Dynamical Systems on Networks: From Epidemics to Flight of Drones

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Outline of Talks

Part I: Global Stability Problems in Heterogeneous Epidemic Models

1.1: Modeling Infectious Diseases in Heterogeneous Populations
  - Simple epidemic models and their dynamics
  - Basic reproduction number and the threshold theorem
  - Multi-group models for heterogeneous populations

1.2: Global-Stability Problem in Multi-Group Models
  - Global-stability problem and Lyapunov functions
  - A Lyapunov function for multi-group models
  - Why is global-stability difficult to prove?

1.3: Matrix-Tree Theorem in Graph Theory
  - Rooted directed trees and unicyclic graphs
  - Kirchhoff’s Matrix-Tree Theorem

1.4: How do all of these come together?
  - Global-stability result for multi-group models.
Outline of Talks

Part II: Dynamical Systems on Networks

II.1: Dynamical systems on networks
- A network as a directed graph
- Dynamical systems on networks
- Examples

1.2: Global-stability problem
- Global-stability problem and Lyapunov functions
- A general theorem
- Applications

1.3: Flight formation control for drones
- Network of autonomous robotic agents
- Flight formation problems and HPC control protocol
- Simulations

1.4: Synchronization problems
- Synchronization of metronomes, a video
- Global synchronization of coupled oscillators.
How to Model an Epidemic?

An Epidemic Curve
How to Model an Epidemic?

\[ I(t) \]

- new infection
- recovery
- death

\[ I'(t) = \text{Incidence Rate} - \text{Recovery Rate} - \text{Death Rate} = f(I(t), S(t), N(t)) - \gamma I(t) - d I(t) \]

- \( f(I, S, N) = \beta IS \): bilinear incidence
- \( f(I, S, N) = \lambda IN \): proportionate incidence
How to Model an Epidemic?

\[ I'(t) = \text{Incidence Rate} - \text{Recovery Rate} - \text{Death Rate} \]

\[ = f(I(t), S(t), N(t)) - \gamma I(t) - d I(t) \]
How to Model an Epidemic?

\[ I'(t) = \text{Incidence Rate} - \text{Recovery Rate} - \text{Death Rate} \]

\[ = f(I(t), S(t), N(t)) - \gamma I(t) - d I(t) \]

\[ f(I, S, N) = \beta I S : \text{bilinear incidence} \]

\[ f(I, S, N) = \lambda \frac{I S}{N} : \text{proportionate incidence} \]
A Single-Group SIR Model

\[ S : \text{Susceptibles} \quad I : \text{Infectious} \quad R : \text{Removed} \]

\[
\begin{align*}
S' &= \Lambda - \beta IS - dS \\
I' &= \beta IS - (\gamma + d)I \\
R' &= \gamma I - dR
\end{align*}
\]
A Single-Group SIR Model

\[ S : \text{Susceptibles} \quad I : \text{Infectious} \quad R : \text{Removed} \]

\[ S' = \Lambda - \beta IS - dS \]
\[ I' = \beta IS - (\gamma + d)I \]
\[ R' = \gamma I - dR \]
A Single-Group SIR Model

Numerical output I: epidemic case
A Single-Group SIR Model

Numerical output I: epidemic case

Numerical output II: endemic case
Threshold Theorem

The basic reproduction number is

\[ R_0 = \frac{\beta \Lambda}{(\gamma + d)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d} \]

The average secondary infections produced by a single infective during its entire infectious period.
Threshold Theorem

The basic reproduction number is

\[ R_0 = \frac{\beta \Lambda}{(\gamma + d)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d} \]

The average secondary infections produced by a single infective during its entire infectious period.

Theorem (Threshold Theorem)

1. If \( R_0 \leq 1 \), then the disease-free equilibrium \( P_0 = (\Lambda/d, 0) \) is stable and attracts all solutions in \( \mathbb{R}^2_+ \).
2. If \( R_0 > 1 \), then \( P_0 \) is unstable, and a unique endemic (positive) equilibrium \( P^* \) is stable and attracts all positive solutions in \( \mathbb{R}^2_+ \).

Proof uses the Poincaré-Bendixson theory for 2d systems.
Threshold Theorem

The basic reproduction number is

\[ R_0 = \frac{\beta \Lambda}{(\gamma + d)d} = \beta \cdot \frac{1}{\gamma + d} \cdot \frac{\Lambda}{d} \]

The average secondary infections produced by a single infective during its entire infectious period.

Theorem (Threshold Theorem)

1. If \( R_0 \leq 1 \), then the disease-free equilibrium \( P_0 = (\Lambda/d, 0) \) is stable and attracts all solutions in \( R_2^+ \).
2. If \( R_0 > 1 \), then \( P_0 \) is unstable, and a unique endemic (positive) equilibrium \( P^* \) is stable and attracts all positive solutions in \( R_2^+ \).

Proof uses the Poincaré-Bendixson theory for 2d systems.
Each circled number represents a homogeneous group.

\( \beta_{jk} \) : transmission coefficient between \( I_j \) and \( S_k \).
$n$-Group Models for Heterogeneous Populations

Transmission Matrix $B$ is irreducible.
n-Group Models for Heterogeneous Populations

\[
B = \begin{bmatrix}
\beta_{11} & 0 & 0 & 0 \\
\beta_{21} & \beta_{22} & \beta_{23} & 0 \\
0 & \beta_{32} & \beta_{33} & \beta_{34} \\
\beta_{41} & \beta_{42} & 0 & \beta_{44}
\end{bmatrix}
\]

Transmission Matrix $B$ is reducible.
A Two-Group SIR Model

Incidence terms (bilinear):
- Group 1: $\beta_{11}I_1S_1$
- Group 2: $\beta_{22}I_2S_2$
A Two-Group SIR Model

Incidence terms (bilinear):
- Group 1: $\beta_{11} I_1 S_1 + \beta_{21} I_2 S_1$
- Group 2: $\beta_{22} I_2 S_2$
A Two-Group SIR Model

Incidence terms (bilinear):

- **Group 1:** $\beta_{11} l_1 S_1 + \beta_{21} l_2 S_1$
- **Group 2:** $\beta_{22} l_2 S_2 + \beta_{12} l_1 S_2$
An $n$-Group SIR Model

\[
\begin{align*}
S_k' &= \Lambda_k - d_k S_k - \sum_{j=1}^{n} \beta_{jk} I_j S_k, \\
I_k' &= \sum_{j=1}^{n} \beta_{jk} I_j S_k - (d_k + \gamma_k) I_k,
\end{align*}
\]

$k = 1, \ldots, n.$

Mathematical Questions:

- If $R_0 > 1$, is $P^*$ unique?
An $n$-Group SIR Model

\begin{align*}
S'_k &= \Lambda_k - d_k S_k - \sum_{j=1}^{n} \beta_{jk} I_j S_k, \\
I'_k &= \sum_{j=1}^{n} \beta_{jk} I_j S_k - (d_k + \gamma_k) I_k,
\end{align*}

$k = 1, \ldots, n$.

Mathematical Questions:

- If $R_0 > 1$, is $P^*$ unique?
- When $P^*$ is unique, is it globally stable?
Previous Results on GAS of $P^*$

For Models using bilinear incidence:

- **Lajmanovich and Yorke (1976)**
  - $n$-group SIS model, by Lyapunov function
  - later extended by Nold, Hirsch
- **Hethcote (1975)**
  - $n$-group SIR model with no vital dynamics
- **Thieme (1983)**
  - $n$-group SEIRS model, small latent and immune periods
- **Beretta and Capasso (1986)**
  - $n$-group SIR model, constant group sizes
- **Lin and So (1993)**
  - $n$-group SIRS model, constant group sizes
  - $\beta_{ij} \ (i \neq j)$ small
Non-uniqueness of $P^*$ when $R_0 > 1$

- **Lin (1992)**
  - $n$-group model for HIV

- **Huang, Cooke, Castillo-Chavez (1992)**
  - $n$-group model for HIV with delay

These models use *proportionate incidence*. 
Global-Stability and Lyapunov Functions

Consider a general system of ODE

$$ \dot{x} = F(x), \quad x \in D \subset \mathbb{R}^d. $$

$\bar{x}$ is an equilibrium if $F(\bar{x}) = 0$.

An equilibrium $\bar{x}$ is globally stable in $D$ if it is locally stable and all solutions in $D$ converge to $\bar{x}$ as $t \to \infty$. 
Global-Stability and Lyapunov Functions

Consider a general system of ODE

\[ x' = F(x), \quad x \in D \subset \mathbb{R}^d. \]

\( \bar{x} \) is an equilibrium if \( F(\bar{x}) = 0 \).

An equilibrium \( \bar{x} \) is \textbf{globally stable in} \( D \) if it is locally stable and all solutions in \( D \) converge to \( \bar{x} \) as \( t \to \infty \).
Global-Stability and Lyapunov Functions

Theorem (Lyapunov)

Suppose \( \exists \) a Lipschitz function \( V(x) \) such that

(1) \( V(x) \geq V(\bar{x}) \) and \( V(x) = V(\bar{x}) \iff x = \bar{x} \).

(2) \( \star V(x) = \nabla V(x) \cdot F(x) \leq 0, \quad x \in D, \text{ and} \)

\( \star V(x) = 0 \iff x = \bar{x} \).

Then \( \bar{x} \) is globally stable in \( D \).
Global-Stability and Lyapunov Functions

Theorem (Lyapunov)

Suppose \( \exists \) a Lipschitz function \( V(x) \) such that

1. \( V(x) \geq V(\bar{x}) \) and \( V(x) = V(\bar{x}) \iff x = \bar{x} \).

2. \( \star V(x) = \nabla V(x) \cdot F(x) \leq 0, \quad x \in D, \text{ and } \)
   \[ \star V(x) = 0 \iff x = \bar{x}. \]

Then \( \bar{x} \) is globally stable in \( D \).

\( V(x(t)) \) strictly decreasing along a solution \( x(t) \)
Constructing a Lyapunov Function for the $n$-Group Model

Consider a candidate

$$V = \sum_{k=1}^{n} v_k \left[ (S_k - S_k^* \ln S_k) + (I_k - I_k^* \ln I_k) \right]$$

A Lyapunov function for a single-group model
Constructing a Lyapunov Function for the $n$-Group Model

Consider a candidate

$$V = \sum_{k=1}^{n} v_k \left[ (S_k - S_k^* \ln S_k) + (I_k - I_k^* \ln I_k) \right]$$

A Lyapunov function for a single-group model

Choose appropriate $v_k$ so that $V(x)$ is negative definite.
Derivative of $V$

\[ V' = \sum_{k=1}^{n} \nu_k \left[ (S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{I_k} I_k') \right] \]
Derivative of $V$

\[ V' = \sum_{k=1}^{n} v_k \left[ (S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{I_k} I_k') \right] \]

\[ = \sum_{k=1}^{n} v_k \left[ d_k S_k^* \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right] \]

\[ + \sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right] \]

\[ + \sum_{j,k=1}^{n} v_k \beta_{kj} I_k^* S_j^* \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j^*}{l_j} \frac{l_k^*}{l_k} \right) \]
Derivative of $V$

$$V' = \sum_{k=1}^{n} v_k \left[ (S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{l_k}) I_k' \right]$$

$$= \sum_{k=1}^{n} v_k \left[ d_k S_k^* \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right] \leq 0$$

$$+ \sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S_k^* I_j - (d_k + \gamma_k) I_k \right]$$

$$+ \sum_{j,k=1}^{n} v_k \beta_{kj} I_k^* S_j^* \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j}{l_j^*} \frac{l_k}{l_k^*} \right)$$
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Michael Li

Derivative of $V$

\[ V' = \]
\[ \sum_{k=1}^{n} v_k \left[ (S'_k - \frac{S_k^*}{S_k} S'_k) + (I'_k - \frac{l_k^*}{l_k}) \right] \]
\[ = \sum_{k=1}^{n} v_k \left[ d_k S_k^* \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right] \]
\[ + \sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S_k^* l_j - (d_k + \gamma_k) l_k \right] \equiv 0 \]
\[ + \sum_{j,k=1}^{n} v_k \beta_{kj} l_k^* S_j^* \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j^*}{l_k} \right) \]
Derivative of $V$

\[
V' = \\
\sum_{k=1}^{n} v_k \left[ (S'_k - \frac{S^*_k}{S_k} S'_k) + (I'_k - \frac{I^*_k}{I_k} I'_k) \right] \\
= \sum_{k=1}^{n} v_k \left[ d_k S^*_k \left( 2 - \frac{S^*_k}{S_k} - \frac{S_k}{S^*_k} \right) \right] \\
+ \sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S^*_k I_j - (d_k + \gamma_k) I_k \right] \\
+ \sum_{j,k=1}^{n} v_k \beta_{kj} I^*_k S^*_j \left( 2 - \frac{S^*_j}{S_j} - \frac{S_j}{S^*_j} \frac{I_j}{I^*_j} \frac{I^*_j}{I_k} \right) \\
H_n := \sum_{j,k=1}^{n} v_k \bar{\beta}_{kj} \left( 2 - \frac{S^*_j}{S_j} - \frac{S_j}{S^*_j} \frac{I_j}{I^*_j} \frac{I^*_j}{I_k} \right)
\]
Derivative of $V$

$$V' =$$

$$\sum_{k=1}^{n} v_k \left[ (S_k' - \frac{S_k^*}{S_k} S_k') + (I_k' - \frac{I_k^*}{I_k} I_k') \right]$$

$$= \sum_{k=1}^{n} v_k \left[ d_k S_k^* \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \right]$$

$$+ \sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S_k^* l_j - (d_k + \gamma_k) I_k \right]$$

$$+ \sum_{j,k=1}^{n} v_k \beta_{kj} l_k^* S_j^* \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j^*}{l_j} \frac{l_k^*}{l_k} \right)$$

$$H_n := \sum_{j,k=1}^{n} v_k \beta_{kj} \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j^*}{l_j} \frac{l_k^*}{l_k} \right)$$
Choosing Constants $v_k$

Choose $v_k$ so that

$$\sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S_j^* I_j - (d_k + \gamma_k) I_k \right] \equiv 0$$

for all $I_1, \cdots, I_n > 0$. 

Choosing Constants $v_k$

Choose $v_k$ so that

$$\sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S_{k}^* l_j - (d_k + \gamma_k) l_k \right] \equiv 0$$

for all $l_1, \cdots, l_n > 0$. This is equivalent to

$$\begin{bmatrix} \beta_{11} l_1^* S_1^* & \cdots & \beta_{n1} l_n^* S_1^* \\ \vdots & \ddots & \vdots \\ \beta_{1n} l_1^* S_n^* & \cdots & \beta_{nn} l_n^* S_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} \beta_{j1} l_j^* S_1^* v_1 \\ \vdots \\ \sum_{j=1}^{n} \beta_{jn} l_j^* S_n^* v_n \end{bmatrix}$$
Choosing Constants $v_k$

Choose $v_k$ so that

$$\sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S^*_k I_j - (d_k + \gamma_k) I_k \right] \equiv 0$$

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since, at $P^*$,

$$(d_k + \gamma_k) = \sum_{j=1}^{n} \beta_{jk} l_j^* S_k^*.$$
Choosing Constants $v_k$

Choose $v_k$ so that

$$
\sum_{k=1}^{n} v_k \left[ \sum_{j=1}^{n} \beta_{jk} S_k^* l_j - (d_k + \gamma_k) l_k \right] \equiv 0
$$

for all $l_1, \cdots, l_n > 0$. This is equivalent to

$$
\begin{bmatrix}
\beta_{11} l_1^* S_1^* & \cdots & \beta_{n1} l_1^* S_1^* \\
\vdots & \ddots & \vdots \\
\beta_{1n} l_n^* S_n^* & \cdots & \beta_{nn} l_n^* S_n^*
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
= 
\begin{bmatrix}
\sum_{j=1}^{n} \beta_{j1} l_j^* S_1^* v_1 \\
\vdots \\
\sum_{j=1}^{n} \beta_{jn} l_j^* S_n^* v_n
\end{bmatrix}
$$

since, at $P^*$,

$$
(d_k + \gamma_k) = \sum_{j=1}^{n} \beta_{jk} l_j^* S_k^* .
$$
Set \( \bar{\beta}_{jk} = \beta_{jk} l_1^* S_1^* \). Then \((v_1, \cdots, v_k)\) are determined by the linear system

\[
\begin{bmatrix}
\sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\
-\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]
Set $\bar{\beta}_{jk} = \beta_{jk} l_j^* S_k^*$. Then $(v_1, \cdots, v_k)$ are determined by the linear system

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\end{bmatrix}
\begin{bmatrix}
v_1 \\
\vdots \\
v_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
$$

The solution space is 1d and a basis is given by

$$
v_k = C_{kk}, \quad \text{the } k\text{-th principal minor}, \quad k = 1, \cdots, n.
$$
Set $\beta_{jk} = l_j^* S_k^*$. Then \((v_1, \cdots, v_k)\) are determined by the linear system

\[
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\vdots & \vdots & \ddots & \vdots \\
-\beta_{1n} & -\beta_{2n} & \cdots & \sum_{l \neq n} \beta_{nl}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

The solution space is 1d and a basis is given by

\[v_k = C_{kk}, \quad \text{the } k\text{-th principal minor, } \quad k = 1, \cdots, n.\]

Need to show

\[V' \leq H_n = \sum_{j,k=1}^{n} v_k \beta_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k}\right) \leq 0,
\]

for all $S_1, I_1, \cdots, S_n, I_n \geq 0$. 
Directed Graphs and Rooted Spanning Trees

Let $G$ be a directed graph with vertex set $V(G) = \{1, \cdots, n\}$ and weight matrix $B = (\beta_{ij})_{n \times n}$.
Directed Graphs and Rooted Spanning Trees

Let $G$ be a directed graph with vertex set $V(G) = \{1, \cdots, n\}$ and weight matrix $B = (\beta_{ij})_{n \times n}$.

A spanning tree $T$ of $G$ is a sub-tree of $G$ of $n - 1$ edges.

A rooted spanning tree is oriented towards a vertex.
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The weight of tree $T$ is $w(T) = \prod \beta_{ij}$ over all edges $(i,j)$. 
Directed Graphs and Rooted Spanning Trees

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The Matrix-Tree Theorem

Let $B = (\bar{\beta}_{ij})_{n \times n}$ be the weight matrix of graph $G$.

The Kirchhoff Matrix (combinatorial Laplacian) of $B$ is

$$
\bar{B} = \begin{bmatrix}
\sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\
-\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl}
\end{bmatrix}.
$$

Note that all column sums of $\bar{B}$ are 0.
The Matrix-Tree Theorem

Let $B = (\tilde{\beta}_{ij})_{n \times n}$ be the weight matrix of graph $G$.

The Kirchhoff Matrix (combinatorial Laplacian) of $B$ is

\[
\tilde{B} = \begin{bmatrix}
\sum_{l \neq 1} \tilde{\beta}_{1l} & -\tilde{\beta}_{21} & \cdots & -\tilde{\beta}_{n1} \\
-\tilde{\beta}_{12} & \sum_{l \neq 2} \tilde{\beta}_{2l} & \cdots & -\tilde{\beta}_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
-\tilde{\beta}_{1n} & -\tilde{\beta}_{2n} & \cdots & \sum_{l \neq n} \tilde{\beta}_{nl}
\end{bmatrix}.
\]

Note that all column sums of $\tilde{B}$ are 0.

Theorem (Matrix Tree Theorem, Kirchhoff 1847)

\[C_{kk} = \sum_{T \in \mathbb{T}_k} w(T).\]

$\mathbb{T}_k$: The set of spanning trees rooted at vertex $k$. 
Solving System $\bar{B} \nu = 0$ : $n = 3$

$$\nu_1 = C_{11} = \sum_{T \in \mathcal{T}_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:
Solving System $\bar{B} \nu = 0$: $n = 3$

\[ \nu_1 = C_{11} = \sum_{T \in T_1} w(T) = \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{21} \bar{\beta}_{31} + \bar{\beta}_{23} \bar{\beta}_{31} \]

All possible spanning trees rooted at vertex 1:

```
3 2 3
\downarrow \bar{\beta}_{32} \downarrow \bar{\beta}_{21} \downarrow \bar{\beta}_{31}
2 1 2
\downarrow \bar{\beta}_{21} \downarrow \bar{\beta}_{23} \downarrow \bar{\beta}_{31}
1
```

Furthermore

\[ \nu_1 \bar{\beta}_{13} = \bar{\beta}_{32} \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{21} \bar{\beta}_{31} \bar{\beta}_{13} + \bar{\beta}_{23} \bar{\beta}_{31} \bar{\beta}_{13} \]
Solving System $\bar{B} \nu = 0 : \ n = 3$

$$\nu_1 = C_{11} = \sum_{T \in T_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

Furthermore

$$\nu_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$
Solving System $\tilde{B} \mathbf{v} = 0 : \ n = 3$

$$\mathbf{v}_1 = C_{11} = \sum_{T \in T_1} w(T) = \tilde{\beta}_{32} \tilde{\beta}_{21} + \tilde{\beta}_{21} \tilde{\beta}_{31} + \tilde{\beta}_{23} \tilde{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

Furthermore

$$\mathbf{v}_1 \tilde{\beta}_{13} = \tilde{\beta}_{32} \tilde{\beta}_{21} \tilde{\beta}_{13} + \tilde{\beta}_{21} \tilde{\beta}_{31} \tilde{\beta}_{13} + \tilde{\beta}_{23} \tilde{\beta}_{31} \tilde{\beta}_{13}$$
Solving System $\bar{B} \, \nu = 0 : \, n = 3$

$$\nu_1 = C_{11} = \sum_{T \in T_1} w(T) = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{21}\bar{\beta}_{31} + \bar{\beta}_{23}\bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

```
          3
         ↓\bar{\beta}_{32}
        2       1
         ↓\bar{\beta}_{21}  ↓\bar{\beta}_{31}
        2       3
         ↓\bar{\beta}_{31}
        1
```

Furthermore

$$\nu_1\bar{\beta}_{13} = \bar{\beta}_{32}\bar{\beta}_{21}\bar{\beta}_{13} + \bar{\beta}_{21}\bar{\beta}_{31}\bar{\beta}_{13} + \bar{\beta}_{23}\bar{\beta}_{31}\bar{\beta}_{13}$$
Solving System $\bar{B} \mathbf{v} = 0$ : $n = 3$

$$\mathbf{v}_1 = C_{11} = \sum_{T \in T_1} w(T) = \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{21} \bar{\beta}_{31} + \bar{\beta}_{23} \bar{\beta}_{31}$$

All possible spanning trees rooted at vertex 1:

Furthermore

$$\mathbf{v}_1 \bar{\beta}_{13} = \bar{\beta}_{32} \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{21} \bar{\beta}_{31} \bar{\beta}_{13} + \bar{\beta}_{23} \bar{\beta}_{31} \bar{\beta}_{13}$$

Each product is the weight of a **unicyclic graph** with a cycle of length $1 \leq r \leq 3$. 
How does a unicyclic graph correspond to products in $v_k \beta_{kj}$?

\[ v_1 \beta_{13} = \beta_{32} \beta_{21} \beta_{13} + \beta_{21} \beta_{13} \beta_{31} + \beta_{23} \beta_{31} \beta_{13} \]
How does a unicyclic graph correspond to products in $v_k \beta_{kj}$?

\[ v_1 \beta_{13} = \beta_{32} \beta_{21} \beta_{13} + \beta_{21} \beta_{13} \beta_{31} + \beta_{23} \beta_{31} \beta_{13} \]
Unicyclic Graphs and Rooted Trees

How does a unicyclic graph correspond to products in $v_k \beta_{kj}$?

\[
v_1 \bar{\beta}_{13} = \bar{\beta}_{32} \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{21} \bar{\beta}_{13} \bar{\beta}_{31} + \bar{\beta}_{23} \bar{\beta}_{31} \bar{\beta}_{13}
\]

\[
v_3 \bar{\beta}_{31} = \bar{\beta}_{12} \bar{\beta}_{23} \bar{\beta}_{31} + \bar{\beta}_{21} \bar{\beta}_{13} \bar{\beta}_{31} + \bar{\beta}_{13} \bar{\beta}_{23} \bar{\beta}_{31}
\]
Another Unicyclic Graph

\[ v_1 \bar{\beta}_{13} = \bar{\beta}_{32} \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{21} \bar{\beta}_{31} \bar{\beta}_{13} + \bar{\beta}_{23} \bar{\beta}_{31} \bar{\beta}_{13} \]
Another Unicyclic Graph

\[ v_1 \beta_{13} = \bar{\beta}_{32} \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{21} \bar{\beta}_{31} \bar{\beta}_{13} + \bar{\beta}_{23} \bar{\beta}_{31} \bar{\beta}_{13} \]
\[ v_2 \beta_{21} = \bar{\beta}_{13} \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{12} \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{31} \bar{\beta}_{12} \bar{\beta}_{21} \]
Another Unicyclic Graph

\[ v_1 \beta_{13} = \beta_{32} \beta_{21} \beta_{13} + \beta_{21} \beta_{31} \beta_{13} + \beta_{23} \beta_{31} \beta_{13} \]

\[ v_2 \beta_{21} = \beta_{13} \beta_{32} \beta_{21} + \beta_{12} \beta_{32} \beta_{21} + \beta_{31} \beta_{12} \beta_{21} \]

\[ v_3 \beta_{32} = \beta_{12} \beta_{23} \beta_{32} + \beta_{13} \beta_{23} \beta_{32} + \beta_{21} \beta_{13} \beta_{32} \]
Another Unicyclic Graph

\[ v_1 \bar{\beta}_{13} = \bar{\beta}_{32} \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{21} \bar{\beta}_{31} \bar{\beta}_{13} + \bar{\beta}_{23} \bar{\beta}_{31} \bar{\beta}_{13} \]

\[ v_2 \bar{\beta}_{21} = \bar{\beta}_{13} \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{12} \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{31} \bar{\beta}_{12} \bar{\beta}_{21} \]

\[ v_3 \bar{\beta}_{32} = \bar{\beta}_{12} \bar{\beta}_{23} \bar{\beta}_{32} + \bar{\beta}_{13} \bar{\beta}_{23} \bar{\beta}_{32} + \bar{\beta}_{21} \bar{\beta}_{13} \bar{\beta}_{32} \]
Another Unicyclic Graph

\[
\begin{align*}
3 & \downarrow \quad 1 & \downarrow \quad 2 & \downarrow \\
2 & \downarrow \quad 3 & \downarrow \quad 2 \downarrow \quad 1 \\
1 & \downarrow \\
& 2 & \downarrow \quad 3
\end{align*}
\]

\[
\begin{align*}
\nu_1 \bar{\beta}_{13} &= \bar{\beta}_{32} \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{21} \bar{\beta}_{31} \bar{\beta}_{13} + \bar{\beta}_{23} \bar{\beta}_{31} \bar{\beta}_{13} \\
\nu_2 \bar{\beta}_{21} &= \bar{\beta}_{13} \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{12} \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{31} \bar{\beta}_{12} \bar{\beta}_{21} \\
\nu_3 \bar{\beta}_{32} &= \bar{\beta}_{12} \bar{\beta}_{23} \bar{\beta}_{32} + \bar{\beta}_{13} \bar{\beta}_{23} \bar{\beta}_{32} + \bar{\beta}_{21} \bar{\beta}_{13} \bar{\beta}_{32}
\end{align*}
\]

\[
V' \leq H_n = \sum_{j,k=1}^{n} \nu_k \bar{\beta}_{kj} \left(2 - \frac{S_{j}^*}{S_j} - \frac{S_{j}^*}{S_j} \frac{l_j l_k^*}{l_j l_k^*} \right)
\]
$H_n$ is Summed over all Unicyclic Graphs

$$H_n = \sum_{j,k=1}^{n} v_k \bar{\beta}_{kj} \left( 2 - \frac{S^*_j}{S_j} - \frac{S_j}{S^*_j} \frac{l_j}{l^*_j} \frac{l_k^*}{l_k} \right)$$

$$= \sum_Q H_Q$$
\( H_n \) is Summed over all Unicyclic Graphs

\[
H_n = \sum_{j,k=1}^{n} v_k \bar{\beta}_{kj} \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j^*}{l_j^* l_k^*} \right)
\]

\[
= \sum_Q H_Q
\]

and

\[
H_Q = w(Q) \cdot \sum_{(p,q) \in E(C_Q)} \left( 2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{l_p^* l_q^*}{l_p^* l_q^*} \right)
\]

\[
= w(Q) \cdot \left[ 2r - \sum_{(p,q) \in E(C_Q)} \left( \frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{l_p^* l_q^*}{l_p^* l_q^*} \right) \right]
\]
$H_n$ is Summed over all Unicyclic Graphs

$$H_n = \sum_{j,k=1}^{n} v_k \bar{\beta}_{kj} \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j}{l_j^*} \frac{l_k}{l_k^*} \right)$$

$$= \sum_Q H_Q$$

and

$$H_Q = w(Q) \cdot \sum_{(p,q) \in E(C_Q)} \left( 2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{l_p}{l_p^*} \frac{l_q}{l_q^*} \right)$$

$$= w(Q) \cdot \left[ 2r - \sum_{(p,q) \in E(C_Q)} \left( \frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{l_p}{l_p^*} \frac{l_q}{l_q^*} \right) \right]$$

Finally, because $C_Q$ is a cycle,

$$\prod_{(p,q) \in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{l_p}{l_p^*} \cdot \frac{l_q}{l_q^*} = \prod_{(p,q) \in E(C_Q)} \frac{l_p}{l_p^*} \cdot \frac{l_q}{l_q^*} = 1.$$
$H_n$ is Summed over all Unicyclic Graphs

$$H_n = \sum_{j,k=1}^{n} \nu_k \bar{\beta}_{kj} \left(2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j}{l_j^*} \frac{l_k}{l_k^*}\right)$$

$$= \sum_{Q} H_Q$$

and

$$H_Q = w(Q) \cdot \sum_{(p,q) \in E(C_Q)} \left(2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{l_p}{l_p^*} \frac{l_q}{l_q^*}\right)$$

$$= w(Q) \cdot \left[2r - \sum_{(p,q) \in E(C_Q)} \left(\frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{l_p}{l_p^*} \frac{l_q}{l_q^*}\right)\right] \leq 0$$

Finally, because $C_Q$ is a cycle,

$$\prod_{(p,q) \in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{l_p}{l_p^*} \cdot \frac{l_q}{l_q^*} = \prod_{(p,q) \in E(C_Q)} \frac{l_p}{l_p^*} \cdot \frac{l_q}{l_q^*} = 1.$$
$V' \leq H_n = \sum_{j,k=1}^{n} v_k \bar{\beta}_{kj} \left( 2 - \frac{S_j^*}{S_j} - \frac{S_j}{S_j^*} \frac{l_j}{l^*_j} \frac{l_k}{l^*_k} \right)$

$= \sum_Q H_Q \leq 0$

and

$H_Q = w(Q) \cdot \sum_{(p,q) \in E(C_Q)} \left( 2 - \frac{S_p^*}{S_p} - \frac{S_p}{S_p^*} \frac{l_p}{l^*_p} \frac{l_q}{l^*_q} \right)$

$= w(Q) \cdot \left[ 2r - \sum_{(p,q) \in E(C_Q)} \left( \frac{S_p^*}{S_p} + \frac{S_p}{S_p^*} \frac{l_p}{l^*_p} \frac{l_q}{l^*_q} \right) \right]$

Finally, because $C_Q$ is a cycle,

$$\prod_{(p,q) \in E(C_Q)} \frac{S_p^*}{S_p} \cdot \frac{S_p}{S_p^*} \cdot \frac{l_p}{l^*_p} \cdot \frac{l_q}{l^*_q} = \prod_{(p,q) \in E(C_Q)} \frac{l_p}{l^*_p} \cdot \frac{l_q}{l^*_q} = 1.$$
Main Result

For the $n$-group SIR model with bilinear incidence,

**Theorem (Guo, Li, Shuai, 2007)**

*Assume that transmission matrix $B$ is irreducible.*

*If $R_0 > 1$, then $P^*$ is unique and is globally stable in $\mathbb{R}^{2n}_+$.***
The same graph-theoretical approach can be used to:

Build Lyapunov function $V$ for a large-scale system

$$V = \sum_{k=1}^{n} c_k V_k$$

using the known Lyapunov function $V_k$ for each component.