1 Project 1: The Willmore conjecture (theorem)

Recall the Willmore conjecture [4]. This conjecture, now a theorem [3], states that for a 2-torus \( T \rightarrow \mathbb{R}^3 \), the Willmore energy obeys

\[
W := \int H^2 dA e^{2\pi}.
\]

The integral here is a 2-dimensional integral over the torus, and \( dA \) is the surface area element (often called the 2-dimensional volume element of the torus or the 2-volume element). The mean curvature \( H \) is the average of the two principal curvatures

\[
H = \frac{1}{2} (\kappa_1 + \kappa_2) = \frac{1}{2} \left( \frac{1}{r_1} + \frac{1}{r_2} \right),
\]

where \( r_1 \) is the radius of the smallest “osculating circle” tangent to the torus at a point and \( r_2 \) is the radius of the largest such circle at the same point. (These two circles always lie in orthogonal planes in \( \mathbb{R}^3 \)).

The purpose of this project is to test the conjecture on so-called tube tori. In doing so, we will discover the minimizer of the Willmore energy. This is the torus for which the above inequality becomes an equality \( W := \int H^2 dA = 2\pi \). In other words, the minimizer is a certain tube torus.

To construct a tube torus, first draw a simple closed curve in \( \mathbb{R}^3 \). Then take all points which lie a fixed distance \( b > 0 \) from this curve. That’s a tube torus (for a general torus, \( b \) could be a function that varies along the curve). A tube torus can self-intersect. If so, it is immersed in \( \mathbb{R}^3 \). If it doesn’t self-intersect, it’s embedded in \( \mathbb{R}^3 \).

Let’s start with a planar circle, say the circle \( r^2 := x^2 + y^2 = a^2 \) in the \( z = 0 \) plane of \( \mathbb{R}^3 \). A tube torus about this circle can be written as the parametrized surface

\[
x = (a - b \cos \theta) \cos \phi
\]

\[
y = (a - b \cos \theta) \sin \phi
\]

\[
z = b \sin \theta,
\]

Here \( \theta, \phi \in [0, 2\pi] \). Sketch this torus to get the idea.

Compute the two principal curvatures at each point by finding the radii of the corresponding osculating circles. You should get \( \kappa_1 = \frac{1}{b}, \kappa_2 = \frac{\cos \theta}{b \cos \theta - a} \). Then compute the Willmore energy and find the minimizing torus.

Can you generalize this result by working out the Willmore energy for the tube torus about an arbitrary simple closed curve in the \((x,y)\)-plane? Willmore’s article [5] actually gives the computation for a tube torus about
arbitrary simple closed space curve in $\mathbb{R}^3$. You can understand it if you know a little theory of space curves. All you have to know is how to compute the \textit{torsion} and \textit{curvature} of a space curve (see section 3 of his paper, beginning at the top of page 309, but note that his terminology and notation are a bit difficult than what we used).

Can you write a computer program that tests Willmore’s conjecture for certain classes of torii? The two possible classes might be the tube torii about a general plane curve (see the above paragraph) or the tori of rotation. To construct a torus of rotation, first draw a simple closed curve in the $(x, z)$-plane of $\mathbb{R}^3$. To get an embedded torus, the curve should lie entirely to the right of the $z$-axis in this plane. Now rotate this curve about the $z$-axis. The resulting surface is a \textit{torus of rotation}. It can be rather hard to figure out the mean curvature for these tori, so start with as simple an example as possible (perhaps try it if your closed curve is an ellipse).

For a review of the Willmore conjecture, see [2].