frac{1}{1+r}\Big(qf_u + (1-q)f_d\Big)$$
$$= \frac{1}{(1+r)^2}\Big(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}\Big).$$

This means that the price equals $\frac{1}{(1+r)^2}E^Q[f]$, where $f = \max\{S_2 - K, 0\}$ is the option payoff and Q is the probability measure with probabilities q^2, $2q(1-q)$, $(1-q)^2$ corresponding to the different states $u^2 S_0$, udS_0, $d^2 S_0$, respectively, of the stock at time 2.

- One can also show that

$$\frac{1}{(1+r)^2}\Big(q^2u^2S_0 + 2q(1-q)udS_0 + (1-q)^2d^2S_0\Big)$$

equals S_0 so that Q is a risk-neutral measure.

3. Multiperiod binomial model

- We can further extend the model to n periods so that we have

	time n	probability
stock:		
	$u^n S_0$	p^n
	$u^{n-1} d S_0$	$np^{n-1}(1-p)$
	$\vdots$	$\vdots$
	$u^{n-j} d^j S_0$	$\binom{n}{j} p^{n-j}(1-p)^j$
	$\vdots$	$\vdots$
	$u d^{n-1} S_0$	$np(1-p)^{n-1}$
	$d^n S_0$	$(1-p)^n$
bank account:		
	$(1+r)^n$	

- A European call with maturity n and strike K has the payoff $\max\{S_n - K, 0\}$, which means

$$\begin{cases} f_{u^n} = \max\{u^n S_0 - K, 0\} & \text{if } S_n = u^n S_0 \\ \vdots & \vdots \\ f_{u^{n-j}d^j} = \max\{u^{n-j} d^j S_0 - K, 0\} & \text{if } S_n = u^{n-j} d^j S_0 \\ \vdots & \vdots \\ f_{d^n} = \max\{d^n S_0 - K, 0\} & \text{if } S_n = d^n S_0 \end{cases}$$

- Extending the pattern of the two-period, the fair price of the option is given by

$$\frac{1}{(1+r)^n}\Big(q^n f_{u^n} + \cdots + \binom{n}{j} q^{n-j}(1-q)^j f_{u^{n-j}d^j} + \cdots + (1-q)^n f_{d^n}\Big),$$

  where

$$q = \frac{1+r-d}{u-d}.$$

- Associating to q the corresponding measure Q, we can write the option price as

$$\frac{1}{(1+r)^n}E^Q[f] = \frac{1}{(1+r)^n}E^Q[\max\{S_n - K, 0\}],$$

  where we emphasize that it is the expectation under Q and not under the historical probability.

- Remark: Under the historical probability, S_n is related to a binomial distribution with parameters p and n. Under the probability measure Q, S_n is still related to a binomial distribution but with parameters $q = \frac{1+r-d}{u-d}$ and n. So for the option pricing, we just change the parameters of the distribution of S_n and take then expectations of discounted values.

Pricing a call option by writing a MATLAB function **callnperiod.m**:

```
function price = callnperiod(u,d,r,S0,K,n)
% calculate the price of a call option with ...
  strike K in an n period binomial model
if d<1+r && 1+r<u
    price=0;
    q = (1+r-d)/(u-d);
  for j=0:n
      price = price + ...
        nchoosek(n,j)*q^(n-j)*(1-q)^j*...
          max(u^(n-j)*d^j*S0-K,0)/(1+r)^n;
  end
else
error('wrong parameters')
end
```

```
% plot the call price in dependence of the ...
  strike price
K = 0:0.05:10;
price = callnperiod(1.2,.95,.05,1,K,20);
plot(K,price,'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
```

```
ylabel('option price','fontsize',14);
axis([0 10 0 1]) % choosing suitable range ...
  for axes
```

4. Transition to continuous time

- The binomial model can be used as approximation for a model with continuous trading possibilities on some time interval $[0, T]$.

To show convergence, one lets tend the number n of periods to infinity and, simultaneously, the length of each period tend to zero. This means that one makes specific choices for u, d and r depending on n; for details, please see the Appendix.

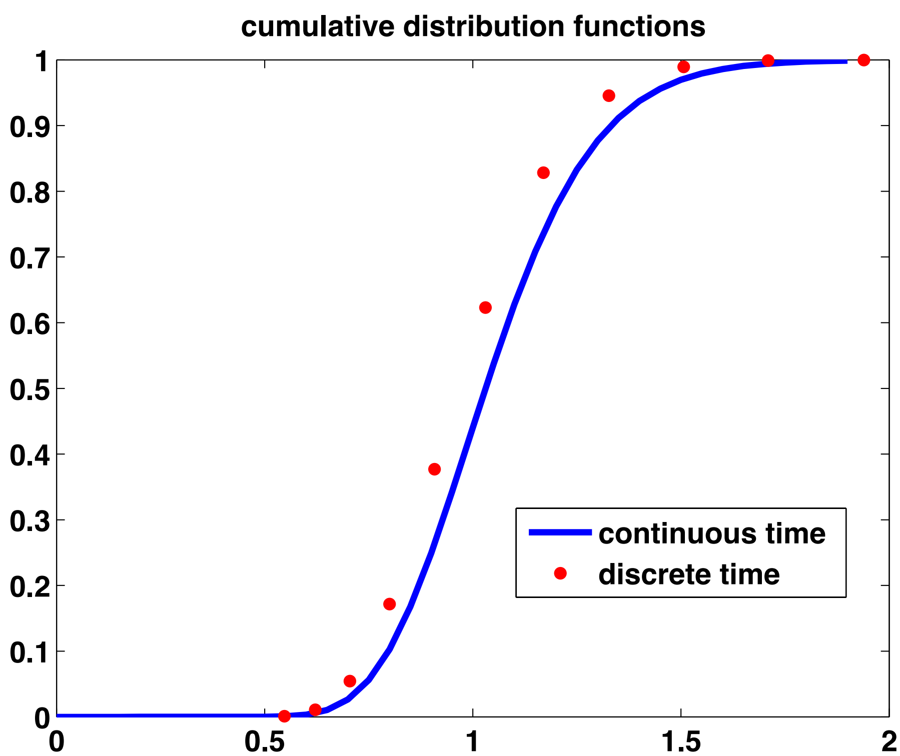
- The resulting continuous-time model has a bank account whose value at time T equals $\exp(\rho T)$ and a stock whose price at time T is given by

$$S_T = S_0 \exp\left((\mu - \sigma^2/2)T + \sigma\sqrt{T}N\right),$$

where ρ, μ and $\sigma > 0$ are constants and N is a standard normally distributed random variable.

```
function convergenceS(mu,sigma,T,n)
% compares the cumulative distribution ...
  function of S_n in a binomial model with ...
  that of the corresponding log-normal ...
  distribution

u = exp((mu-sigma^2/2)*T/n+sigma*(T/n)^.5);
% appropriate choice of u
d = exp((mu-sigma^2/2)*T/n-sigma*(T/n)^.5);
% appropriate choice of d
j=0:n;
```

```
Sn = d.^(n-j).*u.^j; % S_n for p = 1/2
bin=binocdf(0:n,n,1/2); % cumulative ...
  distribution function of S_n
points = 0:.05:Sn(n+1); % choose equidistant ...
  points for plot
lognorm=logncdf(points,(mu-sigma^2/2)*T,...
    sigma*T^.5); % cumulative distribution ...
      function of S
plot(points,lognorm,Sn,bin,'r.','LineWidth',...
    3,'MarkerSize',18) % r = red, . = point
set(gca,'fontsize',14,'FontWeight','bold');
title('cumulative distribution functions');
legend('continuous time','discrete ...
  time','location','best');
```



- Similarly to the binomial model, the price of a European call option with strike K and maturity T is given by

$$\frac{1}{\exp(\rho T)} E^{Q}[\max\{S_T - K, 0\}]$$

for some probability measure Q. This probability measure is such that

$$S_T = S_0 \exp\left((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\right)$$

for a random variable $\tilde{N}$ that is normally distributed under Q. Using this fact, we can rewrite the price of the European call as

$$c = S_0 \Phi(d_1) - K \exp(-\rho T) \Phi\left(d_1 - \sigma\sqrt{T}\right), \quad (\star)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\, \mathrm{d}u$$

is the standard-normal distribution function and $d_1 = \frac{\log \frac{S_0}{K} + \rho T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$.

Comments:

- $(\star)$ is the famous Black-Scholes formula. Note c depends on S_0, K, ρ, σ and T, but not on μ.

- In continuous time, the underlying process of the stock price dynamics is related to a Brownian motion.

- The partial derivatives of the Black-Scholes formula ($\star$) with respect to its parameters are called Greeks.

  1. Delta $= \frac{\partial c}{\partial S_0} = \Phi(d_1) \in (0, 1)$ is the amount of the risky asset held in the replicating portfolio.
  2. Gamma $= \frac{\partial^2 c}{\partial S_0^2} = \Phi'(d_1) \frac{1}{S_0 \sigma \sqrt{T}} > 0$;
     if Gamma is big, frequent adjustments of the replicating portfolio are necessary.
  3. Theta $= -\frac{\partial c}{\partial T}$
     $= -\frac{S_0 \sigma \Phi'(d_1)}{2\sqrt{T}} - K\rho \exp(-\rho T)\Phi\left(d_1 - \sigma\sqrt{T}\right)$
     < 0.
  4. Rho $= \frac{\partial c}{\partial \rho} = KT \exp(-\rho T)\Phi\left(d_1 - \sigma\sqrt{T}\right) > 0$.
  5. Vega $= \frac{\partial c}{\partial \sigma} = S_0\sqrt{T}\Phi'(d_1) > 0$.

- The principle of valuation under Q holds generally. The price of a derivative with payoff $f(S_T)$ is

$$\exp(-\rho T)\, E^Q[f(S_T)]$$
$$= e^{-\rho T} E^Q\Big[f\Big(S_0 \exp\big((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\big)\Big)\Big]$$

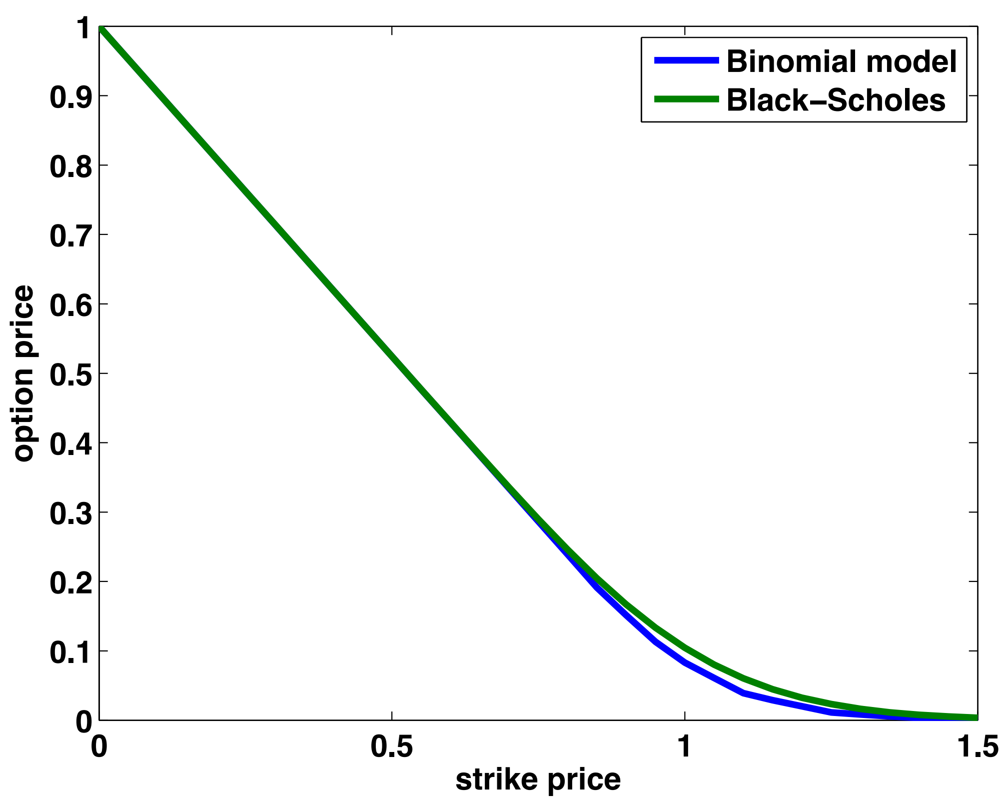
for a normally distributed $\tilde{N}$ under Q.

Comparison of Black-Scholes with Binomial model:

```
function [priceBin,priceBS] = ...
  compareCall(rho,mu,sigma,T,K,n)
% compares the call option price in a ...
  binomial model with the continuous-time ...
  analogue from the Black-Scholes formula
u = exp((mu-sigma^2/2)*T/n+sigma*(T/n)^.5); ...
  % appropriate choice of u
d = exp((mu-sigma^2/2)*T/n-sigma*(T/n)^.5); ...
  % appropriate choice of d
r = rho*T/n; % appropriate choice of r
priceBin = callnperiod(u,d,r,1,K,n);
d1 = (log(1./K) + rho*T)/sigma/T^.5 + ...
  sigma*T^.5/2;
priceBS = normcdf(d1) - ...
  K.*exp(-rho*T).*normcdf(d1-sigma*T^.5);

% or, alternatively, by applying the ...
  Financial Toolbox, we could use
% priceBS = blsprice(1,K,rho,T,sigma);
```

Plot the comparison of the Call option prices using the script `compareCallPlot.m`:

```
% plot comparison of Call option prices in ...
  binomial model and Black—Scholes model
K=0:.05:1.5;
[a,b] = compareCall(.05,.5,.2,1,K,10);
plot(K,a,K,b,'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
ylabel('option price','fontsize',14);
legend('Binomial model','Black—Scholes')
```



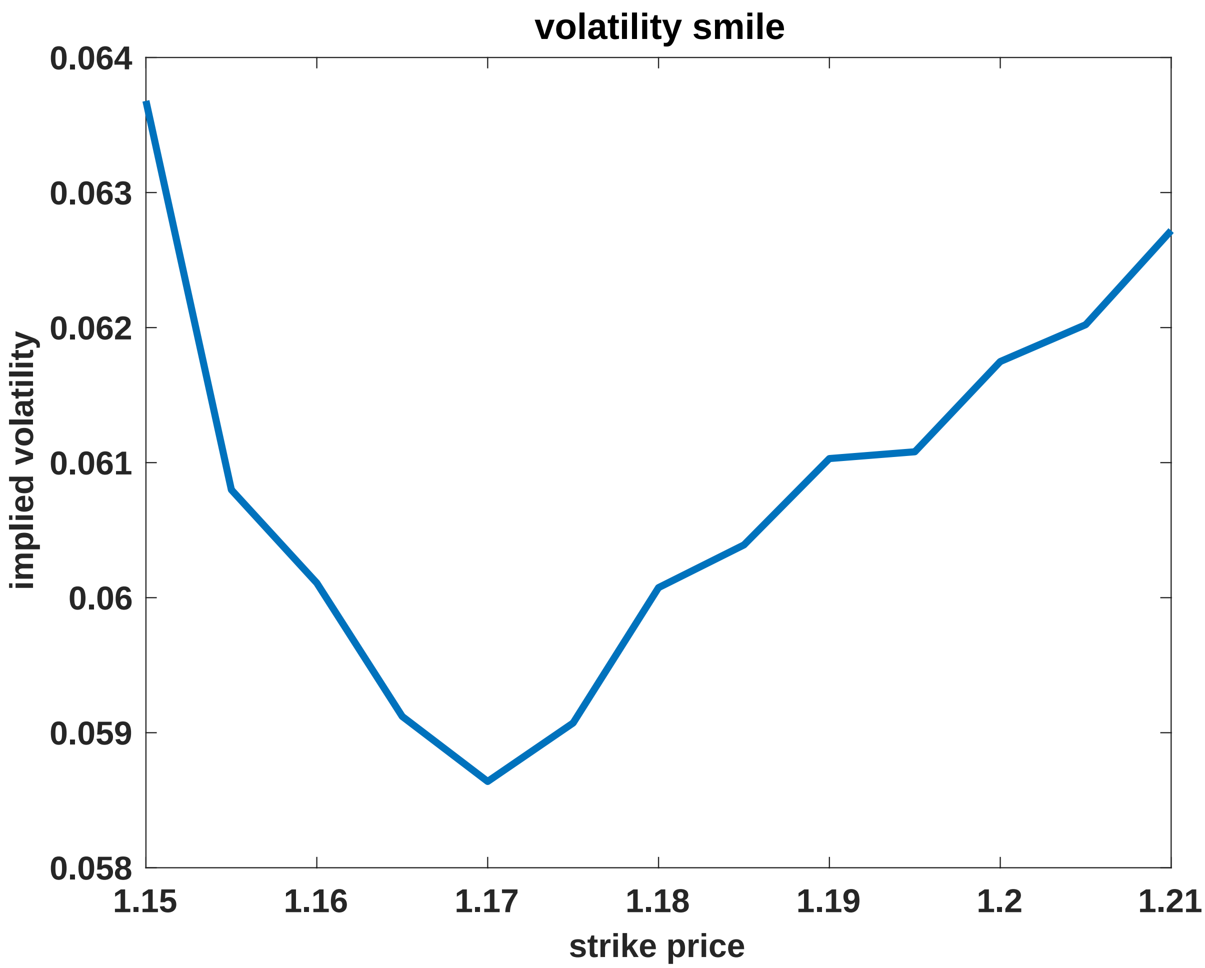
5. Implied volatility

- The value of σ is hard to determine $\rightarrow$ idea: find σ by inverting the Black-Scholes formula and using the market price of the option.
- The implied volatility σ_{impl} is defined as the unique σ such $c_{\text{BS}}(\sigma) = c_{\text{market}}$, where c_{market} is the market price of the option and c_{BS} is the value of the Black-Scholes formula $(\star)$ depending on σ.
- If the Black-Scholes model is correct, σ_{impl} does not depend on K, S_0, T and ρ. But in reality, one sees a strong dependence on K (volatility smile/skew).

```
% The financial toolbox has the function  ...
  blsimpv to calculate implied volatility: ...
  blsimpv(Current price of Stock S_0,  ...
  Strike K, Interest rate rho, Time to ...
  maturity T, Option price)

>> blsimpv(100, 95, 0, 0.25, 10)

ans =
    0.3722
```

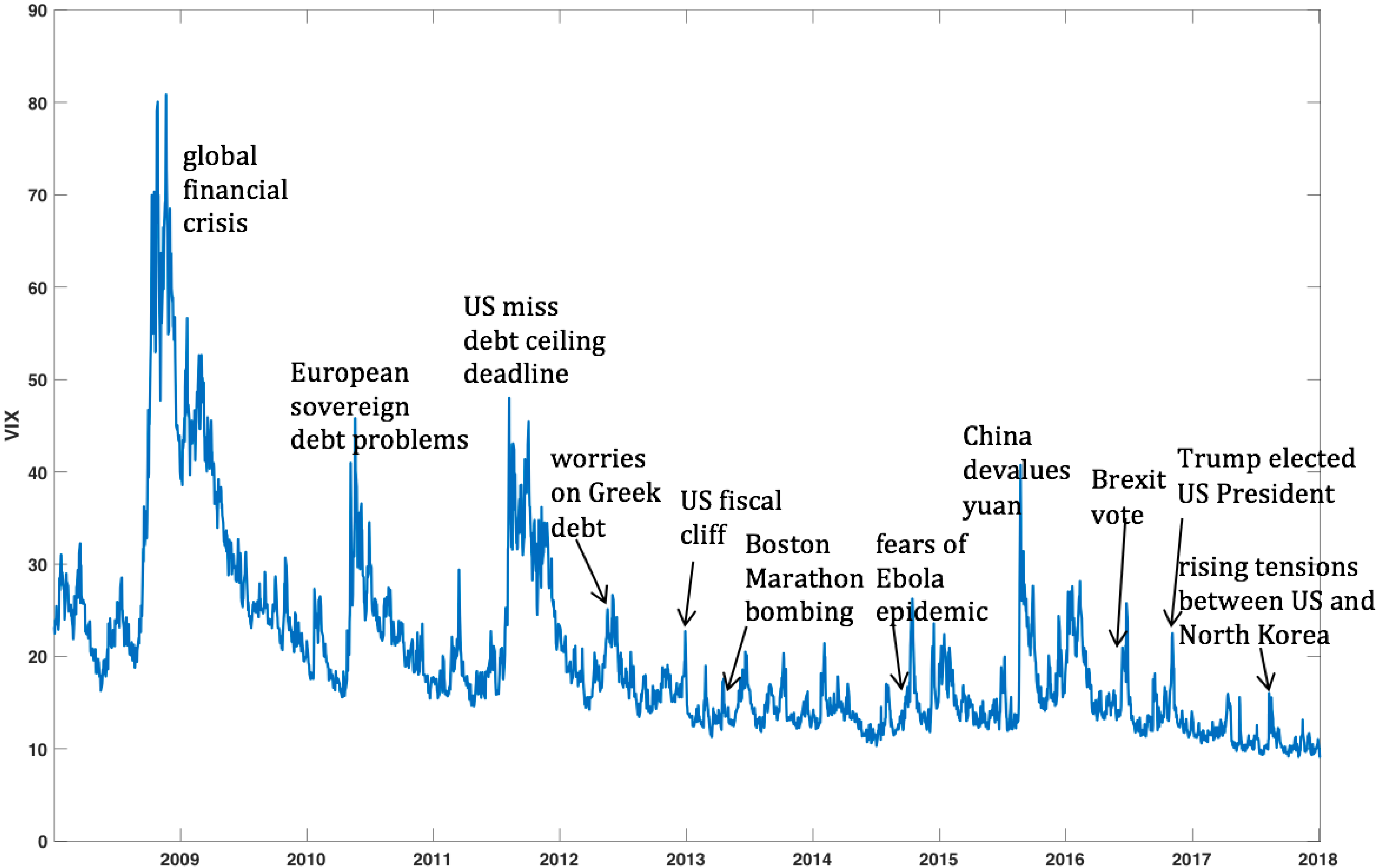
We now calculate the implied volatility on EUR/USD Call options, writing a script `volaEURUSD.m`. The resulting plot shows a volatility smile.

```
% volaEURUSD.m needs financial toolbox
%
% Implied volatility of currency options.  ...
  We have the following data: 1 EUR = 1.1678 ...
  USD (July 1st, 2018); the matrix A gives ...
  the prices (in USD) of call options with ...
  maturity end of September [such data can ...
  be found at http://www.cmegroup.com/]
 A = [1.15 0.0253; 1.155 0.0214; 1.16 ...
   0.0182; 1.165 0.0152; 1.17 0.0126; 1.175 ...
   0.0105; 1.18 0.0088; 1.185 0.0072; 1.19 ...
   0.0059; 1.195 0.0047; 1.20 0.0038; 1.205 ...
   0.0030; 1.21 0.0024];

% calculate the implied volatilities:
A(:,3) = blsimpv(1.1678,A(:,1),0,3/12,A(:,2));
% until end of September = 3 months
% A(:,2) means all numbers of the 2nd column
plot(A(:,1),A(:,3),'LineWidth',3);
set(gca,'fontsize',14,'FontWeight','bold');
xlabel('strike price','fontsize',14);
ylabel('implied volatility','fontsize',14);
xlim([A(1,1),A(end,1)])
title('volatility smile')
```



There exist indices which measure the implied volatility. A popular measure is VIX, which reflects the implied volatility of options on the stock index S&P 500. VIX is often referred to as “fear index”, because a high level of VIX means a lot of uncertainty in the market; see the development of VIX on the next page.

VIX over the last ten years



Appendix: additional explanations and proofs to Section 4

A.1 Choice in the continuous-time model

In the continuous-time situation, we model the terminal value of the bank account as $B_T = \exp(\rho T)$ and the terminal value of the stock as

$$S_T = S_0 \exp\Big((\mu - \sigma^2/2)T + \sigma\sqrt{T}N\Big), \qquad (1)$$

where ρ, μ and $\sigma > 0$ are constants and N is a standard normally distributed random variable.

A.2 Explanations behind choice

The reason behind these choices is as follows. In continuous time, the bank account models continuous interest, which means

$$\mathrm{d}B_t = \rho B_t \, \mathrm{d}t.$$

We can interpret this as that the infinitesimal change $\mathrm{d}B_t$ in the bank account is equal to the continuous interest rate ρ times the capital B_t. This equation is equivalent to $\frac{\mathrm{d}B_t}{\mathrm{d}t} = \rho B_t$, which yields $B_T = \exp(\rho T)$ using that $B_0 = 1$.

To explain the form (1) of the stock price, we can say that on average (which means in expectation) the stock should have a similar growth form than the bank account. Hence, $E[S_T] = S_0 \exp(\mu T)$ for some constant μ (typically μ will be bigger than ρ to compensate for the risk in the stock), using that S starts at S_0 and not necessarily at 1, in contrast to the bank account. Now, S_T will not just be equal to the deterministic value $S_0 \exp(\mu T)$, but will also reflect some random factor because we do not know future prices. Hence, S_T is of the form

$$S_T = S_0 \exp(\mu T) \times (\text{positive random factor}). \quad (2)$$

The reason for this positive random factor is related to the so-called Brownian motion. At the moment, you should just accept that we can model it with a normally distributed random variable, but because it should be positive, we take the exponential of this normally distributed random variable so that

$$\text{positive random factor} = \exp(cN) \quad (3)$$

where c is some constant and N is a standard normally distributed random variable. The bigger T, the longer the time horizon is and more uncertain S_T is. Therefore, c should depend on T, and we will again see later that the right form is $c = \sigma\sqrt{T}$, hence it grows like square root in T times some constant σ, which gives us how big the fluctuation in S_T is. Combining this with (2) and (3), we get

$$S_T = S_0 \exp\left(\mu T + \sigma\sqrt{T}N\right) \tag{4}$$

for some normally distributed N. Recall we wanted to have $E[S_T] = S_0 \exp(\mu T)$ so that μ has the interpretation of the mean growth rate, but we can calculate

$$\begin{aligned} & E\Big[S_0 \exp\left(\mu T + \sigma\sqrt{T}N\right)\Big] \\ & = S_0 \exp(\mu T) E\Big[\exp\left(\sigma\sqrt{T}N\right)\Big] \\ & = S_0 \exp\left(\mu T + \sigma^2 T/2\right), \end{aligned}$$

using the formula that $E[\exp(\alpha N)] = \exp(\alpha^2/2)$ for any constant α and standard normally distributed N. Therefore, to get $E[S_T] = S_0 \exp(\mu T)$, we need to divide (3) by $\exp(\sigma^2 T/2)$, which leads to (1).

A.3 Convergence proofs

We show now that under suitable choices of r_n, d_n and u_n, the terminal values of the bank account and stock in the binomial model converge to $B_T = \exp(\rho T)$ and S_T given in (1).

Proposition 1 *For $r_n = \rho T/n$, we have*

$$\lim_{n\to\infty}(1 + r_n)^n = \exp(\rho T).$$

Proof.

$$\begin{aligned}
\lim_{n\to\infty}(1+\rho T/n)^n &= \exp\Big(\ln\Big(\lim_{n\to\infty}(1+\rho T/n)^n\Big)\Big)\\
&= \exp\Big(\lim_{n\to\infty}\ln(1+\rho T/n)^n\Big)\\
&= \exp\Big(\lim_{n\to\infty} n\ln(1+\rho T/n)\Big)\\
&= \exp\left(\lim_{n\to\infty}\frac{\ln(1+\rho T/n)}{1/n}\right),
\end{aligned}$$

which equals

$$\begin{aligned}
\exp\left(\lim_{n\to\infty}\frac{\ln(1+\rho T/n)}{1/n}\right) &\overset{(*)}{=} \exp\left(\lim_{s\searrow 0}\frac{\ln(1+\rho T s)}{s}\right)\\
&\overset{(**)}{=} \exp\left(\lim_{s\searrow 0}\frac{\rho T}{1+\rho T s}\right)\\
&= \exp(\rho T)
\end{aligned}$$

(∗) set $s = 1/n$, then $n \to \infty \Longleftrightarrow s \searrow 0$
(∗∗) L'Hôpital's rule using $\frac{d}{ds}\ln(1+\rho T s) = \frac{\rho T}{1+\rho T s}$ ■

Proposition 2 *Set* $p = 1/2$ *and define*

$$u_n = \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right),$$
$$d_n = \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right)$$

then S_n *in the* n*-period binomial model converges to* S_T *in* (1)*.*

Proof. If S_n reflects j times u_n and $n-j$ times d_n, we have

$$\begin{aligned}
S_n &= S_0 u_n^j d_n^{n-j} \\
&= S_0 \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n}j + \sigma\sqrt{\frac{T}{n}}j\right) \\
&\quad \times \exp\left(\left(\mu - \sigma^2/2\right)\frac{T}{n}(n-j) - \sigma\sqrt{\frac{T}{n}}(n-j)\right) \\
&= S_0 \exp\left(\left(\mu - \sigma^2/2\right)T + \sigma\sqrt{T}\frac{2j-n}{\sqrt{n}}\right).
\end{aligned}$$

Comparing this with (1), it remains to show that $\frac{2j-n}{\sqrt{n}}$ converges to a standard normally distributed random variable. Define random variables X_i by

$$X_i = \begin{cases} 1, & \text{if we have } u_n \text{ in period } i \\ -1, & \text{if we have } d_n \text{ in period } i \end{cases} \tag{5}$$

and note that if we have j times u_n and $n-j$ times d_n, then

$$\sum_{i=1}^{n} X_i = j + (n-j)(-1) = 2j - n.$$

Therefore, we can write

$$\frac{2j-n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i. \tag{6}$$

We now apply the Central Limit Theorem, which says that for independent and identically distributed random variables $X_1, X_2, \ldots$ with mean $\mu = E[X_i]$ and finite variance $\sigma^2 = \mathrm{Var}(X_i)$,

$$\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right) \tag{7}$$

converges (in distribution) to a standard normally distributed random variable. In our case of X_i given by

(5) with equal probability $1/2$ for the two cases (because $p = 1/2$ by assumption), we have

$$\mu = E[X_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0,$$
$$\sigma^2 = \mathrm{Var}(X_i) = E\Big[X_i^2\Big] = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot (-1)^2 = 1.$$

Therefore, (7) simplifies in our case to $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. Because of (6), this shows that $\frac{2j-n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ converges (in distribution) to a standard normally distributed random variable. ■

A.4 Derivation of the Black-Scholes formula

Similarly to the binomial model, the price for a payoff f in the Black-Scholes model is given by $\frac{1}{\mathrm{e}^{\rho T}} E^Q[f]$ where the terminal value of the stock price is

$$S_T = S_0 \exp\Big((\rho - \sigma^2/2)T + \sigma\sqrt{T}\tilde{N}\Big)$$

with $\tilde{N}$ standard normally distributed under Q. In the case of a call option with strike price K, the price equals

$$
\begin{aligned}
c &= \frac{1}{\mathrm{e}^{\rho T}} E^Q[\max\{S_T - K, 0\}] \\
&= \frac{1}{\mathrm{e}^{\rho T}} E^Q\Big[\max\Big\{S_0 \mathrm{e}^{(\rho-\sigma^2/2)T+\sigma\sqrt{T}\tilde{N}} - K, 0\Big\}\Big] \\
&= \frac{1}{\mathrm{e}^{\rho T}} \int_{-\infty}^{\infty} \max\Big\{S_0 \mathrm{e}^{(\rho-\sigma^2/2)T+\sigma\sqrt{T}x} - K, 0\Big\} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \max\Big\{S_0 \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T}, 0\Big\} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x.
\end{aligned}
$$

Now, we use the equivalences

$$
\begin{aligned}
& S_0 \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T} \geq 0 \\
& \iff S_0 \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} \geq K\mathrm{e}^{-\rho T} \\
& \iff \mathrm{e}^{-\sigma^2 T/2+\sigma\sqrt{T}x} \geq \frac{K}{S_0}\mathrm{e}^{-\rho T} \\
& \iff -\sigma^2 T/2 + \sigma\sqrt{T}x \geq \log\left(\frac{K}{S_0}\right) - \rho T \\
& \iff x \geq \frac{\log(K/S_0) - \rho T}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2.
\end{aligned}
$$

Therefore, defining $d = \frac{\log(K/S_0)-\rho T}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2$ allows us to write

$$c = \int_{-\infty}^{\infty} \max\left\{S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T}, 0\right\} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= \int_{d}^{\infty} \left(S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} - K\mathrm{e}^{-\rho T}\right) \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= \int_{d}^{\infty} S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$- \int_{d}^{\infty} K\mathrm{e}^{-\rho T} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x.$$

For the first term, we calculate

$$\int_{d}^{\infty} S_0 \mathrm{e}^{-\sigma^2 T/2 + \sigma\sqrt{T}x} \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= S_0 \mathrm{e}^{-\sigma^2 T/2} \int_{d}^{\infty} \frac{\mathrm{e}^{\sigma\sqrt{T}x - x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= S_0 \mathrm{e}^{-\sigma^2 T/2} \int_{d}^{\infty} \frac{\mathrm{e}^{-(x-\sigma\sqrt{T})^2/2}\mathrm{e}^{\sigma^2 T/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= S_0 \int_{d-\sigma\sqrt{T}}^{\infty} \frac{\mathrm{e}^{-y^2}}{\sqrt{2\pi}}\,\mathrm{d}y$$
$$= S_0\Big(1 - \Phi\Big(d - \sigma\sqrt{T}\Big)\Big)$$
$$= S_0 \Phi\Big(-d + \sigma\sqrt{T}\Big).$$

For the second term, we have

$$\int_d^\infty K\mathrm{e}^{-\rho T}\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = K\mathrm{e}^{-\rho T}\int_d^\infty \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= K\mathrm{e}^{-\rho T}\Big(1-\Phi(d)\Big)$$
$$= K\mathrm{e}^{-\rho T}\Phi(-d).$$

Defining

$$d_1 = -d + \sigma\sqrt{T} = \frac{\log(K/S_0)+\rho T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

we obtain

$$c = \int_d^\infty S_0\mathrm{e}^{-\sigma^2T/2+\sigma\sqrt{T}x}\ \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$-\int_d^\infty K\mathrm{e}^{-\rho T}\ \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x$$
$$= S_0\Phi(d_1) - K\mathrm{e}^{-\rho T}\Phi\Big(d_1-\sigma\sqrt{T}\Big),$$

which is the Black-Scholes formula.