Algorithms of Wavelets and Framelets

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Outline of Tutorial

- Goal: introduction to wavelet theory through wavelet transforms.
- Algorithms for discrete wavelet/framelet transform
- Perfect reconstruction, sparsity, stability
- Multi-level fast wavelet transform
- Oblique extension principle
- Framelet transform for signals on bounded intervals

Declaration: Some figures and graphs in this talk are from various sources from Internet, or from published papers, or produced by `matlab`, `maple`, or `C` programming. [Details and sources of all graphs can be provided upon request of the audience.]

Notation

- $l(\mathbb{Z})$ for signals: all $v = \{v(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$.
- $l_0(\mathbb{Z})$ for filters: all finitely supported sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ on $\mathbb{Z}$.
- For $v = \{v(k)\}_{k\in\mathbb{Z}} \in l(\mathbb{Z})$, define

$$v^\star(k) := \overline{v(-k)}, \qquad k \in \mathbb{Z},$$
$$\widehat{v}(\xi) := \sum_{k\in\mathbb{Z}} v(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}.$$

- Convolution $u * v$ and inner product:

$$[u * v](n) := \sum_{k\in\mathbb{Z}} u(k)v(n-k), \quad n \in \mathbb{Z},$$
$$\langle v, w\rangle := \sum_{k\in\mathbb{Z}} v(k)\overline{w(k)}, \quad v, w \in l_2(\mathbb{Z})$$

Subdivision and Transition Operators

- The subdivision operator $\mathcal{S}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$:

$$[\mathcal{S}_u v](n) := 2 \sum_{k \in \mathbb{Z}} v(k) u(n-2k), \quad n \in \mathbb{Z}$$

- The transition operator $\mathcal{T}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$ is

$$[\mathcal{T}_u v](n) := 2 \sum_{k \in \mathbb{Z}} v(k) \overline{u(k-2n)}, \quad n \in \mathbb{Z}.$$

Subdivision and Transition Operators

- The upsampling operator $\uparrow\mathrm{d} : l(\mathbb{Z}) \to l(\mathbb{Z})$:

$$[v\uparrow\mathrm{d}](n) := \begin{cases} v(n/\mathrm{d}), & \text{if } n/\mathrm{d} \text{ is an integer;} \\ 0, & \text{otherwise,} \end{cases} \qquad n \in \mathbb{Z}$$

- the downsampling (or decimation) operator $\downarrow\mathrm{d} : l(\mathbb{Z}) \to l(\mathbb{Z})$:

$$[v\downarrow\mathrm{d}](n) := v(\mathrm{d}n), \qquad n \in \mathbb{Z}.$$

- Subdivision and transition operators:

$$\mathcal{S}_u v = 2u * (v\uparrow 2) \quad \text{and} \quad \mathcal{T}_u v = 2(u^\star * v)\downarrow 2.$$

Discrete Framelet Transform (DFrT): Decomposition

- Let $u_0, \ldots, u_s \in l_0(\mathbb{Z})$ be filters for decomposition.
- For data $v \in l(\mathbb{Z})$, a 1-level framelet decomposition:

$$w_\ell := \frac{\sqrt{2}}{2}\,\mathcal{T}_{u_\ell} v, \qquad \ell = 0, \ldots, s,$$

  where w_ℓ are called framelet coefficients.
- Grouping together, a framelet decomposition operator $\mathcal{W} : l(\mathbb{Z}) \to (l(\mathbb{Z}))^{1\times(s+1)}$:

$$\mathcal{W}v := \frac{\sqrt{2}}{2}(\mathcal{T}_{u_0} v, \ldots, \mathcal{T}_{u_s} v), \qquad v \in l(\mathbb{Z}).$$

Reconstruction

- Let $\tilde{u}_0, \ldots, \tilde{u}_s \in l_0(\mathbb{Z})$ be filters for reconstruction.
- A one-level framelet reconstruction by $\mathcal{V} : (l(\mathbb{Z}))^{1\times(s+1)} \to l(\mathbb{Z})$:

$$\mathcal{V}(w_0, \ldots, w_s) := \frac{\sqrt{2}}{2} \sum_{\ell=0}^{s} \mathcal{S}_{\tilde{u}_\ell} w_\ell, \quad w_0, \ldots, w_s \in l(\mathbb{Z}).$$

- prefect reconstruction: $\mathcal{V}\mathcal{W}v = v$ for any data v.
- A filter bank $(\{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots \tilde{u}_s\})$ has the perfect reconstruction (PR) if $\mathcal{V}\mathcal{W} = \mathrm{Id}_{l(\mathbb{Z})}$.

Perfect Reconstruction (PR) Property

Theorem

A filter bank $(\{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots, \tilde{u}_s\})$ *has the perfect reconstruction property, that is,*

$$v = \mathcal{V}\mathcal{W}v = \frac{1}{2}\sum_{\ell=0}^{s} \mathcal{S}_{\tilde{u}_\ell}\mathcal{T}_{u_\ell}v, \qquad \forall\, v \in l(\mathbb{Z}),$$

if and only if, for all $\xi \in \mathbb{R}$,

$$\overline{\widehat{u_0}(\xi)}\widehat{\tilde{u}_0}(\xi) + \overline{\widehat{u_1}(\xi)}\widehat{\tilde{u}_1}(\xi) + \cdots + \overline{\widehat{u_s}(\xi)}\widehat{\tilde{u}_s}(\xi) = 1,$$
$$\overline{\widehat{u_0}(\xi+\pi)}\widehat{\tilde{u}_0}(\xi) + \overline{\widehat{u_1}(\xi+\pi)}\widehat{\tilde{u}_1}(\xi) + \cdots + \overline{\widehat{u_s}(\xi+\pi)}\widehat{\tilde{u}_s}(\xi) = 0.$$

Matrix form of Perfect Reconstruction

The perfect reconstruction (PR) condition can be equivalently rewritten into the following matrix form:

$$\begin{bmatrix} \widehat{\tilde{u}_0}(\xi) & \cdots & \widehat{\tilde{u}_s}(\xi) \\ \widehat{\tilde{u}_0}(\xi+\pi) & \cdots & \widehat{\tilde{u}_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{u_0}(\xi) & \cdots & \widehat{u_s}(\xi) \\ \widehat{u_0}(\xi+\pi) & \cdots & \widehat{u_s}(\xi+\pi) \end{bmatrix}^\star = I_2,$$

where I_2 denotes the 2×2 identity matrix and $A^\star := \overline{A}^\mathsf{T}$.

a filter bank $(\{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots, \tilde{u}_s\})$ has PR is called a dual framelet filter bank.

Biorthogonal Wavelet Filter Bank

A dual framelet filter bank with $s = 1$ is called a biorthogonal wavelet filter bank, a nonredundant filter bank.

Proposition

Let $(\{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots, \tilde{u}_s\})$ *be a dual framelet filter bank. Then the following statements are equivalent:*

(i) $\mathcal{W}$ *is onto or* $\mathcal{V}$ *is one-one;*

(ii) $\mathcal{V}\mathcal{W} = \mathrm{Id}_{l(\mathbb{Z})}$ *and* $\mathcal{W}\mathcal{V} = \mathrm{Id}_{(l(\mathbb{Z}))^{1\times(s+1)}}$*, that is,* $\mathcal{V}$ *and* $\mathcal{W}$ *are inverse operators to each other;*

(iii) $s = 1$.

Duality in $l_2(\mathbb{Z})$

Lemma

Let $u \in l_0(\mathbb{Z})$ be a finitely supported filter on $\mathbb{Z}$. Then $\mathcal{S}_u : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$ is the adjoint operator of $\mathcal{T}_u : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$. More precisely,

$$\langle \mathcal{S}_u v, w \rangle = \langle v, \mathcal{T}_u w \rangle, \qquad \forall\, v, w \in l_2(\mathbb{Z}). \tag{1}$$

The space $(l_2(\mathbb{Z}))^{1\times(s+1)}$ has inner product:

$$\langle (w_0, \ldots, w_s), (\tilde{w}_0, \ldots, \tilde{w}_s) \rangle := \langle w_0, \tilde{w}_0 \rangle + \cdots + \langle w_s, \tilde{w}_s \rangle, \qquad w_0, \ldots, w_s,$$

and

$$\|(w_0, \ldots, w_s)\|^2_{(l_2(\mathbb{Z}))^{1\times(s+1)}} := \|w_0\|^2_{l_2(\mathbb{Z})} + \cdots + \|w_s\|^2_{l_2(\mathbb{Z})}.$$

Role of $\frac{\sqrt{2}}{2}$ in DFrT

Theorem

Let $u_0, \ldots, u_s \in l_0(\mathbb{Z})$. *Then TFAE:*

(i) $\|\mathcal{W}v\|^2_{(l_2(\mathbb{Z}))^{1\times(s+1)}} = \|v\|^2_{l_2(\mathbb{Z})}$ *for all* $v \in l_2(\mathbb{Z})$, *that is,*

$$\|\mathcal{T}_{u_0} v\|^2_{l_2(\mathbb{Z})} + \cdots + \|\mathcal{T}_{u_s} v\|^2_{l_2(\mathbb{Z})} = 2\|v\|^2_{l_2(\mathbb{Z})}, \quad \forall\, v \in l_2(\mathbb{Z});$$

(ii) $\langle \mathcal{W}v, \mathcal{W}\tilde{v}\rangle = \langle v, \tilde{v}\rangle$ *for all* $v, \tilde{v} \in l_2(\mathbb{Z})$;

(iii) *the filter bank* $(\{u_0, \ldots, u_s\}, \{u_0, \ldots, u_s\})$ *has PR:*

$$\begin{bmatrix} \widehat{u_0}(\xi) & \cdots & \widehat{u_s}(\xi) \\ \widehat{u_0}(\xi+\pi) & \cdots & \widehat{u_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{u_0}(\xi) & \cdots & \widehat{u_s}(\xi) \\ \widehat{u_0}(\xi+\pi) & \cdots & \widehat{u_s}(\xi+\pi) \end{bmatrix}^{\star} = I_2,$$

$\{u_0, \ldots, u_s\}$ with PR is called a tight framelet filter bank.

Orthogonal Wavelet Filter Bank

A tight framelet filter bank with $s = 1$ is called an orthogonal wavelet filter bank.

Proposition

Let $\{u_0, \dots, u_s\}$ be a tight framelet filter bank. Then the following are equivalent:

1. $\mathcal{W}$ *is an onto orthogonal mapping satisfying* $\langle \mathcal{W}v, \mathcal{W}\tilde{v} \rangle = \langle v, \tilde{v} \rangle$ *for all* $v, \tilde{v} \in l_2(\mathbb{Z})$;
2. *for all* $w_0, \dots, w_s, \tilde{w}_0, \dots, \tilde{w}_s \in l_2(\mathbb{Z})$,

$$\langle \mathcal{V}(w_0, \dots, w_s), \mathcal{V}(\tilde{w}_0, \dots, \tilde{w}_s) \rangle = \langle (w_0, \dots, w_s), (\tilde{w}_0, \dots, \tilde{w}_s) \rangle;$$

3. $s = 1$.

Examples

To list a filter $u = \{u(k)\}_{k\in\mathbb{Z}}$ with support $[m, n]$,

$$u = \{u(m), \ldots, u(-1), \underline{\mathbf{u(0)}}, u(1), \ldots, u(n)\}_{[m,n]},$$

- $\{u_0, u_1\}$ is the Haar orthogonal wavelet filter bank:

$$u_0 = \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]}, \qquad u_1 = \{\underline{\mathbf{\tfrac{1}{2}}}, -\tfrac{1}{2}\}_{[0,1]}. \tag{2}$$

- $(\{u_0, u_1\}, \{\tilde{u}_0, \tilde{u}_1\})$ is a biorthogonal wavelet filter bank, where

$$u_0 = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}, \ u_1 = \{\underline{\mathbf{\tfrac{1}{4}}}, -\tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]},$$
$$\tilde{u}_0 = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \ \tilde{u}_1 = \{\tfrac{1}{8}, \underline{\mathbf{\tfrac{1}{4}}}, -\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{8}\}_{[-1,3]}.$$

Illustration: I

Apply the Haar orthogonal filter bank to

$$v = \{1, 0, -1, -1, -4, 60, 58, 56\}_{[0,7]}$$

Note that

$$[\mathcal{T}_{u_0} v](n) = v(2n) + v(2n+1), \quad [\mathcal{T}_{u_1} v](n) = v(2n) - v(2n+1), \; n \in \mathbb{Z}.$$

We have the wavelet coefficients:

$$w_0 = \tfrac{\sqrt{2}}{2}\{1, -2, 56, 114\}_{[0,3]}, \quad w_1 = \tfrac{\sqrt{2}}{2}\{1, 0, -64, -2\}_{[0,3]}.$$

Note that

$$[\mathcal{S}_{u_0} \mathring{w}_0](2n) = \mathring{w}_0(n), \; [\mathcal{S}_{u_0} \mathring{w}_0](2n+1) = \mathring{w}_0(n), \; n \in \mathbb{Z}$$
$$[\mathcal{S}_{u_1} \mathring{w}_1](2n) = \mathring{w}_1(n), \; [\mathcal{S}_{u_1} \mathring{w}_1](2n+1) = -\mathring{w}_1(n), \; n \in \mathbb{Z}.$$

Illustration: II

Hence, we have

$$\frac{\sqrt{2}}{2}\mathcal{S}_{u_0}w_0 = \frac{1}{2}\{1,1,-2,-2,56,56,114,114\}_{[0,7]},$$
$$\frac{\sqrt{2}}{2}\mathcal{S}_{u_1}w_1 = \frac{1}{2}\{1,-1,0,0,-64,64,-2,2\}_{[0,7]}.$$

Clearly, we have the perfect reconstruction of v:

$$\frac{\sqrt{2}}{2}\mathcal{S}_{u_0}w_0 + \frac{\sqrt{2}}{2}\mathcal{S}_{u_1}w_1 = \{1,0,-1,-1,-4,60,58,56\}_{[0,7]} = v$$

and the following energy-preserving identity

$$\|w_0\|^2_{l_2(\mathbb{Z})} + \|w_1\|^2_{l_2(\mathbb{Z})} = \frac{16137}{2} + \frac{4101}{2} = 10119 = \|v\|^2_{l_2(\mathbb{Z})}.$$

Diagram of 1-level DFrTs

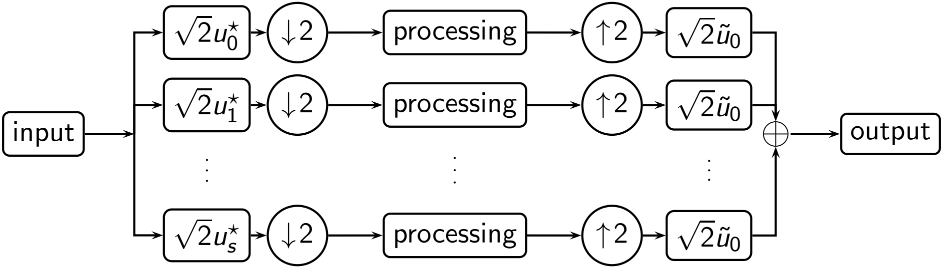

Figure: Diagram of a one-level discrete framelet transform using a dual framelet filter bank $(\{u_0, \ldots, u_s\}, (\tilde{u}_0, \ldots, \tilde{u}_s\})$.

Sparsity of DFrT

- One key feature of DFrT is its sparse representation for smooth or piecewise smooth signals.
- It is desirable to have as many as possible negligible framelet coefficients for smooth signals.
- Smooth signals are modeled by polynomials. Let $\mathsf{p} : \mathbb{R} \to \mathbb{C}$ be a polynomial: $\mathsf{p}(x) = \sum_{n=0}^{m} p_n x^n$.
- a polynomial sequence $\mathsf{p}|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ by $[\mathsf{p}|_{\mathbb{Z}}](k) = \mathsf{p}(k), k \in \mathbb{Z}$.
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Polynomial Differentiation Operator

- Polynomial differentiation operator:

$$\mathsf{p}\big(x - i\tfrac{d}{d\xi}\big)\mathbf{f}(\xi) := \sum_{n=0}^{\infty} p_n \big(x - i\tfrac{d}{d\xi}\big)^n \mathbf{f}(\xi).$$

$$\mathsf{p}\big(x - i\tfrac{d}{d\xi}\big)\mathbf{f}(\xi) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \mathsf{p}^{(n)}(x)\mathbf{f}^{(n)}(\xi)$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathsf{p}^{(n)}\big(-i\tfrac{d}{d\xi}\big)\mathbf{f}(\xi).$$

- Generalized product rule for differentiation:

$$\mathsf{p}\big(x - i\tfrac{d}{d\xi}\big)\big(\mathbf{g}(\xi)\mathbf{f}(\xi)\big) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \mathbf{g}^{(n)}(\xi)\mathsf{p}^{(n)}\big(x - i\tfrac{d}{d\xi}\big)\mathbf{f}(\xi).$$

$$\Big[\mathsf{p}\big(-i\tfrac{d}{d\xi}\big)\big(e^{ix\xi}\mathbf{f}(\xi)\big)\Big]\Big|_{\xi=0} = \Big[\mathsf{p}\big(x - i\tfrac{d}{d\xi}\big)\mathbf{f}(\xi)\Big]\Big|_{\xi=0}.$$

Convolution with Polynomials

Lemma

Let $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$. *Then for any polynomial* $\mathsf{p} \in \Pi$, $\mathsf{p} * u$ *is a polynomial with* $\deg(\mathsf{p} * u) \leqslant \deg(\mathsf{p})$,

$$[\mathsf{p} * u](x) = \sum_{k\in\mathbb{Z}} \mathsf{p}(x-k)u(k) = \Big[\mathsf{p}\big(x - i\tfrac{d}{d\xi}\big)\widehat{u}(\xi)\Big]\Big|_{\xi=0}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\mathsf{p}^{(n)}(x)\widehat{u}^{(n)}(0) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\Big[\mathsf{p}^{(n)}\big(-i\tfrac{d}{d\xi}\big)\widehat{u}(\xi)\Big]\Big|_{\xi=0}.$$

Moreover, $\mathsf{p} * (u\!\uparrow\!2) = [\mathsf{p}(2\cdot) * u](2^{-1}\cdot)$,

$$\mathsf{p}^{(n)} * u = [\mathsf{p} * u]^{(n)}, \quad \mathsf{p}(\cdot - y) * u = [\mathsf{p} * u](\cdot - y),\ \forall\, y \in \mathbb{R}.$$

Big $\mathscr{O}$ Notation

For smooth functions **f** and **g**, it is often convenient to use the following big $\mathscr{O}$ notation:

$$\mathbf{f}(\xi) = \mathbf{g}(\xi) + \mathscr{O}(|\xi - \xi_0|^m), \qquad \xi \to \xi_0$$

to mean that the derivatives of **f** and **g** at $\xi = \xi_0$ agree to the orders up to $m - 1$:

$$\mathbf{f}^{(n)}(\xi_0) = \mathbf{g}^{(n)}(\xi_0), \qquad \forall\, n = 0, \ldots, m-1.$$

Transition Operator Acting on Polynomials

Theorem

Let $u \in l_0(\mathbb{Z})$. Then for any polynomial $\mathsf{p} \in \Pi$,

$$\mathcal{T}_u \mathsf{p} = 2[\mathsf{p} * u^\star](2\cdot) = \mathsf{p}(2\cdot) * \mathring{u} = \sum_{n=0}^{\infty} \frac{2(-i)^n}{n!} \mathsf{p}^{(n)}(2\cdot)\overline{\widehat{u}^{(n)}(0)},$$

where $\mathring{u}$ is a finitely supported sequence on $\mathbb{Z}$ such that

$$\widehat{\mathring{u}}(\xi) = 2\overline{\widehat{u}(\xi/2)} + \mathscr{O}(|\xi|^{\deg(\mathsf{p})+1}), \qquad \xi \to 0.$$

In particular, for any integer $m \in \mathbb{N}$, TFAE:

1. $\mathcal{T}_u \mathsf{p} = 0$ *for all polynomial sequences* $\mathsf{p} \in \Pi_{m-1}$;
2. $\mathcal{T}_u \mathsf{q} = 0$ *for some* $\mathsf{q} \in \Pi$ *with* $\deg(\mathsf{q}) = m - 1$;
3. $\widehat{u}(\xi) = \mathscr{O}(|\xi|^m)$ *as* $\xi \to 0$;
4. $\widehat{u}(\xi) = (1 - e^{-i\xi})^m \mathbf{Q}(\xi)$ *for some 2π-periodic trigonometric polynomial* $\mathbf{Q}$.

Vanishing Moments

- We say that a filter u has m vanishing moments if any of items (1)–(4) in Theorem holds.
- Most framelet coefficients are zero for any input signal which is a polynomial to certain degree.
- If u has m vanishing moments. For a signal v, if v agrees with some polynomial of degree less than m on the support of $u(\cdot - 2n)$, then $[\mathcal{T}_u v](n) = 0$.

Coset Sequences

For $u = \{u(k)\}_{k\in\mathbb{Z}}$ and $\gamma \in \mathbb{Z}$, we define the associated coset sequence $u^{[\gamma]}$ of u at the coset $\gamma + 2\mathbb{Z}$ to be

$$\widehat{u^{[\gamma]}}(\xi) := \sum_{k\in\mathbb{Z}} u(\gamma + 2k)e^{-ik\xi},$$

that is,

$$u^{[\gamma]} = u(\gamma + \cdot)\downarrow 2 = \{u(\gamma + 2k)\}_{k\in\mathbb{Z}}.$$

Subdivision Operator on Polynomials

- $\mathcal{S}_u \mathrm{p}$ is not always a polynomial for $\mathrm{p} \in \Pi$.
- For example, for $\mathrm{p} = 1$ and $u = \{1\}_{[0,0]}$, we have $[\mathcal{S}_u \mathrm{p}](2k) = 2$ and $[\mathcal{S}_u \mathrm{p}](2k+1) = 0$ for all $k \in \mathbb{Z}$.

Lemma

Let $u \in l_0(\mathbb{Z})$ and q be a polynomial. TFAE:

(i) $\sum_{k\in\mathbb{Z}} \mathrm{q}(-\frac{1}{2} - k)u(1+2k) = \sum_{k\in\mathbb{Z}} \mathrm{q}(-k)u(2k)$, *that is,* $(\mathrm{q} * u^{[1]})(-\frac{1}{2}) = (\mathrm{q} * u^{[0]})(0)$;

(ii) $[\mathrm{q}(-i\frac{d}{d\xi})(e^{-i\xi/2}\widehat{u^{[1]}}(\xi))]|_{\xi=0} = [\mathrm{q}(-i\frac{d}{d\xi})\widehat{u^{[0]}}(\xi)]|_{\xi=0}$;

(iii) $[\mathrm{q}(-\frac{i}{2}\frac{d}{d\xi})\widehat{u}(\xi)]|_{\xi=\pi} = 0$.

Subdivision Operator on Polynomials

Theorem

Let $u = \{u(k)\}_{k\in\mathbb{Z}}$. *For* $m \in \mathbb{N}$, *TFAE:*

1. $\mathcal{S}_u \Pi_{m-1} \subseteq \Pi$;
2. $\mathcal{S}_u \mathsf{q} \in \Pi$ *for some* $\mathsf{q} \in \Pi$ *with* $\deg(\mathsf{q}) = m-1$;
3. $\mathcal{S}_u \Pi_{m-1} \subseteq \Pi_{m-1}$;
4. *u has m sum rules:*

$$\widehat{u}(\xi+\pi) = \mathscr{O}(|\xi|^m), \qquad \xi \to 0;$$

5. $\widehat{u}(\xi) = (1+e^{-i\xi})^m \mathbf{Q}(\xi)$ *for some* 2π*-periodic* $\mathbf{Q}$;
6. $e^{-i\xi/2}\widehat{u^{[1]}}(\xi) = \widehat{u^{[0]}}(\xi) + \mathscr{O}(|\xi|^m)$, $\xi \to 0$.

Moreover, $\mathcal{S}_u \mathsf{p} = 2^{-1}\mathsf{p}(2^{-1}\cdot) * u$.

Linear-phase Moments (LPM)

Lemma

Let $u \in l_0(\mathbb{Z})$*. Let* p *be a polynomial and define* $m := \deg(\mathrm{p})$*. For* $c \in \mathbb{R}$, $\mathrm{p} * u = \mathrm{p}(\cdot - c)$ $\iff$ *if* u *has* $m+1$ *linear-phase moments with phase* c*:*

$$\hat{u}(\xi) = e^{-ic\xi} + \mathscr{O}(|\xi|^{m+1}), \qquad \xi \to 0.$$

Proposition

Let $u \in l_0(\mathbb{Z})$ *and* $c \in \mathbb{R}$*. Then* u *has* $m+1$ *linear-phase moments with phase* c $\iff$ $\mathcal{T}_u \mathrm{p} = 2\mathrm{p}(2\cdot + c)$ *for all* $\mathrm{p} \in \Pi_m$*. Similarly,* u *has* $m+1$ *sum rules and* $m+1$ *linear-phase moments with phase* c $\iff$ $\mathcal{S}_u \mathrm{p} = \mathrm{p}(2^{-1}(\cdot - c))$ *for all* $\mathrm{p} \in \Pi_m$*.*

Symmetry

We say that $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ has symmetry if

$$u(2c-k) = \epsilon u(k), \qquad \forall\, k \in \mathbb{Z}$$

with $2c \in \mathbb{Z}$ and $\epsilon \in \{-1, 1\}$.
u is symmetric if $\epsilon = 1$; antisymmetric if $\epsilon = -1$.
c is the symmetry center of the filter u.
A symmetry operator S to record the symmetry type:

$$[\mathsf{S}\widehat{u}](\xi) := \frac{\widehat{u}(\xi)}{\widehat{u}(-\xi)}, \qquad \xi \in \mathbb{R}.$$

$[\mathsf{S}\widehat{u}](\xi) = \epsilon e^{-i2c\xi}$.

LPM and Symmetry

Proposition

Suppose that $u \in l_0(\mathbb{Z})$ has m but not $m+1$ linear-phase moments with phase $c \in \mathbb{R}$. If $m > 1$, then the phase c is uniquely determined by u through

$$c = i\hat{u}'(0) = \sum_{k \in \mathbb{Z}} u(k)k.$$

Moreover, if u has symmetry: $u(2c_0 - k) = u(k)$ for all $k \in \mathbb{Z}$ for some $c_0 \in \frac{1}{2}\mathbb{Z}$, then $c = c_0$ (that is, the phase c agrees with the symmetry center c_0 of u).

Example: B-spline filters

- B-spline filter of order m: $\widehat{a_m^B}(\xi) := 2^{-m}(1+e^{-i\xi})^m$
- $\widehat{a_4^B}(0) = 1, \quad \widehat{a_4^B}'(0) = -2i, \quad \widehat{a_4^B}''(0) = -5, \quad \widehat{a_4^B}'''(0) = 14i.$
- For $\mathsf{p} \in \Pi_3$,

$$[\mathsf{p} * a_4^B](x) = \mathsf{p}(x) - 2\mathsf{p}'(x) + \frac{5}{2}\mathsf{p}''(x) - \frac{7}{3}\mathsf{p}'''(x).$$

$$[\mathcal{T}_{a_4^B}\mathsf{p}](x) = 2\mathsf{p}(2x) + 4\mathsf{p}'(2x) + 5\mathsf{p}''(2x) + \frac{14}{3}\mathsf{p}'''(2x).$$

$$[\mathcal{S}_{a_4^B}\mathsf{p}](x) = \mathsf{p}(x/2) - \mathsf{p}'(x/2) + \frac{5}{8}\mathsf{p}''(x/2) - \frac{7}{24}\mathsf{p}'''(x/2).$$

- Let a be a primal low-pass filter and $b_1, \dots, b_s$ be primal high-pass filters for decomposition.
- For a positive integer J, a J-level discrete framelet decomposition is given by

$$v_{j-1} := \tfrac{\sqrt{2}}{2}\mathcal{T}_a v_j, \quad w_{j-1;\ell} := \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_\ell} v_j,$$
$$\ell = 1, \dots, s, \quad j = J, \dots, 1,$$

where $v_J : \mathbb{Z} \to \mathbb{C}$ is an input signal.
- decomposition operator $\mathcal{W}_J : l(\mathbb{Z}) \to (l(\mathbb{Z}))^{1\times(sJ+1)}$:

$$\mathcal{W}_J v_J := (w_{J-1;1}, \dots, w_{J-1;s}, \dots, w_{0;1}, \dots, w_{0;s}, v_0),$$

Thresholding and Quantization

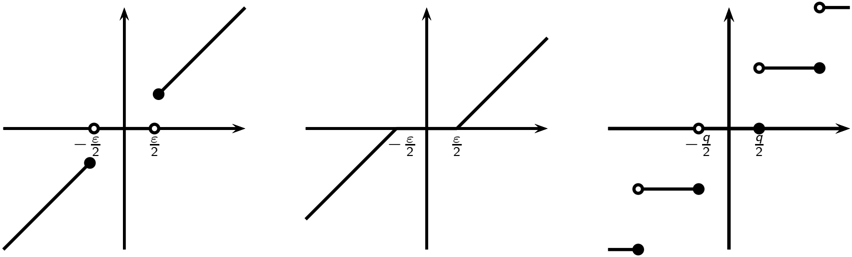

Figure: The hard thresholding function, the soft thresholding function, and the quantization function, respectively. Both thresholding and quantization operations are often used to process the framelet coefficients in a discrete framelet transform.

Multi-level Reconstruction

- Let $\tilde{a}$ be a dual low-pass filter and $\tilde{b}_1, \ldots, \tilde{b}_s$ be dual high-pass filters for reconstruction.
- a J-level discrete framelet reconstruction is

$$\mathring{v}_j := \frac{\sqrt{2}}{2}\mathcal{S}_{\tilde{a}}\mathring{v}_{j-1} + \frac{\sqrt{2}}{2}\sum_{\ell=1}^{s}\mathcal{S}_{\tilde{b}_\ell}\mathring{w}_{j-1;\ell}, \quad j = 1, \ldots, J.$$

- a J-level discrete reconstruction operator $\mathcal{V}_J : (l(\mathbb{Z}))^{1\times(sJ+1)} \to l(\mathbb{Z})$ is defined by

$$\mathcal{V}_J\big(\mathring{w}_{J-1;1}, \ldots, \mathring{w}_{J-1;s}, \ldots, \mathring{w}_{0;1}, \ldots, \mathring{w}_{0;s}, \mathring{v}_0\big) = \mathring{v}_J,$$

- A fast framelet transform with $s = 1$ is called a fast wavelet transform.

Diagram of Multi-level FFrT

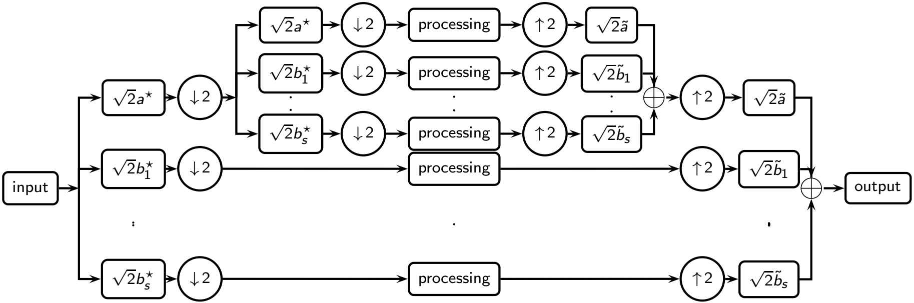

Figure: Diagram of a two-level discrete framelet transform using a dual framelet filter bank $(\{a; b_1, \ldots, b_s\}, (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})$.

Stability

A multi-level discrete framelet transform employing a dual framelet filter bank $\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})$ has stability in the space $l_2(\mathbb{Z})$ if there exists $C > 0$ such that for $J \in \mathbb{N}_0$,

$$\|\mathcal{W}_J v\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}} \leqslant C\|v\|_{l_2(\mathbb{Z})}, \qquad \forall\, v \in l_2(\mathbb{Z})$$

and

$$\|\mathcal{V}_J \vec{w}\|_{l_2(\mathbb{Z})} \leqslant C\|\vec{w}\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}}, \qquad \forall\, \vec{w} \in (l_2(\mathbb{Z}))^{1\times(sJ+1)}.$$

Stability

Proposition

Suppose that a multi-level discrete framelet transform has stability in the space $l_2(\mathbb{Z})$. Then all the wavelet decomposition operators must have uniform stability in space $l_2(\mathbb{Z})$:

$$\frac{1}{C}\|v\|_{l_2(\mathbb{Z})} \leqslant \|\mathcal{W}_J v\|_{l_2(\mathbb{Z})} \leqslant C\|v\|_{l_2(\mathbb{Z})}, \ \forall v \in l_2(\mathbb{Z}), J \in \mathbb{N}.$$

Moreover, wavelet reconstruction operators have uniform stability: for all $J \in \mathbb{N}$,

$$\frac{1}{C}\|\vec{w}\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}} \leqslant \|\mathcal{V}_J \vec{w}\|_{l_2(\mathbb{Z})} \leqslant C\|\vec{w}\|_{(l_2(\mathbb{Z}))^{1\times(sJ+1)}}, \vec{w} \in (l_2(\mathbb{Z}))^{1\times(sJ+1)},$$

if and only if $s = 1$.

Stability of Multi-level FFrT

Theorem

Let $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})$ *be a dual framelet filter bank with* $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$*. Define*

$$\varphi(\xi) := \prod_{j=1}^{\infty} \hat{a}(2^{-j}\xi), \quad \tilde{\varphi}(\xi) := \prod_{j=1}^{\infty} \hat{\tilde{a}}(2^{-j}\xi), \qquad \xi \in \mathbb{R}.$$

Then a multi-level discrete framelet transform has stability in the space $l_2(\mathbb{Z}) \iff \varphi, \tilde{\varphi} \in L_2(\mathbb{R})$ *and*

$$\widehat{b_1}(0) = \cdots = \widehat{b_s}(0) = \widehat{\tilde{b}_1}(0) = \cdots = \widehat{\tilde{b}_s}(0) = 0.$$

Multi-level FFrT Again

$$v_j = \frac{\sqrt{2}}{2}\mathcal{T}_a v_{j+1} = \cdots = (\frac{\sqrt{2}}{2})^{J-j}\mathcal{T}_{a*(a\uparrow 2)*\cdots*(a\uparrow 2^{J-j-1}),2^{J-j}} v$$

$$w_{j;\ell} = \frac{\sqrt{2}}{2}\mathcal{T}_{b_\ell} v_{j+1} = (\frac{\sqrt{2}}{2})^{J-j}\mathcal{T}_{a*(a\uparrow 2)*\cdots*(a\uparrow 2^{J-j-2})*(b_\ell\uparrow 2^{J-j-1}),2^{J-j}} v$$

$$w_{j;k}(k) = \langle v, 2^{(J-j)/2}[a*(a\uparrow 2)*\cdots*(a\uparrow 2^{J-j-2})*(b_\ell\uparrow 2^{J-j-1})](\cdot - 2^{J-j}k)\rangle,$$

Similarly, we have

$$\mathcal{V}_J(0,\ldots,0,v_0) = (\frac{\sqrt{2}}{2})^{J-j}\mathcal{S}_{\tilde{a}*(\tilde{a}\uparrow 2)*\cdots*(\tilde{a}\uparrow 2^{J-1}),2^J} v_0$$

$$\mathcal{V}_J(0,\ldots,0,w_{j;\ell},0,\ldots,0) = (\frac{\sqrt{2}}{2})^{J-j}\mathcal{S}_{\tilde{a}*(\tilde{a}\uparrow 2)*\cdots*(\tilde{a}\uparrow 2^{J-j-2})*(\tilde{b}_\ell\uparrow 2^{J-j-1}),2^{J-j}} w_{j;\ell}.$$

Define filters a_j and $\tilde{a}_j$ by

$$a_j = a * (a\uparrow 2) * \cdots * (a\uparrow 2^{j-1})$$

and

$$\tilde{a}_j := \tilde{a} * (\tilde{a}\uparrow 2) * \cdots * (\tilde{a}\uparrow 2^{j-1}).$$

Now a J-level discrete framelet transform employing $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})$ is equivalent to

$$v = \sum_{k\in\mathbb{Z}} \langle v, 2^{J/2} a_J(\cdot - 2^J k)\rangle 2^{J/2} \tilde{a}_J(\cdot - 2^J k) + \sum_{j=0}^{J-1} \sum_{\ell=1}^{s} \sum_{k\in\mathbb{Z}}$$

$$\langle v, [2^{(J-j)/2} a_{J-j-1} * (b_\ell \uparrow 2^{J-j-1})](\cdot - 2^{J-j} k)\rangle$$

$$[2^{(J-j)/2} \tilde{a}_{J-j-1} * (\tilde{b}_\ell \uparrow 2^{J-j-1})](\cdot - 2^{J-j} k).$$

Lack of Vanishing Moments

Lemma

Let $(\{a; b_1, \dots, b_s\}, \{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\})$ be a dual framelet filter bank. If all $b_1, \dots, b_s$ have $\tilde{m}$ vanishing moments and all $\tilde{b}_1, \dots, \tilde{b}_s$ have m vanishing moments, then

(i) *the primal low-pass filter a must have m sum rules, that is, $\hat{a}(\xi + \pi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$;*

(ii) *the dual low-pass filter $\tilde{a}$ must have $\tilde{m}$ sum rules, that is, $\hat{\tilde{a}}(\xi + \pi) = \mathscr{O}(|\xi|^{\tilde{m}})$ as $\xi \to 0$;*

(iii) *$\overline{\hat{a}}\hat{\tilde{a}}$ has $m + \tilde{m}$ linear-phase moments with phase 0:*

$$1 - \overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) = \mathscr{O}(|\xi|^{m+\tilde{m}}), \qquad \xi \to 0.$$

Example of B-spline Filters

Let $\hat{a}(\xi) = \widehat{a_m^B}(\xi) = 2^{-m}(1 + e^{-i\xi})^m$ and $\hat{\tilde{a}}(\xi) = \widehat{a_{\tilde{m}}^B}(\xi) = 2^{-\tilde{m}}(1 + e^{-i\xi})^{\tilde{m}}$ be two B-spline filters.

$$\overline{\hat{a}(0)}\hat{\tilde{a}}(0) = 1,$$

$$[\overline{\hat{a}}\hat{\tilde{a}}]'(0) = \frac{i(m - \tilde{m})}{2},$$

$$[\overline{\hat{a}}\hat{\tilde{a}}]''(0) = \frac{(m - \tilde{m})^2 + m + \tilde{m}}{4}.$$

Oblique Extension Principle

Theorem

$(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})_\Theta$ *has the following generalized perfect reconstruction property:*

$$\Theta * v = \frac{1}{2}\mathcal{S}_{\tilde{a}}(\Theta * \mathcal{T}_a v) + \frac{1}{2}\sum_{\ell=1}^{s} \mathcal{S}_{\tilde{b}_\ell}\mathcal{T}_{b_\ell} v, \qquad \forall\, v \in l(\mathbb{Z}),$$

$\Longleftrightarrow$ *for all* $\xi \in \mathbb{R}$,

$$\hat{\Theta}(2\xi)\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) + \overline{\widehat{b_1}(\xi)}\widehat{\tilde{b}_1}(\xi) + \cdots + \overline{\widehat{b_s}(\xi)}\widehat{\tilde{b}_s}(\xi) = \hat{\Theta}(\xi),$$

$$\hat{\Theta}(2\xi)\overline{\hat{a}(\xi+\pi)}\hat{\tilde{a}}(\xi) + \overline{\widehat{b_1}(\xi+\pi)}\widehat{\tilde{b}_1}(\xi) + \cdots + \overline{\widehat{b_s}(\xi+\pi)}\widehat{\tilde{b}_s}(\xi) = 0.$$

Vanishing Moments of OEP

Lemma

Let $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})_\Theta$ *be an OEP-based filter bank. Suppose that all* $b_1, \ldots, b_s$ *have* $\tilde{m}$ *vanishing moments and all* $\tilde{b}_1, \ldots, \tilde{b}_s$ *have* m *vanishing moments, where* $m, \tilde{m} \in \mathbb{N}$*. Then*

$$\hat{\Theta}(\xi) - \hat{\Theta}(2\xi)\overline{\hat{a}(\xi)}\hat{\tilde{a}}(\xi) = \mathscr{O}(|\xi|^{m+\tilde{m}}), \qquad \xi \to 0.$$

If in addition $\hat{\Theta}(0) \neq 0$*, then the primal low-pass filter* a *must have* m *sum rules and the dual low-pass filter* $\tilde{a}$ *must have* $\tilde{m}$ *sum rules.*

OEP-based Tight Framelets

Proposition

Let $\theta, a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$. *Then*

$$\|\theta * \mathcal{T}_a v\|^2_{l_2(\mathbb{Z})} + \|\mathcal{T}_{b_1} v\|^2_{l_2(\mathbb{Z})} + \cdots + \|\mathcal{T}_{b_s} v\|^2_{l_2(\mathbb{Z})} = 2\|\theta * v\|^2_{l_2(\mathbb{Z})}, \qquad \forall\, v \in l_2(\mathbb{Z}$$

$\iff$ $(\{a; b_1, \ldots, b_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ *has PR:*

$$\begin{bmatrix} \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \widehat{b_1}(\xi) & \cdots & \widehat{b_s}(\xi) \\ \widehat{b_1}(\xi+\pi) & \cdots & \widehat{b_s}(\xi+\pi) \end{bmatrix}^\star = \mathcal{M}_{\Theta,a},$$

where $\Theta := \theta * \theta^\star$ *and*

$$\mathcal{M}_{\Theta,a} := \begin{bmatrix} \hat{\Theta}(\xi) - \hat{\Theta}(2\xi)|\hat{a}(\xi)|^2 & -\hat{\Theta}(2\xi)\overline{\hat{a}(\xi+\pi)}\hat{a}(\xi) \\ -\hat{\Theta}(2\xi)\overline{\hat{a}(\xi)}\hat{a}(\xi+\pi) & \hat{\Theta}(\xi+\pi) - \hat{\Theta}(2\xi)|\hat{a}(\xi+\pi)|^2 \end{bmatrix}.$$

OEP-based Dual Framelets

- $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})_\Theta$ having PR is called an OEP-based dual framelet filter bank.
- $\{a; b_1, \ldots, b_s\}_\Theta$ having PR is called an OEP-based tight framelet filter bank.

Fejér-Riesz Lemma

Lemma

Let Θ *be a* 2π*-periodic trigonometric polynomial with real coefficients (or with complex coefficients) such that* $\Theta(\xi) \geqslant 0$ *for all* $\xi \in \mathbb{R}$*. Then there exists a* 2π*-periodic trigonometric polynomial* $\boldsymbol{\theta}$ *with real coefficients (or with complex coefficients) such that* $|\boldsymbol{\theta}(\xi)|^2 = \Theta(\xi)$ *for all* $\xi \in \mathbb{R}$*. Moreover, if* $\Theta(0) \neq 0$*, then we can further require* $\boldsymbol{\theta}(0) = \sqrt{\Theta(0)}$*.*

OEP-based Tight Framelet

Lemma

Let $\{a; b_1, \ldots, b_s\}_\Theta$ *be an OEP-based tight framelet filter bank. If* $\hat{\Theta}(0) > 0$*, then* $\hat{\Theta}(\xi) \geqslant 0$ *for all* $\xi \in \mathbb{R}$ *and consequently, there exists* $\theta \in l_0(\mathbb{Z})$ *such that* $\Theta = \theta * \theta^\star$ *holds and* $\hat{\theta}(0) = \sqrt{\hat{\Theta}(0)}$.

OEP with $s = 1$

Theorem

Let $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ *be an OEP-based dual framelet filter bank with* $\hat{\Theta}(0) \neq 0$. *Then*

$$\hat{\Theta}(2\xi)\hat{\Theta}(\pi) = \hat{\Theta}(\xi)\hat{\Theta}(\xi + \pi)$$

$$\begin{bmatrix} \hat{\mathring{a}}(\xi) & \hat{\mathring{b}}(\xi) \\ \hat{\mathring{a}}(\xi+\pi) & \hat{\mathring{b}}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \hat{a}(\xi) & \hat{b}(\xi) \\ \hat{a}(\xi+\pi) & \hat{b}(\xi+\pi) \end{bmatrix}^\star = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Continued...

Theorem

where all filters are finitely supported and are given by

$$\hat{\mathring{a}}(\xi) := \hat{\tilde{\mathring{a}}}(\xi)\hat{\Theta}(2\xi)/\hat{\Theta}(\xi) \qquad \textit{and} \qquad \hat{\mathring{b}}(\xi) := \hat{\tilde{\mathring{b}}}(\xi)/\hat{\Theta}(\xi).$$

That is, $(\{a; b\}, \{\mathring{a}; \mathring{b}\})$ *is a biorthogonal wavelet filter bank. Moreover,*

$$\hat{b}(\xi) = ce^{i(2n-1)\xi}\overline{\hat{\mathring{a}}(\xi+\pi)}, \quad \hat{\mathring{b}}(\xi) = \overline{c^{-1}}e^{i(2n-1)\xi}\overline{\hat{a}(\xi+\pi)} \qquad \textit{for some}$$

DFrT by OEP Filter Bank

- $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})_\Theta$.
- An OEP-based J-level discrete framelet decomposition is exactly the same as the one before.
- For given low-pass framelet coefficients $\mathring{v}_0$ and high-pass framelet coefficients $\mathring{w}_{j;\ell}, \ell = 1, \ldots, s$ and $j = 0, \ldots, J-1$, an OEP-based J-level discrete framelet reconstruction is

$$\breve{v}_0 := \Theta * \mathring{v}_0,$$

$$\breve{v}_j := \frac{\sqrt{2}}{2}\mathcal{S}_{\tilde{a}}\breve{v}_{j-1} + \frac{\sqrt{2}}{2}\sum_{\ell=1}^{s}\mathcal{S}_{\tilde{b}_\ell}\mathring{w}_{j-1;\ell}, \qquad j = 1, \ldots, J,$$

recover $\mathring{v}_J$ from $\breve{v}_J$ via the relation $\mathring{v}_J = \Theta * \breve{v}_J$.

Diagram of DFrT using OEP

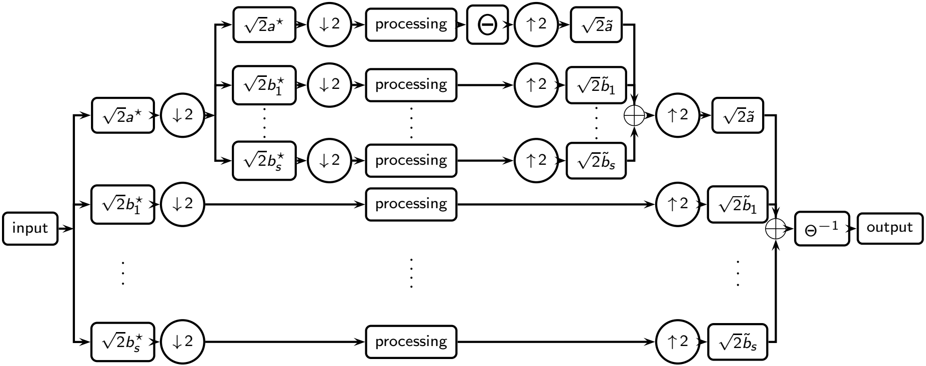

Figure: Diagram of a two-level discrete framelet transform using an OEP-based dual framelet filter bank $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})_\Theta$.

Avoiding Deconvolution using OEP

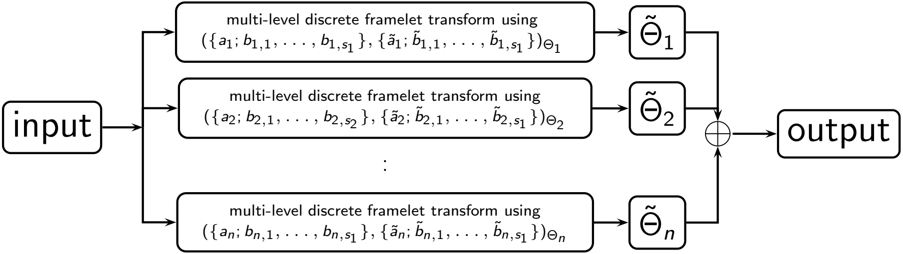

Figure: Diagram of an n-branch multi-level discrete averaging framelet transform using dual framelet filter banks $\{(\{a_m; b_{m,1}, \ldots, b_{m,s_m}\}, (\tilde{a}_m; \tilde{b}_{m,1}, \ldots, \tilde{b}_{m,s_m}\})_{\Theta_m}, \; m = 1, \ldots, n$. Therefore, deconvolution is completely avoided.

Signals on Intervals

- Signals: $v^b = \{v^b(k)\}_{k=0}^{N-1} : [0, N-1] \cap \mathbb{Z} \to \mathbb{C}$.
- Extend v^b from interval $[0, N-1]$ to $\mathbb{Z}$.
- $\mathcal{T}_{u(\cdot - 2m)} v = [\mathcal{T}_u v](\cdot - m), \quad m \in \mathbb{Z}$.
- extend v^b from $[0, N-1] \cap \mathbb{Z}$ to a sequence v on $\mathbb{Z}$ by any method that the reader prefers.
- zero-padding: $v(k) = 0$ for $k \in \mathbb{Z} \backslash [0, N-1]$.
- Must recover $v^b(0), \ldots, v^b(N-1)$ exactly.
- Keep all $[\mathcal{S}_{\tilde{u}_\ell} \mathcal{T}_{u_\ell} v](n), n = 0, \ldots, N-1$.
- $\operatorname{fsupp}(\tilde{u}) = [n_-, n_+]$ with $n_- \leqslant 0$ and $n_+ \geqslant 0$.

- $\frac{1}{2}[\mathcal{S}_{\tilde{u}}\mathcal{T}_u v](n) = \sum_{k\in\mathbb{Z}}[\mathcal{T}_u v](k)\tilde{u}(n-2k), n = 0, \ldots, N-1.$
- Record all the framelet coefficients:

$$[\mathcal{T}_u v](k), \qquad k = [\tfrac{-n_+}{2}], \ldots, [\tfrac{N-1-n_-}{2}].$$

- we always have $[\frac{-n_+}{2}] \leqslant 0$ and $[\frac{N-1-n_-}{2}] \geqslant \frac{N}{2} - 1$.
- framelet coefficients $\{[\mathcal{T}_u v](k)\}_{k=0}^{\frac{N}{2}-1}$ must be recorded.
- extra: $[\mathcal{T}_u v](k)$, $k = [\frac{-n_+}{2}], \ldots, -1$ and $k = \frac{N}{2}, \ldots, [\frac{N-1-n_-}{2}]$.
- Ideal situation can happen $\Longleftrightarrow$ u vanishes outside $[0, 1]$.

Periodic Extension

Proposition

Let $u \in l_1(\mathbb{Z})$ be a filter and $v^b = \{v^b(k)\}_{k=0}^{N-1}$. Extend v^b into an N-periodic sequence v on $\mathbb{Z}$ as follows:

$$v(Nn+k) := v^b(k), \qquad k = 0, \dots, N-1, \quad n \in \mathbb{Z}.$$

Then the following properties hold:

(i) $u * v$ *is an N-periodic sequence on $\mathbb{Z}$;*

(ii) $\mathcal{S}_u v$ *is a $2N$-periodic sequence on $\mathbb{Z}$;*

(iii) *If N is even, then $\mathcal{T}_u v$ is an $\frac{N}{2}$-periodic;*

(iv) *If N is odd, then $\mathcal{T}_u v$ is an N-periodic.*

DFrT using Periodic Extension

- A one-level discrete periodic framelet decomposition:

$$w_\ell^b = \left\{ w_\ell^b(k) := \tfrac{\sqrt{2}}{2}[\mathcal{T}_{u_\ell} v](k) \right\}_{k=0}^{\frac{N}{2}-1}, \quad \ell = 0, \ldots, s,$$

  where v is the N-periodic extension of v^b.
- Grouping all framelet coefficients,

$$\mathcal{W}^{per}(v^b) = (w_0^b, w_1^b, \ldots, w_s^b)^\mathsf{T}.$$

- a one-level discrete periodic framelet reconstruction:

$$\mathring{v}^b = \mathcal{V}^{per}(\mathring{w}_0^b, \ldots, \mathring{w}_s^b) := \left\{ \mathring{v}^b(k) := \tfrac{\sqrt{2}}{2} \sum_{\ell=0}^{s} [\mathcal{S}_{\tilde{u}_\ell} \mathring{w}_\ell](k) \right\}_{k=0}^{N-1},$$

- $\mathcal{V}^{per}\mathcal{W}^{per} = I_N$.

Tight Framelet Filter Bank

- A tight framelet filter bank by Ron-Shen, where

$$u_0 = \{\tfrac{1}{4}, \underline{\boldsymbol{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]},$$
$$u_1 = \{\tfrac{1}{4}, \underline{-\boldsymbol{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]},$$
$$u_2 = \{-\tfrac{\sqrt{2}}{4}, \underline{\mathbf{0}}, -\tfrac{\sqrt{2}}{4}\}_{[-1,1]}.$$

- A test input data:

$$v = \{1, 0, -1, -1, -4, 60, 58, 56\}_{[0,7]}$$

Example: Tight Framelet Filter Bank

We extend v^b to an 8-periodic sequence v on $\mathbb{Z}$, given by

$$v = \{\ldots, 1, 0, -1, -1, -4, 60, 58, 56, \underline{\mathbf{1}, 0, -1, -1, -4, 60, 58, 56}, 1, 0, -1, -1,$$

Then all sequences $\mathcal{T}_{u_0} v, \mathcal{T}_{u_1} v, \mathcal{T}_{u_2} v$ are 4-periodic and

$$w_0 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_0} v = \frac{\sqrt{2}}{2}\{\ldots, \frac{29}{2}, -\frac{3}{4}, \frac{51}{4}, 58, \underline{\mathbf{\frac{29}{2}}, -\frac{3}{4}, \frac{51}{4}, 58}, \frac{29}{2}, -\frac{3}{4}, \frac{51}{4}, 58, \ldots\},$$

$$w_1 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_1} v = \frac{\sqrt{2}}{2}\{\ldots, \frac{27}{2}, \frac{1}{4}, \frac{67}{4}, 0, \underline{\mathbf{\frac{27}{2}}, \frac{1}{4}, \frac{67}{4}, 0}, \frac{27}{2}, \frac{1}{4}, \frac{67}{4}, 0, \ldots\},$$

$$w_2 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_0} v = \{\ldots, -14, \frac{1}{4}, -\frac{59}{4}, -29, \underline{-\mathbf{14}, \frac{1}{4}, -\frac{59}{4}, -29}, -14, \frac{1}{4}, -\frac{59}{4}, -29$$

It is also easy to check that $\frac{\sqrt{2}}{2}(\mathcal{S}_{u_0} w_0 + \mathcal{S}_{u_1} w_1 + \mathcal{S}_{u_2} w_2) = v$.

Proposition

Let $u \in l_1(\mathbb{Z})$ *such that* $u(2c-k) = \epsilon u(k)$. *Extend* $v^b = \{v^b(k)\}_{k=0}^{N-1}$, *with both endpoints non-repeated (EN), into* $(2N-2)$*-periodic* v *by* $v(k) = v^b(2N-2-k)$.

(i) *Then* $u^\star * v$ *is* $(2N-2)$*-periodic with*

$$[u^\star * v](-2c-k) = [u^\star * v](2N-2-2c-k) = \epsilon[u^\star * v](k),$$

and $[-\lfloor c \rfloor, N-1-\lceil c \rceil]$ *is its control interval.*

(ii) *If* $c \in \mathbb{Z}$, *then* $\mathcal{T}_u v$ *is* $(N-1)$*-periodic with*

$$[\mathcal{T}_u v](-c-k) = [\mathcal{T}_u v](N-1-c-k) = \epsilon[\mathcal{T}_u v](k),$$

and $[\lceil -\frac{c}{2} \rceil, \lfloor \frac{N-1-c}{2} \rfloor]$ *is a control interval of* $\mathcal{T}_u v$.

Proposition

Let $u \in l_1(\mathbb{Z})$ *with* $\epsilon \in \{-1, 1\}$ *and* $c \in \frac{1}{2}\mathbb{Z}$*. Extend* v^b*, with both endpoints repeated (ER), into* $2N$*-periodic* v *by* $v(k) = v^b(2N - 1 - k)$*.*

(i) *Then* $u^\star * v$ *is* $2N$*-periodic with*

$$[u^\star * v](-1 - 2c - k) = [u^\star * v](2N - 1 - 2c - k) = \epsilon[u^\star * v](k),$$

and $[-\lfloor \frac{1}{2} + c \rfloor, N - \lceil \frac{1}{2} + c \rceil]$ *is its control interval.*

(ii) *If* $c - \frac{1}{2} \in \mathbb{Z}$*, then* $\mathcal{T}_u v$ *is an* N*-periodic sequence:*

$$[\mathcal{T}_u v](-\tfrac{1}{2} - c - k) = [\mathcal{T}_u v](N - \tfrac{1}{2} - c - k) = \epsilon[\mathcal{T}_u v](k),$$

and $[\lceil -\frac{1}{4} - \frac{c}{2} \rceil, \lfloor \frac{N}{2} - \frac{1}{4} - \frac{c}{2} \rfloor]$ *is its control interval.*

Table 1: Endpoint Nonrepeated (EN)

filter u	$u^\star * v$ with v extended by EN	$\mathcal{T}_u v$ with v extended by EN
$c=0$ $\epsilon=1$	$(2N-2)$-periodic, symmetric about 0 and $N-1$, a control interval $[0, N-1]$.	$(N-1)$-periodic, symmetric about 0 and $\frac{N-1}{2}$, a control interval $[0, \frac{N}{2}-1]$.
$c=0$ $\epsilon=-1$	$(2N-2)$-periodic, antisymmetric about 0 and $N-1$, a control interval $[0, N-1]$, $[u^\star * v](0) = [u^\star * v](N-1) = 0$.	$(N-1)$-periodic, antisymmetric about 0 and $\frac{N-1}{2}$, a control interval $[0, \frac{N}{2}-1]$, $[\mathcal{T}_u v](0) = 0$.
$c=1$ $\epsilon=1$	$(2N-2)$-periodic, symmetric about -1 and $N-2$, a control interval $[-1, N-2]$.	$(N-1)$-periodic, symmetric about $-\frac{1}{2}$ and $\frac{N}{2}-1$, a control interval $[0, \frac{N}{2}-1]$.
$c=1$ $\epsilon=-1$	$(2N-2)$-periodic, antisymmetric about -1 and $N-2$, a control interval $[-1, N-2]$, $[u^\star * v](-1) = [u^\star * v](N-2) = 0$.	$(N-1)$-periodic, antisymmetric about $-\frac{1}{2}$ and $\frac{N}{2}-1$, a control interval $[0, \frac{N}{2}-1]$, $[\mathcal{T}_u v](\frac{N}{2}-1) = 0$.

The decomposition filter u has the symmetry $\mathrm{S}\hat{u}(\xi) = \epsilon e^{-i2c\xi}$, where $\epsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$. v is a symmetric extension with both endpoints non-repeated (EN) of an input signal $v^b = \{v^b(k)\}_{k=0}^{N-1}$.

Table 2: Endpoints Repeated (ER)

filter u	$u^\star * v$ with v extended by ER	$\mathcal{T}_u v$ with v extended by ER
$c = \frac{1}{2}$ $\epsilon = 1$	$2N$-periodic, symmetric about -1 and $N-1$, a control interval $[-1, N-1]$.	N-periodic, symmetric about $-\frac{1}{2}$ and $\frac{N-1}{2}$, a control interval $[0, \frac{N}{2}-1]$.
$c = \frac{1}{2}$ $\epsilon = -1$	$2N$-periodic, antisymmetric about -1 and $N-1$, a control interval $[-1, N-1]$, $[u^\star * v](-1) = [u^\star * v](N-1) = 0$.	N-periodic, antisymmetric about $-\frac{1}{2}$ and $\frac{N-1}{2}$, . a control interval $[0, \frac{N}{2}-1]$
$c = -\frac{1}{2}$ $\epsilon = 1$	$2N$-periodic, symmetric about 0 and N, a control interval $[0, N]$.	N-periodic, symmetric about 0 and $\frac{N}{2}$, a control interval $[0, \frac{N}{2}]$.
$c = -\frac{1}{2}$ $\epsilon = -1$	$2N$-periodic, antisymmetric about 0 and N, a control interval $[0, N]$, $[u^\star * v](0) = [u^\star * v](N) = 0$.	N-periodic, antisymmetric about 0 and $\frac{N}{2}$, a control interval $[0, \frac{N}{2}]$, $[\mathcal{T}_u v](0) = [\mathcal{T}_u v](\frac{N}{2}) = 0$.

The decomposition filter u has the symmetry $\mathrm{S}\hat{u}(\xi) = \epsilon e^{-i2c\xi}$, where $\epsilon \in \{-1, 1\}$ and $c \in \{-\frac{1}{2}, \frac{1}{2}\}$. v is a symmetric extension with both endpoints repeated (ER) of an input signal $v^b = \{v^b(k)\}_{k=0}^{N-1}$.

Since $S\widehat{u_0} = 1$ and $S\widehat{u_1} = e^{-i2\xi}$, extend v^b by both endpoints non-repeated (EN):

$$v = \{\ldots, 58, 60, -4, -1, -1, 0, \underline{\mathbf{1}, 0, -1, -1, -4, 60, 58, 56}, 58, 60, -4, -1, -1$$

Then $\mathcal{T}_{u_0} v$ is 7-periodic and is symmetric about 0, $7/2$:

$$w_0 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_0} v = \frac{\sqrt{2}}{2}\{\ldots, \tfrac{37}{8}, -\tfrac{5}{8}, \underline{\mathbf{1}, -\tfrac{5}{8}, \tfrac{37}{8}, \tfrac{263}{4}}, \tfrac{263}{4}, \tfrac{37}{8}, \ldots\},$$

and $\mathcal{T}_{u_1} v$ is 7-periodic and is symmetric about $-\frac{1}{2}$, 3:

$$w_1 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_1} v = \frac{\sqrt{2}}{2}\{\ldots, -\tfrac{3}{4}, 0, \underline{\mathbf{0}, -\tfrac{3}{4}, -\tfrac{33}{2}, 1}, -\tfrac{33}{2}, -\tfrac{3}{4}, \ldots\}.$$

Example: Tight Framelet Filter Bank

Since $S\widehat{u_0} = S\widehat{u_1} = S\widehat{u_2} = 1$, extend v^b with both endpoints non-repeated (EN). Then all $\mathcal{T}_{u_0}v, \mathcal{T}_{u_1}v, \mathcal{T}_{u_2}v$ are 7-periodic and symmetric about 0 and $7/2$:

$$w_0 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_0}v = \frac{\sqrt{2}}{2}\{\ldots, 58, \tfrac{51}{4}, -\tfrac{3}{4}, \underline{\boldsymbol{\tfrac{1}{2}}, -\tfrac{3}{4}, \tfrac{51}{4}, 58}, 58, \tfrac{51}{4}, -\tfrac{3}{4}, \ldots\},$$

$$w_1 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_1}v = \frac{\sqrt{2}}{2}\{\ldots, 0, \tfrac{67}{4}, \tfrac{1}{4}, \underline{-\boldsymbol{\tfrac{1}{2}}, \tfrac{1}{4}, \tfrac{67}{4}, 0}, 0, \tfrac{67}{4}, \tfrac{1}{4}, \ldots\}.$$

$$w_2 = \frac{\sqrt{2}}{2}\mathcal{T}_{u_1}v = \{\ldots, -29, -\tfrac{59}{4}, \tfrac{1}{4}, \underline{\mathbf{0}, \tfrac{1}{4}, -\tfrac{59}{4}, -29}, -29, -\tfrac{59}{4}, \tfrac{1}{4}, 0, \ldots\}.$$