

Dynamical Systems on Networks: Part II

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July 27, 2016

Outline

- A network as a directed graph, examples
- Dynamical systems on networks, examples
- Global-stability problems for network dynamics
- Kirchhoff Matrix-Tree Theorem
- A general global stability result
- Application I: flight formation control of drones
- Application II: global synchronization of coupled oscillators

A Network as a Directed Graph

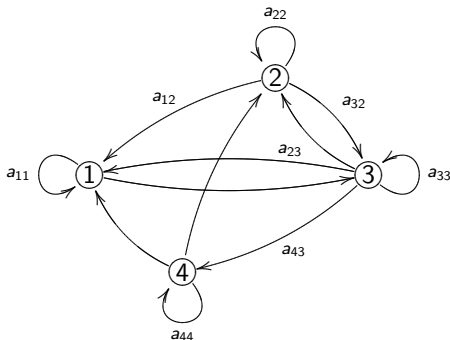
A directed graph $\mathcal{G} = (V, E, A)$

Vertex set: $V = \{1, 2, \dots, n\}$

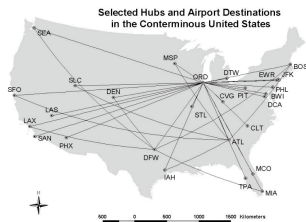
Directed edge: (i, j) from vertex i to j

Weights: $A = (a_{ij})$, $a_{ij} \neq 0 \iff (j, i)$ exists.

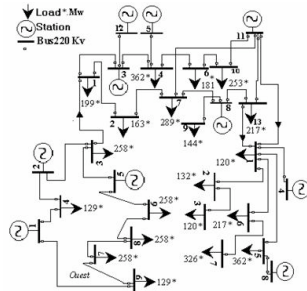
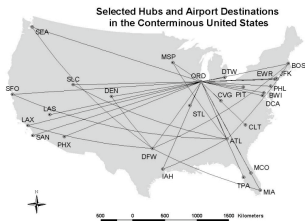
Given a nonnegative matrix, there corresponds a digraph \mathcal{G}_A , for which A is the weight matrix.



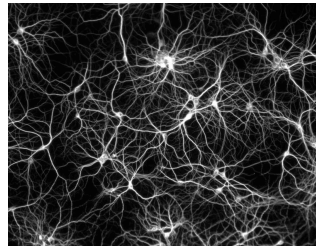
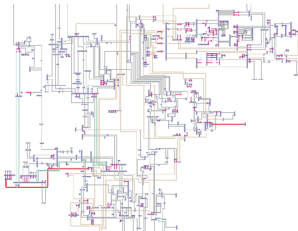
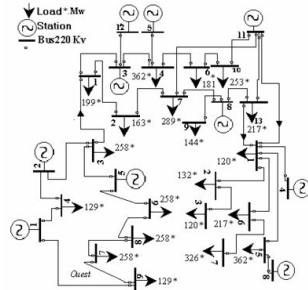
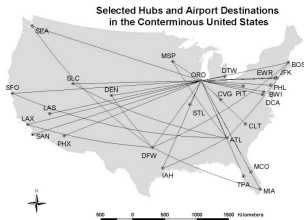
Examples of Networks



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Dynamical Systems on Networks

Given a digraph $G = (V, E)$, a dynamical system can be defined over G .

Vertex dynamics: $u_i' = f_i(t, u_i)$, $i = 1, \dots, n$.
 $u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i}$.

Connections: $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^{m_i}$ influence of j on i
 $g_{ij} \equiv 0 \iff (j, i)$ does not exist.

Coupled system over G :

$$u_i' = f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j), \quad i = 1, 2, \dots, n.$$

Examples of Dynamical Systems on Networks

- **Coupled Oscillators:**

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^n \epsilon_{ij} (\dot{x}_i - \dot{x}_j) = 0,$$

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- **Dispersal of a single species among n patches**

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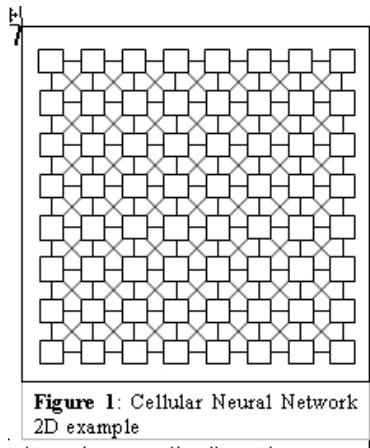
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- **An n -patch predator-prey model**

$$\begin{aligned} x'_i &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij} x_i), \\ y'_i &= y_i(-\gamma_i - \delta_i y_i + \epsilon_i x_i), \end{aligned} \quad i = 1, 2, \dots, n.$$

Examples of Dynamical Systems on Networks cont'd

- Cellular Neural Network and Lattice Dynamical Systems



Examples of Dynamical Systems on Networks cont'ed

- **A Delayed Hopfield-Cohen-Grossberg Model of Neural Networks**

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{j=1}^n J_{ij} f(u_j(t - \tau)), \quad 1 \leq i \leq n$$

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- **An Epidemic Model in Heterogeneous Populations**

$$S'_i = \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j),$$

$$E'_i = \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) - (d_i^E + \epsilon_i) E_i, \quad i = 1, 2, \dots, n.$$

$$I'_i = \epsilon_i E_i - (d_i^I + \gamma_i) I_i.$$

Research Questions

Assume: Independent vertex dynamics are simple or identical

Investigate: If, what, how complex dynamic behaviours emerge through network interactions.

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- Stability and control

Global Stability in Network Dynamics

Given a coupled system over a digraph G :

$$u'_i = f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j), \quad i = 1, 2, \dots, n. \quad (1)$$

Assume: Each vertex $u'_i = f_i(t, u_i)$ is globally stable, as insured by a global Lyapunov function V_i .

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Question: Under what conditions on the underlying network and coupling is the coupled system globally stable?

Of significance in disease control, stability of eco-systems, power distribution grids etc.

Main Result

Theorem [Z. Shuai and ML, 2009] Assume

(1) There exist $F_{ij}(t, u_i, u_j)$ such that

$$\dot{V}_i(u) \leq \sum_{j=1}^n a_{ij} F_{ij}(t, u_i, u_j), \quad t > 0, \quad u_i \in D_i, \quad u_j \in D_j, \quad j = 1, \dots, n. \quad (2)$$

(2) Along each directed cycle \mathcal{C} of G ,

$$\sum_{(r,s) \in E(\mathcal{C})} F_{rs}(t, u_r, u_s) \leq 0, \quad t > 0, \quad u_r \in D_r, \quad u_s \in D_s. \quad (3)$$

Then there exist constants $c_i \geq 0$ such that $V(u) = \sum_{i=1}^n c_i V_i(u)$ satisfies

$$\dot{V}(u) \leq 0, \quad u \in D_1 \times \dots \times D_n.$$

Kirchhoff Matrix-Tree Theorem

Let (G, A) be a weighted digraph with weight matrix $A = (a_{ij})$.
The **Laplacian matrix** of graph G is

$$L = \begin{bmatrix} \sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk} \end{bmatrix}.$$

Let c_i be the **cofactor** of the i -th diagonal element of L .

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Theorem [Kirchhoff (1847)] Assume $n \geq 2$. Then

$$c_i = \sum_{\mathcal{T} \in \mathbb{T}_i} w(\mathcal{T}), \quad i = 1, 2, \dots, n, \quad (4)$$

where \mathbb{T}_i is the set of all spanning trees \mathcal{T} of (\mathcal{G}, A) rooted at vertex i ,
and $w(\mathcal{T})$ is the **weight** of \mathcal{T} .

Reordering of a Double Sum

Proposition [Tree-Cycle-Identity, Z. Shuai and ML 2009] Let c_i be given by the Matrix-Tree Theorem. Then the following identity holds.

$$\sum_{i,j=1}^n c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{Q \in \mathbb{Q}} w(Q) \sum_{(r,s) \in E(\mathcal{C}_Q)} F_{rs}(x_r, x_s), \quad (5)$$

where $F_{ij}(x_i, x_j)$, $1 \leq i, j \leq n$, are arbitrary functions, \mathbb{Q} is the set of all spanning unicyclic graphs Q of (\mathcal{G}, A) , $w(Q)$ is the weight of Q , and \mathcal{C}_Q denotes the oriented cycle of Q .

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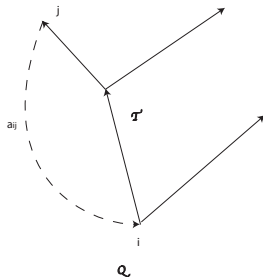
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Proof: Note $w(\mathcal{T}) a_{ij} = w(Q)$,

where Q is the unicyclic graph obtained by adding an arc (j, i) to \mathcal{T} .



Proof of Main Theorem

$$\begin{aligned}\dot{V} &= \sum_{i=1}^n c_i \dot{V}_i \leq \sum_{i,j=1}^n c_i a_{ij} F_{ij}(t, u_i, u_j) \quad (\text{assumption (1)}) \\ &= \sum_{Q \in \mathcal{Q}} w(Q) \sum_{(r,s) \in E(\mathcal{C}_Q)} F_{rs}(t, u_r, u_s) \quad (\text{Proposition}) \\ &\leq 0.\end{aligned}$$

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Is the theorem any good?

Application I: A Network of Coupled Oscillators

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^n \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0, \quad (6)$$

or in systems

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^n \epsilon_{ij}(y_i - y_j). \end{aligned} \quad (7)$$

Each vertex dynamics is given by a damped nonlinear oscillator

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) = 0.$$

Assume that the damping $\alpha_i \geq 0$ and the potential energy $F_i(x_i) = \int^{x_i} f_i(s)ds$ has a strictly global minimum at $x_i = x_i^*$. Then $x = x_i^*$ is globally stable (using the Lyapunov function)

$$V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}.$$

Application I: A Network of Coupled Oscillators

Theorem Assume $\alpha_k > 0$ for some k and digraph \mathcal{G} is strongly connected. Then $E^*(x_1^*, 0, x_2^*, 0, \dots, x_n^*, 0)$ is globally asymptotically stable in \mathbb{R}^{2n} .

Proof. $V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}$

$$\begin{aligned}\dot{V}_i &= -\alpha_i y_i^2 - \sum_{j=1}^n \epsilon_{ij} (y_i - y_j) y_i \\ &\leq \sum_{j=1}^n \epsilon_{ij} \left[-\frac{1}{2} (y_i - y_j)^2 - \frac{1}{2} y_i^2 + \frac{1}{2} y_j^2 \right] \\ &\leq \sum_{j=1}^n \epsilon_{ij} F_{ij}(y_i, y_j)\end{aligned}$$

where

$$F_{ij}(y_i, y_j) = -\frac{1}{2} y_i^2 + \frac{1}{2} y_j^2.$$

Application II: A Single Species Model with Dispersal

$$x'_i = x_i f_i(x_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij} x_i), \quad i = 1, 2, \dots, n. \quad (8)$$

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Theorem [Z. Shuai and ML (2009)] Assume

- (1) matrix (d_{ij}) is irreducible;
- (2) $f'_i(x_i) \leq 0, x_i > 0, i = 1, 2, \dots, n; \exists k, f'_k(x_k) \not\equiv 0$ in any open interval of \mathbb{R}^+ ;
- (3) system (8) is uniformly persistent;
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Then system (8) has a globally asymptotically stable positive equilibrium E^* .

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Note: Lu and Tacheuchi (1993) proved the result under the assumption $f'_i(x_i) < 0, x_i > 0$ for all i , using the theory of monotone dynamical systems.

Application III: An n -Patch Predator-Prey Model

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Kuang and Tacheuchi (1994) proved the two-patch case.

$$V_i(x_i, y_i) = \epsilon_i(x_i - x_i^* \ln x_i) + e_i(y_i - y_i^* \ln y_i)$$

Application IV: A Multi-group Delayed Epidemic Model

$$\begin{aligned} S'_i &= \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j), \\ I'_i &= \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j) - (d_i^I + \gamma_i) I_i, \end{aligned} \quad i = 1, 2, \dots, n. \quad (10)$$

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When $n = 1$, C. McCluskey proved the global stability with Lyapunov function

$$\begin{aligned}
 V_i &= (S_i - S_i^* + S_i^* \ln \frac{S_i}{S_i^*}) + (I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}) + \\
 &\quad \sum_{j=1}^n \beta_{ji} S_i^* \int_0^{\tau_j} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.
 \end{aligned}$$

A video on Youtube

<https://www.youtube.com/watch?v=QmWD76jwjbQ>

GRASP Lab, University of Pennsylvania

Each robotic agent has position vector $r_i = (x_i, y_i) \in \mathbb{R}^2$ and velocity vector $v_i = \dot{r}_i = (\dot{x}_i, \dot{y}_i)$.

The system's evolution is governed by Newton's equation

$$\begin{aligned}\dot{r}_i &= v_i, \\ \dot{v}_i &= u_i,\end{aligned}\quad i = 1, \dots, n. \quad (11)$$

Here

- u_i , $i = 1, \dots, n$, define the **control protocol**
- **Formation control** is achieved through communications among agents
- **Network** represents the communication graph (topology)
- A complete communication graph is too costly.

Hierarchical Potential Clustering (HPC) Protocol

Proposed by J. Maidens and ML:

- 1) Divide the agents into clusters
- 2) Assign a leader to each cluster
- 3) Implement an artificial potential scheme (with a complete graph) within each cluster
- 4) Implement a velocity consensus scheme among the cluster leaders.

HPC Protocol: control within a cluster i

For $j \neq 1$, (i.e. r_{ij} is not a leader in cluster i)

$$u_{ij} = -\nabla_{r_{ij}} P_{ij} - \sum_k \frac{(\theta_{ij} - \theta_{ik}) \|v_{ij}\|}{\|r_{ij} - r_{ik}\|} \hat{n}(ij),$$

where

$$P_{ij} = \sum_{k=1}^{n_i} P_{ij}^{ik}$$

controls distance of agents in the cluster and

$$\theta_{ij} = \tan^{-1} \left(\frac{\dot{y}_{ij}}{\dot{x}_{ij}} \right)$$

is the heading of agent (i, j) .

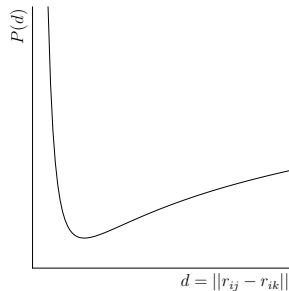


Figure: Potential function P_{ij}^{ik} .

HPC Protocol: control among leaders

For $j = 1$, (i.e., r_{i1} is the leader in cluster i), we add additional force to control there heading

$$u_{i1} = -\nabla_{r_{i1}} P_{i1} - \sum_k \frac{(\theta_{i1} - \theta_{ik}) \|v_{i1}\|}{\|r_{i1} - r_{ik}\|} \hat{n}(i1) + \sum_{h \in N_i} b_{ih} (v_{h1} - v_{i1})$$

where matrix $B = (b_{ij})$ is any nonnegative **irreducible** matrix. The correspond communication graph \mathcal{G}_B among leaders is **strongly connected**.

Formation Stabilization Problem

Definition A control protocol is said to solve the **formation stabilization problem** if solutions of (11) converge asymptotically to a state such that

- (a) the relative positions of each agent (i, j) within a cluster are such that a local minimum of the total vertex potential P_{ij} is achieved,
- (b) the headings of any two agents (i, j) and (h, k) satisfy $\theta_{ij} = \theta_{hk}$.

Main Result

Theorem (J. Maidens and ML, 2013)

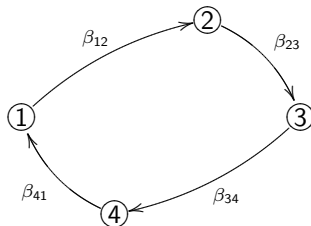
Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph \mathcal{G}_B is **strongly connected**.

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Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph \mathcal{G}_B is **strongly connected**.

An example graph that is strongly connected:

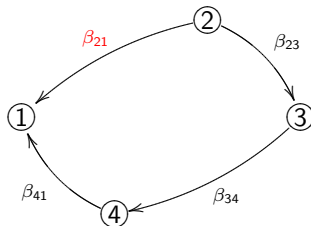


Main Result

Theorem (J. Maidens and ML, 2013)

Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph \mathcal{G}_B is **strongly connected**.

An example graph that is **not** strongly connected:



Simulations

- Clustering without control protocol
 - Video 1
- Clustering without leader control
 - Video 2
 - Video 3
- Clustering with leader control
 - Video 4
 - Video 5

Synchronization

Synchronization of metronomes: a video

<https://www.youtube.com/watch?v=Aaxw4zbULMs>

Coupled Oscillators Revisited

Consider a system of coupled oscillators:

$$\ddot{x}_i + f_i(x_i) + \sum_{j=1}^n \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0,$$

Assume that $f_i(x_i)$ and $F_i(x_i) = \int_i^x f_i(t)dt$ satisfy

(C₁) $f_i(x_i)x_i > 0$, $x_i \neq 0$, $i = 1, 2, \dots, n$,

(C₂) $F_i(x_i) \rightarrow \infty$ as $|x_i| \rightarrow \infty$, $i = 1, 2, \dots, n$.

Both (C₁) and (C₂) are satisfied for $f_i(x_i) = x_i^3$.

Global Synchronization

Definition: System (29) is said to achieve global synchronization if, for every solution $x(t)$ of system (29) and all $1 \leq i, j \leq n$,

$$\dot{x}_i(t) - \dot{x}_j(t) = 0.$$

Question: Under what conditions of matrix $A = (a_{ij})$ does the system (29) achieves global synchronization?

A Theorem

Theorem (P. Du and ML 2015)

In system (29), suppose that the direct graph \mathcal{G}_A is strongly connected, and assumptions (C_1) and (C_2) are satisfied. Then system (29) achieves global synchronization.

Considering the equivalent system

$$\dot{x}_i = y_i$$

$$\dot{y}_i = -f_i(x_i) + \sum_{j=1}^n \epsilon_{ij}(y_j - y_i)$$

Using Lyapunov functions:

$$V_i = \frac{1}{2}y_i^2 + F_i(x_i),$$

and

$$V = \sum_{i=1}^n c_i V_i.$$