

Global-Stability Problem for Coupled Differential Equations on Networks

Michael Y. Li and Zhisheng Shuai

Department of Mathematical and Statistical Sciences, University of Alberta
Edmonton, Alberta, T6G 2G1, Canada

mli@math.ualberta.ca zshuai@math.ualberta.ca

Keywords: Kirchhoff's Matrix Tree Theorem, networks of coupled dynamical systems, Lyapunov functions, global stability, single-species models with dispersal, predator-prey models with dispersal, multigroup epidemic models, delay epidemic models.

2000 Mathematics Subject Classification: 34D23, 34K20, 92D25, 92D30

1 Introduction

2 Graph theoretical results

In this section, we revisit Kirchhoff's Matrix Tree Theorem and prove several related combinatorial identities that we will use in later sections. We begin by recalling some definitions from graph theory. We refer the reader to [13] and [31] for detailed discussions.

A *directed graph* or *digraph* $\mathcal{G} = (V, E)$ contains a set $V = \{1, 2, \dots, n\}$ of vertices and a set E of arcs (i, j) leading from its initial vertex i to its terminal vertex j . A subgraph \mathcal{H} of \mathcal{G} is said to be *spanning* if \mathcal{H} and \mathcal{G} have the same vertex sets. A digraph \mathcal{G} is *weighted* if each arc (i, j) is assigned a positive weight a_{ij} . If \mathcal{H} is a subgraph of a weighted digraph \mathcal{G} , then the weight $w(\mathcal{H})$ of \mathcal{H} is the product of the weights on all arcs of \mathcal{H} .

A *directed path* \mathcal{P} in \mathcal{G} is a subgraph whose vertices can be labelled i_1, i_2, \dots, i_m so that its arcs are of the form (i_k, i_{k+1}) for $k = 1, 2, \dots, m - 1$. A *directed cycle* \mathcal{C} in \mathcal{G} is the subgraph obtained from such an oriented path \mathcal{P} by adding arc (i_m, i_1) if it exists. If $m = 1$, we refer to the directed cycle consisting of a single vertex x_1 and a single arc (x_1, x_1) as a loop. A *rooted tree* \mathcal{T} is a subgraph with no directed cycle in which one vertex, the root, is the initial vertex of no arcs, but each of the remaining vertices is the initial vertex of exactly one arc. A subgraph \mathcal{Q} is *unicyclic* if it is a disjoint union of rooted trees whose roots form a directed cycle. Note that every vertex of \mathcal{Q} is the initial vertex of exactly one arc.

Given a weighted digraph \mathcal{G} with n vertices, define the weight matrix $A = (a_{ij})_{n \times n}$ whose entry a_{ij} equals the weight associated with arc (i, j) if it exists, and 0 otherwise. For our purposes, we denote the weighted digraph as (\mathcal{G}, A) . The *Laplacian matrix* of (\mathcal{G}, A) is

defined as

$$L = \begin{bmatrix} \sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk} \end{bmatrix}. \quad (2.1)$$

Let v_i denote the i -th principal $(n-1)$ minor of L .

Proposition 2.1 (Kirchhoff's Matrix Tree Theorem [16, 27]). *Assume $n \geq 2$. Then*

$$v_i = \sum_{T \in \mathbb{T}_i} w(T), \quad i = 1, 2, \dots, n, \quad (2.2)$$

where \mathbb{T}_i is the set of all spanning trees T of (\mathcal{G}, A) that are rooted at vertex i , and $w(T)$ is the weight of T .

A digraph \mathcal{G} is *strongly connected* if for any pair of distinct vertices, there exists an oriented path from one to the other. In particular, a weighted digraph (\mathcal{G}, A) is strongly connected if and only if the weight matrix A is irreducible [4].

Corollary 2.2. *Assume $n \geq 2$. If (\mathcal{G}, A) is strongly connected, then $v_i > 0$ for $1 \leq i \leq n$.*

Theorem 2.3. *Assume $n \geq 2$. Then the following identity holds.*

$$\sum_{i,j=1}^n v_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q} \in \mathcal{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x_r, x_s), \quad (2.3)$$

where $F_{ij}(x_i, x_j)$, $1 \leq i, j \leq n$, are arbitrary functions, \mathcal{Q} is the set of all spanning unicyclic graphs \mathcal{Q} of (\mathcal{G}, A) , $w(\mathcal{Q})$ is the weight of \mathcal{Q} , and $\mathcal{C}_{\mathcal{Q}}$ denotes the oriented cycle of \mathcal{Q} .

Proof. Note that each term in the expression for v_i given in (2.2) is the weight of a spanning tree T of (\mathcal{G}, A) that is rooted at vertex i . Consequently, each term in $v_i a_{ij}$ is the weight of a spanning unicyclic graph \mathcal{Q} of \mathcal{G} obtained from tree T by adding an arc (i, j) directed from the vertex i to the root vertex j . As a sequence,

$$w(T) a_{ij} = w(\mathcal{Q}),$$

and

$$w(T) a_{ij} F_{ij}(x_i, x_j) = w(\mathcal{Q}) F_{ij}(x_i, x_j),$$

where $(i, j) \in E(\mathcal{C}_{\mathcal{Q}})$. In general, when a directed arc from the root to any non-root vertex is added to a rooted tree, we obtain a unicyclic graph. See Figure 2. When we perform this operation in all possible ways to all rooted trees in \mathcal{G} , we obtain all unicyclic graphs in \mathcal{G} and each unicyclic graph \mathcal{Q} is created as many times as the number of arcs in its cycle $\mathcal{C}_{\mathcal{Q}}$. The identity (2.3) follows if we reorganize the double sum on the left hand side as a sum over all unicyclic graphs in \mathcal{G} . This completes the proof of Theorem 2.3. \square

Theorem 2.4. Assume $n \geq 2$. Then the following identity holds.

$$\sum_{i,j=1}^n v_i a_{ij} G_i(x_i) = \sum_{i,j=1}^n v_i a_{ij} G_j(x_j), \quad (2.4)$$

where $G_i(x_i), 1 \leq i \leq n$, are arbitrary functions.

Proof. Using Theorem 2.3, we know that both sides of (2.4) are equal to

$$\sum_{\mathcal{Q} \in \mathcal{Q}} w(\mathcal{Q}) \sum_{k \in V(\mathcal{C}_{\mathcal{Q}})} G_k(x_k),$$

where $V(\mathcal{C}_{\mathcal{Q}})$ is the vertex set of $\mathcal{C}_{\mathcal{Q}}$. □

We end this section with a lemma that will be used in later sections.

Lemma 2.5. Assume

- (1) a weighted digraph (\mathcal{G}, A) is strongly connected;
- (2) function $H_{ij}(x_i, x_j) \geq 0$ for all $1 \leq i, j \leq n$;
- (3) $H_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$.

Then $\sum_{i,j=1}^n a_{ij} H_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$ for all $1 \leq i, j \leq n$.

Proof. Note that $\sum_{i,j=1}^n a_{ij} H_{ij}(x_i, x_j) = 0$ implies that $H_{ij}(x_i, x_j) = 0$ and thus $x_i = x_j$ if $a_{ij} \neq 0$. For any given $1 \leq k, l \leq n$, $k \neq l$, by the strong connectivity of (\mathcal{G}, A) , there exists a directed path \mathcal{P} in (\mathcal{G}, A) from k to l . Applying the relation $x_i = x_j$ to each arc (i, j) of \mathcal{P} , we can see that $x_l = x_k$. Therefore, $H_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$ for all $1 \leq i, j \leq n$. □

3 Coupled differential equations on networks

Given a digraph \mathcal{G} with n vertices, $n \geq 2$, a network can be built on \mathcal{G} by assigning each vertex its own internal dynamics (called vertex dynamics) and then coupling these vertex dynamics together according to the connection in \mathcal{G} . Assume each vertex dynamics is described by a system of differential equations

$$u'_i = f_i(t, u_i), \quad (3.1)$$

where $u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i}$. Let $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^{m_i}$ represent the influence of vertex j on vertex i , and $g_{ij} \equiv 0$ if there exists no arc from j to i in \mathcal{G} . Then we obtain the following coupled system on \mathcal{G}

$$u'_i = f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j), \quad i = 1, 2, \dots, n. \quad (3.2)$$

Here functions f_i, g_{ij} are assumed to be sufficiently smooth.

One of main mathematical interests in the literature of network dynamics is to establish the collective dynamics of a coupled system like (3.2). In particular, it is a natural question to ask how properties of digraph \mathcal{G} affect dynamical behaviors of the system, specially stability and synchronization. In this paper, we will focus on the global stability problems for the equilibrium of (3.2). Suppose that $E^* = (u_1^*, u_2^*, \dots, u_n^*)$ is an equilibrium of (3.2) in some open set $D = \bigotimes_{i=1}^n D_i \subset \mathbb{R}^m$, where $m = \sum_{i=1}^n m_i$. E^* is said to be *globally asymptotically stable* in D if it is locally stable and attracts all solutions initiated in D . Among other approaches, Lyapunov direct's method is often applied to prove the global stability problems for differential equations arisen in different fields, such as mathematical biology, chemostat, electric circuit, and etc. In the following, a general type of Lyapunov function for system (3.2) will be constructed by using graph theoretic results from the previous section.

Suppose that there exist functions $F_{ij}(t, u_i, u_j)$ defined on $\mathbb{R} \times D_i \times D_j$ and locally Lipschitz continuous functions $V_i(t, u_i)$ defined on $\mathbb{R} \times D_i$ for $1 \leq i, j \leq n$. Define the derivative of each V_i along (3.2) as follows

$$\dot{V}_i := \frac{\partial V_i(t, u_i)}{\partial t} + \frac{\partial V_i(t, u_i)}{\partial u_i} \cdot \left(f_i(t, u_i) + \sum_{j=1}^n c_{ij}^j g_{ij}(t, u_i, u_j) \right). \quad (3.3)$$

Assume

$$\dot{V}_i \leq \sum_{j=1}^n a_{ij} F_{ij}(t, u_i, u_j), \quad t > 0, u_i \in D_i, u_j \in D_j, \quad (3.4)$$

where constant $a_{ij} > 0$ for all $1 \leq i, j \leq n$. Denote $A = (a_{ij})$ and define a weighted digraph (\mathcal{W}, A) with n vertices as follows: there exists an arc from vertex i to vertex j and its weight equals a_{ij} if $a_{ij} > 0$, and no arc from i to j otherwise. Let v_i denote the i -th principal $(n-1)$ minor of the Laplacian matrix L of (\mathcal{W}, A) , as described in Proposition 2.1. Note that if (\mathcal{W}, A) is strongly connected, then $v_i > 0$ for $1 \leq i \leq n$. Denote $u = (u_1, u_2, \dots, u_n)$ and set

$$V(t, u) = \sum_{i=1}^n v_i V_i(t, u_i). \quad (3.5)$$

Theorem 3.1. *Assume that there exist functions $V_i(t, u_i)$ and $F_{ij}(t, u_i, u_j)$ such that (3.4) holds and in every directed cycle \mathcal{C} of (\mathcal{W}, A) ,*

$$\sum_{(r,s) \in E(\mathcal{C})} F_{rs}(t, u_r, u_s) \leq 0, \quad t > 0, u_r \in D_r, u_s \in D_s. \quad (3.6)$$

Then $\dot{V}(t, u) \leq 0$ for $t > 0$ and $u \in D$, namely, V as defined in (3.5) is a Lyapunov function for (3.2).

Proof. Direct calculation gives

$$\dot{V} = \sum_{i=1}^n v_i \dot{V}_i \leq \sum_{i,j=1}^n v_i a_{ij} F_{ij}(t, u_i, u_j).$$

Using Theorem 2.3 with weighted digraph (\mathcal{W}, A) , we obtain

$$\sum_{i,j=1}^n v_i a_{ij} F_{ij}(t, u_i, u_j) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(t, u_r, u_s).$$

Since in every directed cycle \mathcal{C} of (\mathcal{W}, A) , (3.6) holds, we have

$$\sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(t, u_r, u_s) \leq 0.$$

Therefore, $\dot{V} \leq 0$, which completes the proof of Theorem 3.1. \square

In Theorem 3.1, if we further assume that $\dot{V} = 0$ iff $u = E^*$, then E^* is globally asymptotically stable in D . Instead, assume that the largest compact invariant subset of $\{u \in D : \dot{V} = 0\}$ is the singleton $\{E^*\}$, then by the LaSalle Invariance Principle [23], E^* is globally asymptotically stable in D .

Corollary 3.2. *Assume that there exist functions $V_i(t, u_i)$, $F_{ij}(t, u_i, u_j)$, and $G_i(t, u_i)$, such that (3.4) holds and*

$$F_{ij}(t, u_i, u_j) \leq G_i(t, u_i) - G_j(t, u_j), \quad 1 \leq i, j \leq n. \quad (3.7)$$

Then $\dot{V} \leq 0$.

Corollary 3.3. *Assume that there exist functions $V_i(t, u_i)$ and $F_{ij}(t, u_i, u_j)$ such that*

$$\dot{V}_i \leq -c_i V_i(t, u_i) + \sum_{j=1}^n a_{ij} F_{ij}(t, u_i, u_j), \quad 1 \leq i \leq n, \quad (3.8)$$

with constants $c_i \geq 0$, and either (3.6) or (3.7) holds. Then $\dot{V} \leq -cV$, where $c = \min\{c_i : 1 \leq i \leq n\}$.

4 Damped harmonic oscillators on a network

Given a digraph \mathcal{G} with n vertices, $n \geq 2$, a coupled system of damped harmonic oscillators on \mathcal{G} can be built as follows: each vertex i is assigned a damped harmonic oscillator described by

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) = 0,$$

and the interactions from vertex j to vertex i are provided in the form of $\epsilon_{ij}(\dot{x}_i - \dot{x}_j)$, where $\epsilon_{ij} \geq 0$ and $\epsilon_{ij} = 0$ if and only if no arc from j to i in \mathcal{G} . Thus we obtain an example of coupled differential equations on a network

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^n \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0, \quad (4.1)$$

or in the form of a system of differential equations

$$\begin{aligned}\dot{x}_i &= y_i, \\ \dot{y}_i &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^n \epsilon_{ij}(y_i - y_j).\end{aligned}\tag{4.2}$$

Assume that the damping $\alpha_i \geq 0$ and the potential energy $F_i(x_i) = \int^{x_i} f_i(s)ds$ has a strictly global minimum at $x_i = x_i^*$. Then $E^* = (x_1^*, 0, x_2^*, 0, \dots, x_n^*, 0)$ is an equilibrium of system (4.2).

Theorem 4.1. *If $\alpha_k > 0$ for some k and digraph \mathcal{G} is strongly connected, then E^* is globally asymptotically stable in \mathbb{R}^{2n} .*

Proof. Set $V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}$. The derivative along (4.2) gives

$$\dot{V}_i = -\alpha_i y_i^2 - \sum_{j=1}^n \epsilon_{ij}(y_i - y_j)y_i.$$

Let $a_{ij} = \epsilon_{ij}$, $F_{ij}(y_i, y_j) = y_i y_j - y_i^2$, and $G_i(y_i) = -\frac{1}{2}y_i^2$. Then we have

$$\dot{V}_i \leq \sum_{j=1}^n a_{ij} F_{ij}(y_i, y_j)$$

and

$$F_{ij}(y_i, y_j) = -\frac{1}{2}(y_i - y_j)^2 - \frac{1}{2}y_i^2 + \frac{1}{2}y_j^2 \leq G_i(y_i) - G_j(y_j).$$

Thus, conditions (3.4) and (3.7) have been verified. By Theorem 3.1 and Corollary 3.2, the function $V(x_1, y_1, \dots, x_n, y_n) = \sum_{i=1}^n v_i V_i(x_i, y_i)$ as defined in (3.5) is a Lyapunov function for (4.2), namely, $\dot{V} \leq 0$ for all $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$.

Recall that the coefficient v_i in V is the i -th principal $(n-1)$ minor of the Laplacian of weighted digraph (\mathcal{W}, A) . Note that \mathcal{G} is strongly connected, so does (\mathcal{W}, A) . Thus $v_i > 0$ for all $1 \leq i \leq n$. As a sequence, $\dot{V} = 0$ implies $\alpha_i y_i^2 = 0$ and $a_{ij}(y_i - y_j)^2 = 0$ for all i and j . Since $\alpha_k > 0$ for some k , we obtain $y_k = 0$. Let $H_{ij}(y_i, y_j) = (y_i - y_j)^2$, then we have $\sum_{i,j=1}^n a_{ij} H_{ij}(y_i, y_j) = 0$. Using Lemma 2.5, we conclude that $y_i = y_j$ for all i, j . Hence, $\dot{V} = 0$ implies $y_i = 0$ for all i . From the second equation of (4.1), we have $0 = \dot{y}_i = -f_i(x_i)$, thus $x_i = x_i^*$. This implies that the largest compact invariant subset of $\{(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} : \dot{V} = 0\}$ is the singleton $\{E^*\}$. Therefore, by the LaSalle Invariance Principle [23], E^* is globally asymptotically stable in \mathbb{R}^{2n} . \square

5 A single-species model with dispersal

Consider a system that describes dispersal of a single species among n patches ($n \geq 2$)

$$x'_i = x_i f_i(x_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij} x_i), \quad i = 1, 2, \dots, n.\tag{5.1}$$

Here $x_i \in \mathbb{R}^+$ represents the population density of the species in patch i , $f_i \in C^1(\mathbb{R}^+, \mathbb{R})$ represents the growth rate in patch i , and constant $d_{ij} \geq 0$ is the dispersal rate from patch j to patch i . Hastings [14] studied the local stability of a positive equilibrium of (5.1). Sufficient conditions for uniqueness and global stability of the positive equilibrium are derived in Beretta and Takeuchi [2] and later in Lu and Takeuchi [25]. In this section, we reformulate (5.1) as a coupled system on a network. This allows us to apply our general approach in Section 3 and prove a stronger result on the global stability of the positive equilibrium.

A digraph \mathcal{G} with n vertices associated with (5.1) can be constructed as follows: vertex i represents patch i , an arc (j, i) from vertex j to vertex i is assigned if the dispersal rate $d_{ij} > 0$, and no such arc exists if $d_{ij} = 0$. The vertex dynamics at each vertex is defined by the scalar ordinary differential equation $x'_i = x_i f_i(x_i)$. The coupling among vertices are provided by the dispersal among patches dictated by \mathcal{G} . Thus (5.1) is another example of coupled differential equations on a network.

Theorem 5.1. *Assume*

- (1) *matrix (d_{ij}) is irreducible;*
- (2) *$f'_i(x_i) \leq 0, x_i > 0, i = 1, 2, \dots, n$; for some k , $f'_k(x_k) \not\equiv 0$ in any open interval of \mathbb{R}^+ ;*
- (3) *system (5.1) is uniformly persistent;*
- (4) *solutions of (5.1) are uniformly bounded.*

Then system (5.1) has a unique positive equilibrium E^ and E^* is globally asymptotically stable in the positive cone \mathbb{R}_+^n .*

Proof. Uniform persistence, together with uniform boundedness of all solutions of system (5.1), implies that (5.1) has at least one positive equilibrium (see Theorem D.3 in [28] or Theorem 2.8.6 in [5]). Let $E^* = (x_1^*, x_2^*, \dots, x_n^*), x_i^* > 0, i = 1, 2, \dots, n$, denote a positive equilibrium of (5.1). We will show that E^* is globally asymptotically stable in \mathbb{R}_+^n and thus is unique. Set $V_i(x_i) = x_i - x_i^* + x_i^* \ln \frac{x_i}{x_i^*}$. It can be verified that $V_i(x_i) > 0$ for all $x_i > 0$ and $V_i(x_i) = 0$ if and only if $x_i = x_i^*$. Direct calculation yields

$$\begin{aligned}
\dot{V}_i &= (x_i - x_i^*) \left[f_i(x_i) + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] \\
&= (x_i - x_i^*) \left[f_i(x_i^*) + \int_{x_i^*}^{x_i} f'_i(\xi) d\xi \cdot (x_i - x_i^*) + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] \\
&= (x_i - x_i^*) \left[- \sum_{j=1}^n d_{ij} \left(\frac{x_j^*}{x_i^*} - \alpha_{ij} \right) + \int_{x_i^*}^{x_i} f'_i(\xi) d\xi \cdot (x_i - x_i^*) + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] \\
&= \int_{x_i^*}^{x_i} f'_i(\xi) d\xi \cdot (x_i - x_i^*)^2 + \sum_{j=1}^n d_{ij} x_j^* \left(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*} \right).
\end{aligned}$$

Let $a_{ij} = d_{ij} x_j^*$, $F_{ij}(x_i, x_j) = \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*}$, and $G_i(x_i) = -\frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*}$. Then we have

$$\dot{V}_i \leq \sum_{j=1}^n a_{ij} F_{ij}(x_i, x_j),$$

and

$$F_{ij}(x_i, x_j) = G_i(x_i) - G_j(x_j) + 1 - \frac{x_i^* x_j}{x_i x_j^*} + \ln \frac{x_i^* x_j}{x_i x_j^*} \leq G_i(x_i) - G_j(x_j).$$

Here we use two facts that $\int_{x_i^*}^{x_i} f'_i(\xi) d\xi \cdot (x_i - x_i^*)^2 \leq 0$ and $\Phi(a) := 1 - a + \ln a \leq 0$ for $a > 0$ with equality holding iff $a = 1$. Now conditions (3.4) and (3.7) have been verified. By Theorem 3.1 and Corollary 3.2, the function $V(x_1, \dots, x_n) = \sum_{i=1}^n v_i V_i(x_i)$ as defined in (3.5) is a Lyapunov function for (5.1), namely, $\dot{V} \leq 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}_+^n$.

Since (d_{ij}) is irreducible, $A = (a_{ij})$ is also irreducible, or equivalently, (\mathcal{W}, A) is strongly connected. Thus, $v_i > 0$ for $i = 1, 2, \dots, n$. As a result, $\dot{V} = 0$ implies that $\int_{x_i^*}^{x_i} f'_i(\xi) d\xi \cdot (x_i - x_i^*)^2 = 0$ and $a_{ij} \Phi\left(\frac{x_i^* x_j}{x_i x_j^*}\right) = 0$ for all $1 \leq i, j \leq n$. Since for some k , $f'_k(x_k) \neq 0$, we have $x_k = x_k^*$. Note that $\Phi\left(\frac{x_i^* x_j}{x_i x_j^*}\right) = 0$ if and only if $\frac{x_i}{x_i^*} = \frac{x_j}{x_j^*}$. Using Lemma 2.5, we know that $\sum_{i,j=1}^n a_{ij} \Phi\left(\frac{x_i^* x_j}{x_i x_j^*}\right) = 0$ if and only if $\frac{x_i}{x_i^*} = \frac{x_j}{x_j^*}$ for all i, j . Therefore, $\dot{V} = 0$ if and only if $x_i = x_i^*$ for all i . By the classical Lyapunov stability theory, E^* is globally asymptotically stable in \mathbb{R}_+^n . This completes the proof of Theorem 5.1. \square

Under the condition that for each i , $f'_i(x_i) < 0, x_i > 0$, the global stability of E^* was proved in Lu and Takeuchi [25] by using the theory of cooperative systems. Their condition implies our assumption (2) in Theorem 5.1. Theorem 5.1 contains the global stability result in [25] as a special case.

6 An n -patch predator-prey model

In this section, we consider a predator-prey model in which preys disperse among n patches ($n \geq 2$).

$$\begin{aligned} x'_i &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij} x_i), \\ y'_i &= y_i(-\gamma_i - \delta_i y_i + \epsilon_i x_i), \end{aligned} \quad i = 1, 2, \dots, n. \quad (6.1)$$

The parameters in the model are nonnegative constants, and e_i, ϵ_i are positive. We refer the reader to [9, 21] for interpretations of predator-prey models and parameters. When $n = 2$, Kuang and Takeuchi [21] studied (6.1) and proved the uniqueness and global stability of a positive equilibrium by constructing a Lyapunov function. In this section, we regard (6.1) as a coupled system on a network. Then, using a Lyapunov function for a single patch predator-prey model [15] and our main result Theorem 3.1, we establish the global stability of a positive equilibrium for the general n -patch model (6.1).

We first construct a digraph \mathcal{G} with n vertices associated with system (6.1) similarly as the previous section. Namely, $(j, i) \in E(\mathcal{G})$ if and only if $d_{ij} > 0$. At each vertex of \mathcal{G} , the vertex dynamics is described by a predator-prey system. The coupling among these predator-prey systems are provided by dispersal among prey populations. Thus (6.1) is another example of coupled systems on a network.

Theorem 6.1. *Assume that (d_{ij}) is irreducible, and that for some k , either $b_k > 0$ or $\delta_k > 0$. If a positive equilibrium E^* exists, then it is unique and globally asymptotically stable in the positive cone \mathbb{R}_+^{2n} .*

Proof. Let $E^* = (x_1^*, y_1^*, \dots, x_n^*, y_n^*)$, $x_k^*, y_k^* > 0$ for $1 \leq k \leq n$, denote the positive equilibrium. Set $V_i(x_i, y_i) = \epsilon_i(x_i - x_i^* \ln x_i) + e_i(y_i - y_i^* \ln y_i)$. Then the derivative along (6.1) yields

$$\begin{aligned} \dot{V}_i &= \epsilon_i(x_i - x_i^*) \left[r_i - b_i x_i - e_i y_i + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] \\ &\quad + e_i(y_i - y_i^*) (-\gamma_i - \delta_i y_i + \epsilon_i x_i) \\ &= \epsilon_i(x_i - x_i^*) \left[b_i x_i^* + e_i y_i^* - \sum_{j=1}^n d_{ij} \left(\frac{x_j^*}{x_i^*} - \alpha_{ij} \right) - b_i x_i - e_i y_i + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] \\ &\quad + e_i(y_i - y_i^*) (\delta_i y_i^* - \epsilon_i x_i^* - \delta_i y_i + \epsilon_i x_i) \\ &= -\epsilon_i b_i (x_i - x_i^*)^2 - e_i \delta_i (y_i - y_i^*)^2 + \sum_{j=1}^n d_{ij} \epsilon_i x_j^* \left(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_j x_i^*}{x_j^* x_i} \right). \end{aligned}$$

Let $a_{ij} = d_{ij} \epsilon_i x_j^*$, $F_{ij}(x_i, x_j) = \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*}$, and $G_i(x_i) = -\frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*}$. Conditions (3.4) and (3.7) can be verified in the same way as the previous section. Thus, by Theorem 3.1, the function $V(x_1, y_1, \dots, x_n, y_n) = \sum_{i=1}^n v_i V_i(x_i, y_i)$ as defined in (3.5) is a Lyapunov function for (6.1), namely, $\dot{V} \leq 0$ for all $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}_+^{2n}$.

Similarly to the previous section, in the set where $\dot{V} = 0$, we conclude that $-\epsilon_i b_i (x_i - x_i^*)^2 - e_i \delta_i (y_i - y_i^*)^2 = 0$ and $a_{ij} \Phi\left(\frac{x_i^* x_j}{x_i x_j^*}\right) = 0$ for all $1 \leq i, j \leq n$. Since either $b_k > 0$ or $\delta_k > 0$ for some k , $x_k = x_k^*$ or $y_k = y_k^*$. If $y_k = y_k^*$, then $y_k' = 0$, which implies that $-\gamma_k - \delta_k y_k^* + \epsilon_k x_k = 0$, and thus $x_k = x_k^*$. Using a similar argument as one in the previous section, we conclude that $x_i = x_i^*$ for all i . Using the first equation of (6.1), we have $0 = x_i' = x_i^* (r_i - b_i x_i^* - e_i y_i) + \sum_{j=1}^n d_{ij} (x_j^* - x_i^*)$, hence $y_i = y_i^*$. Therefore, the only compact invariant subset of the set where $\dot{V} = 0$ is singleton $\{E^*\}$. Using the LaSalle Invariance Principle [23], we conclude that E^* is globally asymptotically stable in \mathbb{R}_+^{2n} , and thus is unique. This completes the proof of Theorem 6.1. \square

Note that the existence of E^* can be obtained by persistence analysis, which only involves dynamics on the boundary of \mathbb{R}_+^n . When $n = 2$, Theorem 6.1 gives Theorem 3.2 of [21].

7 A multigroup epidemic model with nonlinear incidence

In Section 5 and Section 6, we analyze two dispersal models in the setting of coupled systems on a network where interactions are provided by dispersal among vertices. In this section, we

consider an n -group epidemic model in which interactions are provided by cross-infections.

$$\begin{aligned} S'_i &= \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j), \\ E'_i &= \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) - (d_i^E + \epsilon_i) E_i, \\ I'_i &= \epsilon_i E_i - (d_i^I + \gamma_i) I_i, \end{aligned} \quad i = 1, 2, \dots, n. \quad (7.1)$$

In this model, a heterogeneous population is partitioned into n homogeneous groups. Each group i is further compartmentalized into S_i, E_i , and I_i which denote the subpopulation that are susceptible to the disease, infected but non-infectious, and infectious, respectively. The coupling term $\beta_{ij} f_{ij}(S_i, I_j)$ represents the cross infection from group j to group i . All parameters in (7.1) are nonnegative constants. For detailed discussions of the model and interpretations of parameters, we refer the reader to [11, 29].

Let \mathcal{G} denote a digraph with n vertices in which an arc (j, i) exists if and only if $\beta_{ij} > 0$. System (7.1) can be treated as a coupled differential equation on \mathcal{G} if we assign each vertex a single-group SEIR model and couple them together with cross infection.

then it means that in (\mathcal{G}, B) there exists an arc leading from vertex j to vertex i ; and in network, it means that individuals of I_j can infect individuals of S_i . Therefore, system (7.1) can be treated as a coupled system on (\mathcal{G}, B) , and thus is a special case of (3.2).

Assume that $\epsilon_i > 0$ and $d_i^* > 0$, where $d_i^* = \min\{d_i^S, d_i^E, d_i^I + \gamma_i\}$. Based on biological considerations, we assume that $f_{ji}(I_j, 0) = 0, f_{ji}(0, S_i) = 0$, and $f_{ji}(I_j, S_i) > 0$ for $S_i > 0, I_j > 0$. We also assume that $f_{ji}(I_j, S_i)$ are sufficiently smooth.

For each i , adding the three equations in (7.1) gives $(S_i + E_i + I_i)' \leq \Lambda_i - d_i^*(S_i + E_i + I_i)$. Hence $\limsup_{t \rightarrow \infty} (S_i + E_i + I_i) \leq \Lambda_i / d_i^*$. Similarly, from the S_i equation we obtain $\limsup_{t \rightarrow \infty} S_i \leq \Lambda_i / d_i^S$. Therefore, omega limit sets of system (7.1) are contained in the following bounded region in the nonnegative cone of \mathbb{R}^{3n}

$$\Gamma = \left\{ (S_1, E_1, I_1, \dots, S_n, E_n, I_n) \in \mathbb{R}_+^{3n} \mid S_i \leq \frac{\Lambda_i}{d_i^S}, S_i + E_i + I_i \leq \frac{\Lambda_i}{d_i^*}, 1 \leq k \leq n \right\}. \quad (7.2)$$

It can be verified that region Γ is positively invariant. System (7.1) always has the *disease-free equilibrium* $P_0 = (S_1^0, 0, 0, \dots, S_n^0, 0, 0)$ on the boundary of Γ , where $S_i^0 = \Lambda_i / d_i^S$. An equilibrium $P^* = (S_1^*, E_1^*, I_1^*, \dots, S_n^*, E_n^*, I_n^*)$ in the interior $\overset{\circ}{\Gamma}$ of Γ is called an *endemic equilibrium*, where $S_i^*, E_i^*, I_i^* > 0$ satisfy the equilibrium equations

$$\Lambda_i = d_i^S S_i^* + \sum_{j=1}^n \beta_{ji} f_{ji}(I_j^*, S_i^*), \quad (7.3)$$

$$(d_i^E + \epsilon_i) E_i^* = \sum_{j=1}^n \beta_{ji} f_{ji}(I_j^*, S_i^*), \quad (7.4)$$

$$\epsilon_i E_i^* = (d_i^I + \gamma_i) I_i^*. \quad (7.5)$$

In the rest of this section we make the following basic assumptions on functions $f_{ji}(I_j, S_i)$:

$$(H_1) \quad 0 < \lim_{I_j \rightarrow 0^+} \frac{f_{ji}(I_j, S_i)}{I_j} = C_{ji}(S_i) \leq +\infty, \quad 0 < S_i \leq S_i^0;$$

$$(H_2) \quad f_{ji}(I_j, S_i) \leq C_{ji}(S_i)I_j \text{ for sufficiently small } I_j;$$

$$(H_3) \quad f_{ji}(I_j, S_i) \leq C_{ji}(S_i)I_j \text{ for all } I_j > 0;$$

$$(H_4) \quad C_{ji}(S_i) < C_{ji}(S_i^0), \quad 0 < S_i < S_i^0.$$

Assume that $f_{ji}(I_j, S_i)$ satisfies (H_1) , and let

$$R_0 = \rho(M_0), \tag{7.6}$$

denote the spectral radius of the matrix

$$M_0 = M(S_1^0, S_2^0, \dots, S_n^0) = \left(\frac{\beta_{ji} \epsilon_i C_{ji}(S_i^0)}{(d_i^E + \epsilon_i)(d_i^I + \gamma_i)} \right)_{1 \leq i, j \leq n}.$$

If $C_{ji}(S_i^0) = +\infty$ for some i and j , we set $R_0 = +\infty$. The parameter R_0 is referred to as the basic reproduction number. Its biological significance is that if $R_0 < 1$ the disease dies out while if $R_0 > 1$ the disease becomes endemic [6, 30]. The following results for system (7.1) are standard and can be proved the same way as in [10, 29].

Proposition 7.1. *Assume that $B = (\beta_{ij})$ is irreducible and $f_{ji}(I_j, S_i)$ satisfies (H_1) .*

- (1) *If $R_0 \leq 1$ and assumptions (H_2) and (H_4) hold, then for system (7.1), P_0 is locally asymptotically stable.*
- (2) *If $R_0 \leq 1$ and assumptions (H_3) and (H_4) hold, then P_0 is the unique equilibrium and it is globally asymptotically stable in Γ .*
- (3) *If $R_0 > 1$, then P_0 is unstable and system (7.1) is uniformly persistent. Furthermore, there exists at least one endemic equilibrium P^* for system (7.1).*

A challenging question in mathematical epidemiology is whether a multi-group model such as system (7.1) has a unique endemic equilibrium P^* when $R_0 > 1$, and whether P^* is globally asymptotically stable when it is unique.

Theorem 7.2. *Assume that $B = (\beta_{ij})$ is irreducible and $f_{ji}(I_j, S_i)$ satisfies (H_1) . If $R_0 > 1$ and $f_{ji}(I_j, S_i)$ satisfies the following conditions*

$$(S_i - S_i^*)(f_{ii}(I_i^*, S_i) - f_{ii}(I_i^*, S_i^*)) > 0, \quad S_i \neq S_i^*, \tag{7.7}$$

$$\begin{aligned} & \left(f_{ji}(I_j, S_i) f_{ii}(I_i^*, S_i^*) - f_{ji}(I_j^*, S_i^*) f_{ii}(I_i^*, S_i) \right) \cdot \\ & \left(\frac{f_{ji}(I_j, S_i) f_{ii}(I_i^*, S_i^*)}{I_j} - \frac{f_{ji}(I_j^*, S_i^*) f_{ii}(I_i^*, S_i)}{I_j^*} \right) \leq 0, \quad I_j > 0, \end{aligned} \tag{7.8}$$

then there exists a unique endemic equilibrium P^ for system (7.1), and P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$.*

Proof. The case $n = 1$ is proved in [26]. We only consider $n \geq 2$. Let $P^* = (S_1^*, E_1^*, I_1^*, \dots, S_n^*, E_n^*, I_n^*)$, $S_i^*, E_i^*, I_i^* > 0$ for $1 \leq i \leq n$, denote an endemic equilibrium which exists from Proposition 7.1-(3). We prove that P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$. In particular, this implies that the endemic equilibrium is unique. Let

$$V_i(S_i, E_i, I_i) = \int_{S_i^*}^{S_i} \frac{f_{ii}(I_i^*, \xi) - f_{ii}(I_i^*, S_i^*)}{f_{ii}(I_i^*, \xi)} d\xi + E_i - E_i^* \ln E_i + \frac{d_i^E + \epsilon_i}{\epsilon_i} (I_i - I_i^* \ln I_i).$$

Using equilibrium equations (7.3), (7.4), (7.5), we have

$$\begin{aligned} \dot{V}_i &= \left(1 - \frac{f_{ii}(I_i^*, S_i^*)}{f_{ii}(I_i^*, S_i)}\right) \left(\Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ji} f_{ji}(I_j, S_i)\right) + \left(1 - \frac{E_i^*}{E_i}\right) \left(\sum_{j=1}^n \beta_{ji} f_{ji}(I_j, S_i) \right. \\ &\quad \left. - (d_i^E + \epsilon_i) E_i\right) + \frac{d_i^E + \epsilon_i}{\epsilon_i} \left(1 - \frac{I_i^*}{I_i}\right) \left(\epsilon_i E_i - (d_i^I + \gamma_i) I_i\right) \\ &= \left(1 - \frac{f_{ii}(I_i^*, S_i^*)}{f_{ii}(I_i^*, S_i)}\right) \left(d_i^S S_i^* + \sum_{j=1}^n \beta_{ji} f_{ji}(I_j^*, S_i^*) - d_i^S S_i - \sum_{j=1}^n \beta_{ji} f_{ji}(I_j, S_i)\right) \\ &\quad + \left(1 - \frac{E_i^*}{E_i}\right) \left(\sum_{j=1}^n \beta_{ji} f_{ji}(I_j, S_i) - \sum_{j=1}^n \beta_{ji} f_{ji}(I_j^*, S_i^*) \frac{E_i}{E_i^*}\right) \\ &\quad + \sum_{j=1}^n \beta_{ji} \frac{f_{ji}(I_j^*, S_i^*)}{\epsilon_i E_i^*} \left(1 - \frac{I_i^*}{I_i}\right) \left(\epsilon_i E_i - \frac{\epsilon_i E_i^* I_i}{I_i^*}\right) \\ &= -\frac{d_i^S}{f_{ii}(I_i^*, S_i)} (S_i - S_i^*) (f_{ii}(I_i^*, S_i) - f_{ii}(I_i^*, S_i^*)) + \sum_{j=1}^n \beta_{ji} f_{ji}(I_j^*, S_i^*) \left(3 - \frac{f_{ii}(I_i^*, S_i^*)}{f_{ii}(I_i^*, S_i)} \right. \\ &\quad \left. + \frac{f_{ji}(I_j, S_i) f_{ii}(I_i^*, S_i^*)}{f_{ji}(I_j^*, S_i^*) f_{ii}(I_i^*, S_i)} - \frac{f_{ji}(I_j, S_i) E_i^*}{f_{ji}(I_j^*, S_i^*) E_i} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}\right). \end{aligned}$$

Let $b_{ji} = f_{ji}(I_j^*, S_i^*) > 0$, $G_i(I_i) = \frac{I_i}{I_i^*} - \ln \frac{I_i}{I_i^*}$, and

$$F_{ji} = b_{ji} \left(3 - \frac{f_{ii}(I_i^*, S_i^*)}{f_{ii}(I_i^*, S_i)} + \frac{f_{ji}(I_j, S_i) f_{ii}(I_i^*, S_i^*)}{f_{ji}(I_j^*, S_i^*) f_{ii}(I_i^*, S_i)} - \frac{f_{ji}(I_j, S_i) E_i^*}{f_{ji}(I_j^*, S_i^*) E_i} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}\right).$$

Notice that

$$F_{ji} = b_{ji} \left(G_j(x_j) - G_i(x_i) + \Phi\left(\frac{f_{ii}(I_i^*, S_i^*)}{f_{ii}(I_i^*, S_i)}\right) + \Phi\left(\frac{f_{ji}(I_j, S_i) E_i^*}{f_{ji}(I_j^*, S_i^*) E_i}\right) + \Phi\left(\frac{I_j f_{ji}(I_j^*, S_i^*) f_{ii}(I_i^*, S_i)}{I_j^* f_{ji}(I_j, S_i) f_{ii}(I_i^*, S_i^*)}\right) + M\right),$$

where

$$M = \left(\frac{f_{ji}(I_j, S_i) f_{ii}(I_i^*, S_i^*)}{f_{ji}(I_j^*, S_i^*) f_{ii}(I_i^*, S_i)} - 1\right) \left(1 - \frac{I_j f_{ji}(I_j^*, S_i^*) f_{ii}(I_i^*, S_i)}{I_j^* f_{ji}(I_j, S_i) f_{ii}(I_i^*, S_i^*)}\right).$$

Recall that $\Phi(a) = 1 - a + \ln a \leq 0$, $a > 0$, and $\Phi(a) = 0$ iff $a = 1$. Under assumptions (7.7) and (7.8), (3.4) and (??) can be verified similarly as in Section 5. By Theorem 3.1, the function $V = \sum_{i=1}^n v_i V_i$ as defined in (3.5) is a Lyapunov function for (7.1), namely, $\dot{V} \leq 0$. In the set where $\dot{V} = 0$, we conclude that $S_i = S_i^*$ and $\frac{E_i}{E_i^*} = \frac{I_i}{I_i^*} = \frac{I_j}{I_j^*}$ if $\beta_{ji} \neq 0$. Using a similar argument as in Section 6, we can show that the only compact invariant subset of the

set where $\overset{\bullet}{V} = 0$ is singleton $\{P^*\}$. By the LaSalle Invariance Principle [23], P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$ if $R_0 > 1$. This completes the proof of Theorem 7.2. \square

Remarks

1. Condition (7.7) holds if $f_{ii}(I_i^*, S_i)$ is a strictly monotonically increasing function with respect to S_i .
2. In the special case $f_{ji}(I_j, S_i) = g_j(I_j)h_i(S_i)$, condition (7.8) becomes

$$(g_j(I_j) - g_j(I_j^*)) \left(\frac{g_j(I_j)}{I_j} - \frac{g_j(I_j^*)}{I_j^*} \right) \leq 0. \quad (7.9)$$

If $g_j(I_j)$ is C^1 for $I_j > 0$, then a sufficient condition to (7.9) is

$$0 \leq g'_j(I_j) \leq \frac{g_j(I_j)}{I_j}, \quad I_j > 0. \quad (7.10)$$

Furthermore, if $g_j(I_j)$ is monotonically increasing and concave down, then (7.10) holds, so does (7.9). An example of non-concave and non-monotonic functions which satisfy (7.9) was given in Korobeinikov and Maini [20].

3. In the special case $f_{ji}(I_j, S_i) = I_j S_i$, system (7.1) becomes the standard multigroup SEIR model studied in [11]. Theorem 7.2 generalizes Theorem 1.1 in [11].
4. When $n = 1$, Theorem 7.2 contains early results on single-group SEIR models, see [18, 24].

8 A multigroup epidemic model with delays

In this section, we demonstrate that the general approach described in Section 3 can also be applied to establish global stability of equilibria of delay differential systems. As an example, we consider a multi-group SEIR epidemic model with time delays that is described by the following system of functional differential equations

$$\begin{aligned} S'_i &= \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j), \\ I'_i &= \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j) - (d_i^I + \gamma_i) I_i, \end{aligned} \quad i = 1, 2, \dots, n. \quad (8.1)$$

The parameters in (8.1) are nonnegative and interpreted the same as those of (7.1) in the previous section. Similarly to (7.1), we regard (8.1) as a coupled system on a weighted digraph (\mathcal{G}, B) . Particularly, the vertex dynamics at each vertex is defined by a system of delay differential equations.

Denote $\tau = \max\{\tau_i : i = 1, 2, \dots, n\}$. Let C be the Banach space of continuous functions on $[-\tau, 0]$ with uniform norm. We consider system (8.1) in the phase space

$$X = \prod_{k=1}^n (\mathbb{R} \times C). \quad (8.2)$$

In the rest of this section, we always assume our initial conditions of system (8.1) satisfy

$$S_i(0) = s_{i,0}, \quad I_{i,0} = \phi_i, \quad i = 1, 2, \dots, n, \quad (8.3)$$

where $s_{i,0} \in \mathbb{R}_+$ and $\phi_i \in C$ satisfies $\phi_i(s) \geq 0$ for $-\tau_i \leq s \leq 0$. It is straightforward to see that solutions with initial condition (8.3) remain nonnegative. Moreover, from the first equation of (8.1), we obtain $S_i(t)' \leq \Lambda_i - d_i^S S_i(t)$. Hence, $\limsup_{t \rightarrow \infty} S_i(t) \leq \frac{\Lambda_i}{d_i^S}$. For each i , adding the two equations in (8.1) gives $(S_i(t) + I_{i,t}(0))' \leq \Lambda_i - d_i^*(S_i(t) + I_{i,t}(0))$, which implies that $\limsup_{t \rightarrow \infty} (S_i(t) + I_{i,t}(0)) \leq \frac{\Lambda_i}{d_i^*}$, where $d_i^* = \min\{d_i^S, d_i^I + \gamma_i\}$. Therefore, the following set is positively invariant for system (8.1)

$$\Theta = \left\{ (S_1, I_1(\cdot), \dots, S_n, I_n(\cdot)) \in X \mid 0 \leq S_i \leq \frac{\Lambda_i}{d_i^S}, 0 \leq S_i + I_i(0) \leq \frac{\Lambda_i}{d_i^*}, \right. \\ \left. I_i(s) \geq 0, s \in (-\tau_i, 0], i = 1, \dots, n \right\}. \quad (8.4)$$

Let

$$\overset{\circ}{\Theta} = \left\{ (S_1, I_1(\cdot), \dots, S_n, I_n(\cdot)) \in X \mid 0 < S_i < \frac{\Lambda_i}{d_i^S}, 0 < S_i + I_i(0) < \frac{\Lambda_i}{d_i^*}, \right. \\ \left. I_i(s) > 0, s \in (-\tau_i, 0], i = 1, \dots, n \right\}. \quad (8.5)$$

It can be shown that $\overset{\circ}{\Theta}$ is the interior of Θ . In Θ , system (8.1) has the disease-free equilibrium $P_0 = (S_1^0, 0, \dots, S_n^0, 0)$ where $S_i^0 = \frac{\Lambda_i}{d_i^S}$, and any endemic equilibrium $P^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$ if exists, where $S_i^*, I_i^* > 0$ satisfy

$$\Lambda_i = \sum_{j=1}^n \beta_{ji} S_i^* I_j^* + d_i^S S_i^*, \quad \text{and} \quad \sum_{j=1}^n \beta_{ji} S_i^* I_j^* = (d_i^I + \gamma_i) I_i^*.$$

Let the basic reproduction number $R_0 = \rho(M_0)$ be the spectral radius of the matrix

$$M_0 = \left(\frac{\beta_{ji} S_i^0}{d_i^I + \gamma_i} \right)_{n \times n}.$$

Proposition 8.1. *Assume that $B = (\beta_{ij})$ is irreducible.*

1. *If $R_0 \leq 1$, then P_0 is the unique equilibrium for system (8.1) and it is globally asymptotically stable in Θ .*
2. *If $R_0 > 1$, then P_0 is unstable and there exists a unique endemic equilibrium P^* for system (8.1).*

Theorem 8.2. Assume that $B = (\beta_{ij})$ is irreducible. If $R_0 > 1$, then the unique endemic equilibrium P^* for system (8.1) is globally asymptotically stable in $\overset{\circ}{\Theta}$.

Proof. We consider the case $n \geq 2$. Let $P^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$, where $S_i^*, I_i^* > 0$ for $1 \leq i \leq n$, denote the unique endemic equilibrium of system (8.1). Now we construct a functional $V_i : \mathbb{R} \times C \rightarrow \mathbb{R}^+$ as follows

$$V_i = S_i - S_i^* + S_i^* \ln \frac{S_i}{S_i^*} + I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*} + \sum_{j=1}^n \beta_{ji} S_i^* \int_0^{\tau_j} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.$$

Notice that

$$\int_0^{\tau_j} \frac{\partial}{\partial t} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr = - \int_0^{\tau_j} \frac{\partial}{\partial r} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.$$

Using a similar argument as in the previous section, we can show

$$\dot{V}_i = -\frac{d_i^S}{S_i} (S_i - S_i^*)^2 + \sum_{j=1}^n \beta_{ji} S_i^* I_j^* \left(3 - \frac{S_i^*}{S_i} + \frac{I_j}{I_j^*} - \frac{I_i}{I_i^*} - \frac{S_i I_j(t - \tau_j) E_i^*}{S_i^* I_j^* E_i} - \frac{E_i I_i^*}{E_i^* I_i} + \ln \frac{I_j(t - \tau_j)}{I_j} \right).$$

Let $b_{ji} = S_i^* I_j^* > 0$, $G_i(I_i) = \frac{I_i}{I_i^*} - \ln \frac{I_i}{I_i^*}$, and

$$F_{ji} = b_{ji} \left(3 - \frac{S_i^*}{S_i} + \frac{I_j}{I_j^*} - \frac{I_i}{I_i^*} - \frac{S_i I_j(t - \tau_j) E_i^*}{S_i^* I_j^* E_i} - \frac{E_i I_i^*}{E_i^* I_i} + \ln \frac{I_j(t - \tau_j)}{I_j} \right).$$

As similarly as in the previous section, (3.4) and (??) can be verified. Thus by Theorem 3.1, the functional $V = \sum_{i=1}^n v_i V_i$ as defined in (3.5) is a Lyapunov functional for (8.1), namely, $\dot{V} \leq 0$. In the set where $\dot{V} = 0$, we know that $S_i = S_i^*$ and $\frac{E_i}{E_i^*} = \frac{I_i}{I_i^*} = \frac{I_j(t - \tau_j)}{I_j^*}$ if $\beta_{ji} \neq 0$. Using a similar argument as in Section 6, we can show that the only compact invariant subset of the set where $\dot{V} = 0$ is singleton $\{P^*\}$. By the LaSalle-Lyapunov Theorem (see [23, Theorem 3.4.7] or [12, Theorem 5.3.1]), we conclude that P^* is globally attractive in $\overset{\circ}{\Theta}$ if $R_0 > 1$. Furthermore, it can be verified that P^* is locally stable using the same proof as one for Corollary 5.3.2 in [12]. This completes the proof of Theorem 8.2. \square

Remarks

1. When $\tau_i = 0, i = 1, 2, \dots, n$, system (8.1) becomes the standard multigroup SIR model without delays studied in [10]. Theorem 8.2 generalizes Theorem 3.3 in [10].
2. When $n = 1$, Theorem 8.2 gives a global stability result of McCluskey [26] for a single-group SIR model with delay.

Acknowledgments.

This research was supported in part by grants from the Natural Science and Engineering Research Council of Canada (NSERC) and Canada Foundation for Innovation (CFI). Z. Shuai acknowledges the support of an Izaak Walton Killam Memorial Scholarship at the University of Alberta.

References

- [1] F. M. Atay and O. Karabacak, Stability of coupled map networks with delays, SIAM J. Appl. Dyn. Syst. 5 (2006) 508–527.
- [2] E. Beretta and Y. Takeuchi, Global stability of single-species diffusion Volterra models with continuous time delays, Bull. Math. Biol. 49 (1987) 431–448.
- [3] E. Beretta and Y. Takeuchi, Global stability of an SIR epidemic model with time delays, J. Math. Biol. 33 (1995) 250–260.
- [4] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, 1979.
- [5] N. P. Bhatia and G. P. Szegő, *Dynamical Systems: Stability Theory and Applications*, Lecture Notes in Mathematics, Vol. 35, Springer, Berlin, 1967.
- [6] O. Diekmann, J. A. P. Heesterbeek, and J. A. J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, J. Math. Biol. 28 (1990) 365–382.
- [7] B. Fiedler, M. Belhaq, and M. Houssni, Basins of attraction in strongly damped coupled mechanical oscillators: a global example, Z. angew. Math. Phys. 50 (1999) 282–300.
- [8] M. Forti, On global asymptotic stability of a class of nonlinear systems arising in neural network theory, J. Differential Equations 113 (1994) 246–264.
- [9] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.
- [10] H. Guo, M. Y. Li, and Z. Shuai, Global stability of the endemic equilibrium of multigroup SIR epidemic models, Can. Appl. Math. Q. 14 (2006) 259–284.
- [11] H. Guo, M. Y. Li, and Z. Shuai, A graph-theoretic approach to the method of global Lyapunov functions, Proc. Amer. Math. Soc. 136 (2008) 2793–2802.
- [12] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, Vol. 99, Springer, New York, 1993.
- [13] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.
- [14] A. Hastings, Dynamics of a single species in a spatially varying environment: The stabilizing role of high dispersal rates, J. Math. Biol. 16 (1982) 49–55.
- [15] S. B. Hsu, On global stability of a predator-prey systems, Math. Biosci. 39 (1978) 1–10.
- [16] D. E. Knuth, *The Art of Computer Programming*, Vol. 1, 3rd ed., Addison-Wesley, Reading, 1997.

- [17] A. Korobeinikov, Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission, *Bull. Math. Biol.* 68 (2006) 615–626.
- [18] A. Korobeinikov, Global properties of infectious disease models with nonlinear incidence, *Bull. Math. Biol.* 69 (2007) 1871–1886.
- [19] A. Korobeinikov and P. K. Maini, A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence, *Math. Biosci. Eng.* 1 (2004) 57–60.
- [20] A. Korobeinikov and P. K. Maini, Nonlinear incidence and stability of infectious disease models, *Math. Med. Biol. A J. IMA* 22 (2005) 113–128.
- [21] Y. Kuang and Y. Takeuchi, Predator-prey dynamics in models of prey dispersal in two-patch environments, *Math. Biosci.* 120 (1994) 77–98.
- [22] A. Lajmanovich and J. A. York, A deterministic model for gonorrhea in a nonhomogeneous population, *Math. Biosci.* 28 (1976) 221–236.
- [23] J. P. LaSalle, *The Stability of Dynamical Systems*, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1976.
- [24] M. Y. Li and J. S. Muldowney, Global stability for the SEIR model in epidemiology, *Math. Biosci.* 125 (1995) 155–164.
- [25] Z. Lu and Y. Takeuchi, Global asymptotic behavior in single-species discrete diffusion systems, *J. Math. Biol.* 32 (1993) 67–77.
- [26] C. C. McCluskey, Complete global stability for an SIR epidemic model with delay - distributed or discrete, to appear.
- [27] J. W. Moon, *Counting Labelled Trees*, Canadian Mathematical Congress, Montreal, 1970.
- [28] H. L. Smith and P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, Cambridge, 1995.
- [29] H. R. Thieme, *Mathematics in Population Biology*, Princeton University Press, Princeton, 2003.
- [30] P. van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.* 180 (2002) 29–48.
- [31] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 1996.