Algorithms of Wavelets and Framelets

Bin Han

Department of Mathematical and Statistical Sciences
University of Alberta, Canada

Present at 2017 International Undergraduate Summer Enrichment Program at UofA

July 17, 2017
Outline of Tutorial

- Goal: introduction to wavelet theory through wavelet transforms.
- Algorithms for discrete wavelet/framelet transform
- Perfect reconstruction, sparsity, stability
- Multi-level fast wavelet transform
- Oblique extension principle
- Framelet transform for signals on bounded intervals

Declaration: Some figures and graphs in this talk are from various sources from Internet, or from published papers, or produced by matlab, maple, or C programming. [Details and sources of all graphs can be provided upon request of the audience.]
Notation

- $l(\mathbb{Z})$ for signals: all $v = \{v(k)\}_{k \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{C}$.
- $l_0(\mathbb{Z})$ for filters: all finitely supported sequences $u = \{u(k)\}_{k \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{C}$ on $\mathbb{Z}$.
- For $v = \{v(k)\}_{k \in \mathbb{Z}} \in l(\mathbb{Z})$, define
  
  $v^*(k) := v(-k), \quad k \in \mathbb{Z},$
  
  $\hat{v}(\xi) := \sum_{k \in \mathbb{Z}} v(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}$.

- Convolution $u * v$ and inner product:

  $[u * v](n) := \sum_{k \in \mathbb{Z}} u(k)v(n - k), \quad n \in \mathbb{Z},$

  $\langle v, w \rangle := \sum_{k \in \mathbb{Z}} v(k)\overline{w(k)}, \quad v, w \in l_2(\mathbb{Z})$.
Subdivision and Transition Operators

- The subdivision operator $S_u : l(\mathbb{Z}) \to l(\mathbb{Z})$:

  $$[S_u v](n) := 2 \sum_{k \in \mathbb{Z}} v(k) u(n - 2k), \quad n \in \mathbb{Z}$$

- The transition operator $T_u : l(\mathbb{Z}) \to l(\mathbb{Z})$ is

  $$[T_u v](n) := 2 \sum_{k \in \mathbb{Z}} v(k) u(k - 2n), \quad n \in \mathbb{Z}.$$
Subdivision and Transition Operators

- The upsampling operator \( \uparrow d : l(\mathbb{Z}) \to l(\mathbb{Z}) \):

\[
[v \uparrow d](n) := \begin{cases} 
  v(n/d), & \text{if } n/d \text{ is an integer;} \\
  0, & \text{otherwise},
\end{cases} \quad n \in \mathbb{Z}
\]

- the downsampling (or decimation) operator \( \downarrow d : l(\mathbb{Z}) \to l(\mathbb{Z}) \):

\[
[v \downarrow d](n) := v(dn), \quad n \in \mathbb{Z}.
\]

- Subdivision and transition operators:

\[
S_u v = 2u \ast (v \uparrow 2) \quad \text{and} \quad T_u v = 2(u^\ast \ast v) \downarrow 2.
\]
Discrete Framelet Transform (DFrT): Decomposition

- Let \( u_0, \ldots, u_s \in l_0(\mathbb{Z}) \) be filters for decomposition.
- For data \( v \in l(\mathbb{Z}) \), a 1-level framelet decomposition:
  \[ w_\ell := \sqrt{\frac{2^\ell}{2}} \mathcal{T}_{u_\ell} v, \quad \ell = 0, \ldots, s, \]
  where \( w_\ell \) are called framelet coefficients.
- Grouping together, a framelet decomposition operator \( \mathcal{W} : l(\mathbb{Z}) \rightarrow (l(\mathbb{Z}))^{1 \times (s+1)} \):
  \[ \mathcal{W} v := \sqrt{\frac{2}{2}} (\mathcal{T}_{u_0} v, \ldots, \mathcal{T}_{u_s} v), \quad v \in l(\mathbb{Z}). \]
Reconstruction

- Let \( \tilde{u}_0, \ldots, \tilde{u}_s \in l_0(\mathbb{Z}) \) be filters for reconstruction.
- A one-level framelet reconstruction by \( \mathcal{V} : (l(\mathbb{Z}))^{1 \times (s+1)} \to l(\mathbb{Z}) \):
  \[
  \mathcal{V}(w_0, \ldots, w_s) := \sqrt{\frac{2}{2}} \sum_{\ell=0}^{s} S_{\tilde{u}_\ell} w_\ell, \quad w_0, \ldots, w_s \in l(\mathbb{Z}).
  \]

- Perfect reconstruction: \( \mathcal{V} \mathcal{W} \mathcal{V} = \mathcal{V} \) for any data \( \mathcal{V} \).
- A filter bank \( \{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots, \tilde{u}_s\} \) has the perfect reconstruction (PR) if \( \mathcal{V} \mathcal{W} = \text{Id}_{l(\mathbb{Z})} \).
Perfect Reconstruction (PR) Property

**Theorem**

A filter bank \( \{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots, \tilde{u}_s\} \) has the perfect reconstruction property, that is,

\[
v = \mathcal{W} \mathcal{W} v = \frac{1}{2} \sum_{\ell=0}^{s} S_{\tilde{u}_\ell} T_{u_\ell} v, \quad \forall \ v \in l(\mathbb{Z}),
\]

if and only if, for all \( \xi \in \mathbb{R} \),

\[
\begin{align*}
\hat{u}_0(\xi)\hat{\tilde{u}}_0(\xi) + \hat{u}_1(\xi)\hat{\tilde{u}}_1(\xi) + \cdots + \hat{u}_s(\xi)\hat{\tilde{u}}_s(\xi) &= 1, \\
\hat{u}_0(\xi + \pi)\hat{\tilde{u}}_0(\xi) + \hat{u}_1(\xi + \pi)\hat{\tilde{u}}_1(\xi) + \cdots + \hat{u}_s(\xi + \pi)\hat{\tilde{u}}_s(\xi) &= 0.
\end{align*}
\]
The perfect reconstruction (PR) condition can be equivalently rewritten into the following matrix form:

\[
\begin{bmatrix}
\hat{u}_0(\xi) & \cdots & \hat{u}_s(\xi) \\
\hat{u}_0(\xi + \pi) & \cdots & \hat{u}_s(\xi + \pi)
\end{bmatrix} \begin{bmatrix}
\hat{u}_0(\xi) & \cdots & \hat{u}_s(\xi) \\
\hat{u}_0(\xi + \pi) & \cdots & \hat{u}_s(\xi + \pi)
\end{bmatrix}^* = I_2,
\]

where \( I_2 \) denotes the \( 2 \times 2 \) identity matrix and \( A^* := A^T \).

A filter bank \( (\{u_0, \ldots, u_s\}, \{\tilde{u}_0, \ldots, \tilde{u}_s\}) \) has PR is called a \textit{dual framelet filter bank}.
A dual framelet filter bank with \( s = 1 \) is called a biorthogonal wavelet filter bank, a nonredundant filter bank.

**Proposition**

Let \( \{ u_0, \ldots, u_s \}, \{ \tilde{u}_0, \ldots, \tilde{u}_s \} \) be a dual framelet filter bank. Then the following statements are equivalent:

(i) \( \mathcal{W} \) is onto or \( \mathcal{V} \) is one-one;

(ii) \( \mathcal{VW} = \text{Id}_{L(\mathbb{Z})} \) and \( \mathcal{WV} = \text{Id}_{L(\mathbb{Z})^{1 \times (s+1)}} \), that is, \( \mathcal{V} \) and \( \mathcal{W} \) are inverse operators to each other;

(iii) \( s = 1 \).
Duality in $l_2(\mathbb{Z})$

**Lemma**

Let $u \in l_0(\mathbb{Z})$ be a finitely supported filter on $\mathbb{Z}$. Then $S_u : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$ is the adjoint operator of $T_u : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$. More precisely,

$$\langle S_u v, w \rangle = \langle v, T_u w \rangle, \quad \forall \ v, w \in l_2(\mathbb{Z}). \quad (1)$$

The space $(l_2(\mathbb{Z}))^{1 \times (s+1)}$ has inner product:

$$\langle (w_0, \ldots, w_s), (\tilde{w}_0, \ldots, \tilde{w}_s) \rangle := \langle w_0, \tilde{w}_0 \rangle + \cdots + \langle w_s, \tilde{w}_s \rangle, \quad w_0, \ldots, w_s,$$

and

$$\left\| (w_0, \ldots, w_s) \right\|^2_{(l_2(\mathbb{Z}))^{1 \times (s+1)}} := \left\| w_0 \right\|^2_{l_2(\mathbb{Z})} + \cdots + \left\| w_s \right\|^2_{l_2(\mathbb{Z})}.$$
Role of $\frac{\sqrt{2}}{2}$ in DFrT

**Theorem**

Let $u_0, \ldots, u_s \in l_0(\mathbb{Z})$. Then TFAE:

(i) $\|\mathcal{W}v\|_{l_2(\mathbb{Z})}^2 = \|v\|_{l_2(\mathbb{Z})}^2$ for all $v \in l_2(\mathbb{Z})$, that is,

$$\|\mathcal{T}_{u_0}v\|_{l_2(\mathbb{Z})}^2 + \cdots + \|\mathcal{T}_{u_s}v\|_{l_2(\mathbb{Z})}^2 = 2\|v\|_{l_2(\mathbb{Z})}^2, \quad \forall v \in l_2(\mathbb{Z});$$

(ii) $\langle \mathcal{W}v, \mathcal{W}\tilde{v} \rangle = \langle v, \tilde{v} \rangle$ for all $v, \tilde{v} \in l_2(\mathbb{Z})$;

(iii) the filter bank $(\{u_0, \ldots, u_s\}, \{u_0, \ldots, u_s\})$ has PR:

$$\begin{bmatrix}
\hat{u}_0(\xi) & \cdots & \hat{u}_s(\xi) \\
\hat{u}_0(\xi + \pi) & \cdots & \hat{u}_s(\xi + \pi)
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0(\xi) & \cdots & \hat{u}_s(\xi) \\
\hat{u}_0(\xi + \pi) & \cdots & \hat{u}_s(\xi + \pi)
\end{bmatrix}^* = l_2,$$

$\{u_0, \ldots, u_s\}$ with PR is called a tight framelet filter bank.
A tight framelet filter bank with \( s = 1 \) is called an orthogonal wavelet filter bank.

**Proposition**

Let \( \{u_0, \ldots, u_s\} \) be a tight framelet filter bank. Then the following are equivalent:

1. \( \mathcal{W} \) is an onto orthogonal mapping satisfying \( \langle \mathcal{W}v, \mathcal{W}\tilde{v} \rangle = \langle v, \tilde{v} \rangle \) for all \( v, \tilde{v} \in l_2(\mathbb{Z}) \);

2. for all \( w_0, \ldots, w_s, \tilde{w}_0, \ldots, \tilde{w}_s \in l_2(\mathbb{Z}) \),
\[
\langle \mathcal{V}(w_0, \ldots, w_s), \mathcal{V}(\tilde{w}_0, \ldots, \tilde{w}_s) \rangle = \langle (w_0, \ldots, w_s), (\tilde{w}_0, \ldots, \tilde{w}_s) \rangle;
\]

3. \( s = 1 \).
Examples

To list a filter $u = \{ u(k) \}_{k \in \mathbb{Z}}$ with support $[m, n]$,

$$ u = \{ u(m), \ldots, u(-1), u(0), u(1), \ldots, u(n) \}_{[m,n]}, $$

- $\{ u_0, u_1 \}$ is the Haar orthogonal wavelet filter bank:
  $$ u_0 = \{ \frac{1}{2}, \frac{1}{2} \}_{[0,1]}, \quad u_1 = \{ \frac{1}{2}, -\frac{1}{2} \}_{[0,1]}. $$

- $\{ u_0, u_1 \}, \{ \tilde{u}_0, \tilde{u}_1 \}$ is a biorthogonal wavelet filter bank, where
  $$ u_0 = \{ -\frac{1}{8}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, -\frac{1}{8} \}_{[-2,2]}, \quad u_1 = \{ \frac{1}{4}, -\frac{1}{2}, \frac{1}{4} \}_{[0,2]}, $$
  $$ \tilde{u}_0 = \{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \}_{[-1,1]}, \quad \tilde{u}_1 = \{ \frac{1}{8}, \frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{8} \}_{[-1,3]}. $$
Illustration: I

Apply the Haar orthogonal filter bank to

\[ \nu = \{1, 0, -1, -1, -4, 60, 58, 56\} \] 

Note that

\[ [T_{u_0} \nu](n) = \nu(2n) + \nu(2n + 1), \quad [T_{u_1} \nu](n) = \nu(2n) - \nu(2n + 1), \quad n \in \mathbb{Z}. \]

We have the wavelet coefficients:

\[ w_0 = \frac{\sqrt{2}}{2} \{1, -2, 56, 114\}_{[0,3]}, \quad w_1 = \frac{\sqrt{2}}{2} \{1, 0, -64, -2\}_{[0,3]}. \]

Note that

\[ [S_{u_0} \hat{w}_0](2n) = \hat{w}_0(n), \quad [S_{u_0} \hat{w}_0](2n + 1) = \hat{w}_0(n), \quad n \in \mathbb{Z} \]
\[ [S_{u_1} \hat{w}_1](2n) = \hat{w}_1(n), \quad [S_{u_1} \hat{w}_1](2n + 1) = -\hat{w}_1(n), \quad n \in \mathbb{Z}. \]
Hence, we have

\[
\frac{\sqrt{2}}{2} S_{u_0} w_0 = \frac{1}{2} \{1, 1, -2, -2, 56, 56, 114, 114\} [0,7],
\]

\[
\frac{\sqrt{2}}{2} S_{u_1} w_1 = \frac{1}{2} \{1, -1, 0, 0, -64, 64, -2, 2\} [0,7].
\]

Clearly, we have the perfect reconstruction of \( v \):

\[
\frac{\sqrt{2}}{2} S_{u_0} w_0 + \frac{\sqrt{2}}{2} S_{u_1} w_1 = \{1, 0, -1, -1, -4, 60, 58, 56\} [0,7] = v
\]

and the following energy-preserving identity

\[
\| w_0 \|^2_{l_2(\mathbb{Z})} + \| w_1 \|^2_{l_2(\mathbb{Z})} = \frac{16137}{2} + \frac{4101}{2} = 10119 = \| v \|^2_{l_2(\mathbb{Z})}.
\]
Figure: Diagram of a one-level discrete framelet transform using a dual framelet filter bank ($\{u_0, \ldots, u_s\}, (\tilde{u}_0, \ldots, \tilde{u}_s)$).
One key feature of DFrT is its sparse representation for smooth or piecewise smooth signals.

It is desirable to have as many as possible negligible framelet coefficients for smooth signals.

Smooth signals are modeled by polynomials. Let $p : \mathbb{R} \to \mathbb{C}$ be a polynomial: $p(x) = \sum_{n=0}^{m} p_n x^n$.

a polynomial sequence $p|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ by $[p|_{\mathbb{Z}}](k) = p(k), k \in \mathbb{Z}$.

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. 
Polynomial Differentiation Operator

- Polynomial differentiation operator:

\[ p(x - i \frac{d}{d\xi}) f(\xi) := \sum_{n=0}^{\infty} p_n (x - i \frac{d}{d\xi})^n f(\xi). \]

\[ p(x - i \frac{d}{d\xi}) f(\xi) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} p^{(n)}(x) f^{(n)}(\xi) \]

\[ = \sum_{n=0}^{\infty} \frac{x^n}{n!} p^{(n)} (-i \frac{d}{d\xi}) f(\xi). \]

- Generalized product rule for differentiation:

\[ p(x - i \frac{d}{d\xi}) \left( g(\xi) f(\xi) \right) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} g^{(n)}(\xi) p^{(n)}(x - i \frac{d}{d\xi}) f(\xi). \]

\[ \left[ p(-i \frac{d}{d\xi}) (e^{ix\xi} f(\xi)) \right]_{\xi=0} = \left[ p(x - i \frac{d}{d\xi}) f(\xi) \right]_{\xi=0}. \]
Lemma

Let $u = \{u(k)\}_{k \in \mathbb{Z}} \in l_0(\mathbb{Z})$. Then for any polynomial $p \in \Pi$, $p \ast u$ is a polynomial with $\deg(p \ast u) \leq \deg(p)$,

$$[p \ast u](x) = \sum_{k \in \mathbb{Z}} p(x - k)u(k) = \left[ p(x - i \frac{d}{d\xi}) \hat{u}(\xi) \right]_{\xi = 0}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} p^{(n)}(x) \hat{u}^{(n)}(0) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left[ p^{(n)}(-i \frac{d}{d\xi}) \hat{u}(\xi) \right]_{\xi = 0}.$$  

Moreover, $p \ast (u \uparrow 2) = [p(2 \cdot) \ast u](2^{-1} \cdot)$,

$$p^{(n)} \ast u = [p \ast u]^{(n)}, \quad p(\cdot - y) \ast u = [p \ast u](\cdot - y), \quad \forall \ y \in \mathbb{R}.$$
For smooth functions $f$ and $g$, it is often convenient to use the following big $O$ notation:

$$f(\xi) = g(\xi) + O(|\xi - \xi_0|^m), \quad \xi \to \xi_0$$

to mean that the derivatives of $f$ and $g$ at $\xi = \xi_0$ agree to the orders up to $m - 1$:

$$f^{(n)}(\xi_0) = g^{(n)}(\xi_0), \quad \forall \ n = 0, \ldots, m - 1.$$
Theorem

Let \( u \in l_0(\mathbb{Z}) \). Then for any polynomial \( p \in \Pi \),
\[
\mathcal{T}_u p = 2[p \ast u^*](2 \cdot) = p(2 \cdot) \ast \hat{u} = \sum_{n=0}^{\infty} \frac{\xi(\mathcal{P}_q)^n}{n!} p^{(n)}(2 \cdot) \hat{u}^{(n)}(0),
\]
where \( \hat{u} \) is a finitely supported sequence on \( \mathbb{Z} \) such that
\[
\hat{u}(\xi) = 2\hat{u}(\xi/2) + O(|\xi|^\text{deg}(p)+1), \quad \xi \to 0.
\]

In particular, for any integer \( m \in \mathbb{N} \), \( \text{TFAE}: \)

1. \( \mathcal{T}_u p = 0 \) for all polynomial sequences \( p \in \Pi_{m-1} \);
2. \( \mathcal{T}_u q = 0 \) for some \( q \in \Pi \) with \( \text{deg}(q) = m - 1 \);
3. \( \hat{u}(\xi) = O(|\xi|^m) \) as \( \xi \to 0 \);
4. \( \hat{u}(\xi) = (1 - e^{2\pi i \xi})^m Q(\xi) \) for some \( 2\pi \)-periodic trigonometric polynomial \( Q \).
Vanishing Moments

- We say that a filter $u$ has $m$ vanishing moments if any of items (1)–(4) in Theorem holds.
- Most framelet coefficients are zero for any input signal which is a polynomial to certain degree.
- If $u$ has $m$ vanishing moments. For a signal $v$, if $v$ agrees with some polynomial of degree less than $m$ on the support of $u(\cdot - 2n)$, then $[\mathcal{T}_u v](n) = 0$. 
Coset Sequences

For $u = \{u(k)\}_{k \in \mathbb{Z}}$ and $\gamma \in \mathbb{Z}$, we define the associated coset sequence $u[\gamma]$ of $u$ at the coset $\gamma + 2\mathbb{Z}$ to be

$$\hat{u}[\gamma](\xi) := \sum_{k \in \mathbb{Z}} u(\gamma + 2k)e^{-ik\xi},$$

that is,

$$u[\gamma] = u(\gamma + \cdot) \downarrow 2 = \{u(\gamma + 2k)\}_{k \in \mathbb{Z}}.$$
Subdivision Operator on Polynomials

- $S_up$ is not always a polynomial for $p \in \Pi$.
- For example, for $p = 1$ and $u = \{1\}_{[0,0]}$, we have $[S_up](2k) = 2$ and $[S_up](2k + 1) = 0$ for all $k \in \mathbb{Z}$.

**Lemma**

Let $u \in l_0(\mathbb{Z})$ and $q$ be a polynomial. TFAE:

(i) $\sum_{k \in \mathbb{Z}} q(-\frac{1}{2} - k)u(1 + 2k) = \sum_{k \in \mathbb{Z}} q(-k)u(2k)$, that is, $(q * u^[1])(-\frac{1}{2}) = (q * u^[0])(0)$;

(ii) $[q(-i \frac{d}{d\xi})(e^{-i\xi/2} \hat{u}^[1](\xi))]|_{\xi=0} = [q(-i \frac{d}{d\xi})\hat{u}^[0](\xi)]|_{\xi=0}$;

(iii) $[q(-\frac{i}{2} \frac{d}{d\xi})\hat{u}(\xi)]|_{\xi=\pi} = 0$. 
Subdivision Operator on Polynomials

Theorem

Let \( u = \{u(k)\}_{k \in \mathbb{Z}} \). For \( m \in \mathbb{N} \), TFAE:

1. \( S_u \Pi_{m-1} \subseteq \Pi \);
2. \( S_u q \in \Pi \) for some \( q \in \Pi \) with \( \deg(q) = m - 1 \);
3. \( S_u \Pi_{m-1} \subseteq \Pi_{m-1} \);
4. \( u \) has \( m \) sum rules:
   \[
   \hat{u}(\xi + \pi) = O(|\xi|^m), \quad \xi \to 0;
   \]
5. \( \hat{u}(\xi) = (1 + e^{-i\xi})^m Q(\xi) \) for some \( 2\pi \)-periodic \( Q \);
6. \( e^{-i\xi/2} \hat{u}^{[1]}(\xi) = \hat{u}^{[0]}(\xi) + O(|\xi|^m), \quad \xi \to 0. \)

Moreover, \( S_u p = 2^{-1} p(2^{-1} \cdot) * u \).
**Lemma**

Let $u \in l_0(\mathbb{Z})$. Let $p$ be a polynomial and define $m := \deg(p)$. For $c \in \mathbb{R}$, $p \ast u = p(\cdot - c)$ if $u$ has $m + 1$ linear-phase moments with phase $c$:

$$\hat{u}(\xi) = e^{-ic\xi} + O(|\xi|^{m+1}), \quad \xi \to 0.$$ 

**Proposition**

Let $u \in l_0(\mathbb{Z})$ and $c \in \mathbb{R}$. Then $u$ has $m + 1$ linear-phase moments with phase $c$ if $u$ has $m + 1$ sum rules and $m + 1$ linear-phase moments with phase $c$:

$$T_u p = 2p(2 \cdot + c) \text{ for all } p \in \Pi_m.$$ 

$$S_u p = p(2^{-1}(\cdot - c)) \text{ for all } p \in \Pi_m.$$
Symmetry

We say that \( u = \{u(k)\}_{k \in \mathbb{Z}} : \mathbb{Z} \to \mathbb{C} \) has **symmetry** if

\[
u(2c - k) = \epsilon u(k), \quad \forall \ k \in \mathbb{Z}\]

with \( 2c \in \mathbb{Z} \) and \( \epsilon \in \{-1, 1\} \).

\( u \) is **symmetric** if \( \epsilon = 1 \); **antisymmetric** if \( \epsilon = -1 \).

\( c \) is the **symmetry center** of the filter \( u \).

A symmetry operator \( S \) to record the symmetry type:

\[
[S\hat{u}](\xi) := \frac{\hat{u}(\xi)}{\hat{u}(-\xi)}, \quad \xi \in \mathbb{R}.
\]

\([S\hat{u}](\xi) = \epsilon e^{-i2c\xi} \).
Proposition

Suppose that \( u \in l_0(\mathbb{Z}) \) has \( m \) but not \( m + 1 \) linear-phase moments with phase \( c \in \mathbb{R} \). If \( m > 1 \), then the phase \( c \) is uniquely determined by \( u \) through

\[
c = i\hat{u}'(0) = \sum_{k \in \mathbb{Z}} u(k)k.
\]

Moreover, if \( u \) has symmetry: \( u(2c_0 - k) = u(k) \) for all \( k \in \mathbb{Z} \) for some \( c_0 \in \frac{1}{2}\mathbb{Z} \), then \( c = c_0 \) (that is, the phase \( c \) agrees with the symmetry center \( c_0 \) of \( u \)).
Example: B-spline filters

- B-spline filter of order $m$: $\hat{a}_m^B(\xi) := 2^{-m}(1 + e^{-i\xi})^m$
- $\hat{a}_4^B(0) = 1$, $\hat{a}_4^B(0) = -2i$, $\hat{a}_4^B(0) = -5$, $\hat{a}_4^B(0) = 14i$.
- For $p \in \Pi_3$,
  \[ [p * a_4^B](x) = p(x) - 2p'(x) + \frac{5}{2}p''(x) - \frac{7}{3}p'''(x). \]
  \[ [T_{a_4^B}p](x) = 2p(2x) + 4p'(2x) + 5p''(2x) + \frac{14}{3}p'''(2x). \]
  \[ [S_{a_4^B}p](x) = p(x/2) - p'(x/2) + \frac{5}{8}p''(x/2) - \frac{7}{24}p'''(x/2). \]
Let $a$ be a primal low-pass filter and $b_1, \ldots, b_s$ be primal high-pass filters for decomposition.

For a positive integer $J$, a $J$-level discrete framelet decomposition is given by

$$v_{j-1} := \frac{\sqrt{2}}{2} T_a v_j, \quad w_{j-1;\ell} := \frac{\sqrt{2}}{2} T_{b_\ell} v_j,$$

$\ell = 1, \ldots, s, \quad j = J, \ldots, 1,$

where $v_J : \mathbb{Z} \to \mathbb{C}$ is an input signal.

**decomposition operator** $\mathcal{W}_J : l(\mathbb{Z}) \to (l(\mathbb{Z}))^{1 \times (sJ+1)}$:

$$\mathcal{W}_J v_J := (w_{J-1;1}, \ldots, w_{J-1;s}, \ldots, w_{0;1}, \ldots, w_{0;s}, v_0),$$
Thresholding and Quantization

Figure: The hard thresholding function, the soft thresholding function, and the quantization function, respectively. Both thresholding and quantization operations are often used to process the framelet coefficients in a discrete framelet transform.
Multi-level Reconstruction

- Let $\tilde{a}$ be a dual low-pass filter and $\tilde{b}_1, \ldots, \tilde{b}_s$ be dual high-pass filters for reconstruction.
- A $J$-level discrete framelet reconstruction is
  \[
  \hat{\nu}_j := \frac{\sqrt{2}}{2} S_\tilde{a} \hat{\nu}_{j-1} + \frac{\sqrt{2}}{2} \sum_{\ell=1}^{s} S_{\tilde{b}_\ell} \hat{w}_{j-1;\ell}, \quad j = 1, \ldots, J.
  \]
- A $J$-level discrete reconstruction operator
  \[
  \mathcal{V}_J : (l(\mathbb{Z}))^{1 \times (sJ+1)} \to l(\mathbb{Z})
  \]
  is defined by
  \[
  \mathcal{V}_J(\hat{w}_{J-1;1}, \ldots, \hat{w}_{J-1;s}, \ldots, \hat{w}_{0;1}, \ldots, \hat{w}_{0;s}, \hat{\nu}_0) = \hat{\nu}_J,
  \]
  - A fast framelet transform with $s = 1$ is called a fast wavelet transform.
Diagram of Multi-level FFrT

Figure: Diagram of a two-level discrete framelet transform using a dual framelet filter bank ($\{a; b_1, \ldots, b_s\}, (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}$).
Stability

A multi-level discrete framelet transform employing a dual framelet filter bank \( \{ a; b_1, \ldots, b_s \}, \{ \tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s \} \) has stability in the space \( l_2(\mathbb{Z}) \) if there exists \( C > 0 \) such that for \( J \in \mathbb{N}_0 \),

\[
\| \mathcal{W}_J v \|_{(l_2(\mathbb{Z}))^{1 \times (sJ+1)}} \leq C \| v \|_{l_2(\mathbb{Z})}, \quad \forall \ v \in l_2(\mathbb{Z})
\]

and

\[
\| \mathcal{V}_J \tilde{w} \|_{l_2(\mathbb{Z})} \leq C \| \tilde{w} \|_{(l_2(\mathbb{Z}))^{1 \times (sJ+1)}}, \quad \forall \ \tilde{w} \in (l_2(\mathbb{Z}))^{1 \times (sJ+1)}.
\]
Suppose that a multi-level discrete framelet transform has stability in the space $l_2(\mathbb{Z})$. Then all the wavelet decomposition operators must have uniform stability in space $l_2(\mathbb{Z})$:

$$\frac{1}{C} \| v \|_{l_2(\mathbb{Z})} \leq \| W_J v \|_{l_2(\mathbb{Z})} \leq C \| v \|_{l_2(\mathbb{Z})}, \forall v \in l_2(\mathbb{Z}), J \in \mathbb{N}.$$ 

Moreover, wavelet reconstruction operators have uniform stability: for all $J \in \mathbb{N}$,

$$\frac{1}{C} \| \vec{w} \|_{(l_2(\mathbb{Z}))^{1 \times (sJ+1)}} \leq \| V_J \vec{w} \|_{l_2(\mathbb{Z})} \leq C \| \vec{w} \|_{(l_2(\mathbb{Z}))^{1 \times (sJ+1)}}, \vec{w} \in (l_2(\mathbb{Z}))^{1 \times (sJ+1)},$$

if and only if $s = 1$. 

Bin Han (University of Alberta) Algorithms of Wavelets and Framelets UofA 36 / 64
Stability of Multi-level FFrT

**Theorem**

Let \( \{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\} \) be a dual framelet filter bank with \( \hat{a}(0) = \hat{\tilde{a}}(0) = 1 \). Define

\[
\varphi(\xi) := \prod_{j=1}^{\infty} \hat{a}(2^{-j} \xi), \quad \tilde{\varphi}(\xi) := \prod_{j=1}^{\infty} \hat{\tilde{a}}(2^{-j} \xi), \quad \xi \in \mathbb{R}.
\]

Then a multi-level discrete framelet transform has stability in the space \( l_2(\mathbb{Z}) \iff \varphi, \tilde{\varphi} \in L_2(\mathbb{R}) \) and

\[
\hat{b}_1(0) = \cdots = \hat{b}_s(0) = \hat{\tilde{b}}_1(0) = \cdots = \hat{\tilde{b}}_s(0) = 0.
\]
Multi-level FFrT Again

\[ v_j = \frac{\sqrt{2}}{2} T_a v_{j+1} = \cdots = (\frac{\sqrt{2}}{2})^{J-j} T_{a^*}(a|2) \cdots (a|2^{J-j-1}), 2^{J-j} v \]

\[ w_{j;\ell} = \frac{\sqrt{2}}{2} T_{b_\ell} v_{j+1} = (\frac{\sqrt{2}}{2})^{J-j} T_{a^*}(a|2) \cdots (a|2^{J-j-2}) (b_\ell|2^{J-j-1}), 2^{J-j} v \]

\[ w_{j; k}(k) = \langle v, 2^{(J-j)/2} [a^*(a|2) \cdots (a|2^{J-j-2}) (b_\ell|2^{J-j-1})] \rangle (.-2^{J-j} k), \]

Similarly, we have

\[ \mathcal{V}_J(0, \ldots, 0, v_0) = (\frac{\sqrt{2}}{2})^{J-j} S_{\tilde{a}^*}(\tilde{a}|2) \cdots (\tilde{a}|2^{J-1}), 2^J v_0 \]

\[ \mathcal{V}_J(0, \ldots, 0, w_{j; \ell}, 0, \ldots, 0) = (\frac{\sqrt{2}}{2})^{J-j} S_{\tilde{a}^*}(\tilde{a}|2) \cdots (\tilde{a}|2^{J-j-2}) (\tilde{b}_\ell|2^{J-j-1}), 2^{J-j} w_{j; \ell}. \]
Define filters $a_j$ and $\tilde{a}_j$ by

$$a_j = a \ast (a \uparrow 2) \ast \cdots \ast (a \uparrow 2^{j-1})$$

and

$$\tilde{a}_j := \tilde{a} \ast (\tilde{a} \uparrow 2) \ast \cdots \ast (\tilde{a} \uparrow 2^{j-1}).$$

Now a $J$-level discrete framelet transform employing ($\{a; b_1, \ldots, b_s\}$, $\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}$) is equivalent to

$$v = \sum_{k \in \mathbb{Z}} \langle v, 2^{J/2} a_J (\cdot - 2^J k) \rangle 2^{J/2} \tilde{a}_J (\cdot - 2^J k) + \sum_{j=0}^{J-1} \sum_{\ell=1}^{s} \sum_{k \in \mathbb{Z}} \langle v, [2^{(J-j)/2} a_{J-j-1} \ast (b_\ell \uparrow 2^{J-j-1})](\cdot - 2^{J-j} k) \rangle$$

$$\langle v, [2^{(J-j)/2} \tilde{a}_{J-j-1} \ast (\tilde{b}_\ell \uparrow 2^{J-j-1})](\cdot - 2^{J-j} k) \rangle.$$
Lack of Vanishing Moments

Lemma

Let \((\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})\) be a dual framelet filter bank. If all \(b_1, \ldots, b_s\) have \(\tilde{m}\) vanishing moments and all \(\tilde{b}_1, \ldots, \tilde{b}_s\) have \(m\) vanishing moments, then

(i) the primal low-pass filter \(a\) must have \(m\) sum rules, that is,
\[
\hat{a}(\xi + \pi) = \mathcal{O}(|\xi|^m) \quad \text{as} \quad \xi \to 0;
\]

(ii) the dual low-pass filter \(\tilde{a}\) must have \(\tilde{m}\) sum rules, that is,
\[
\hat{\tilde{a}}(\xi + \pi) = \mathcal{O}(|\xi|^\tilde{m}) \quad \text{as} \quad \xi \to 0;
\]

(iii) \(\hat{a}\hat{\tilde{a}}\) has \(m + \tilde{m}\) linear-phase moments with phase 0:
\[
1 - \hat{a}(\xi)\hat{\tilde{a}}(\xi) = \mathcal{O}(|\xi|^{m+\tilde{m}}), \quad \xi \to 0.
\]
Example of B-spline Filters

Let \( \hat{a}(\xi) = \widehat{a_m^B}(\xi) = 2^{-m}(1 + e^{-i\xi})^m \) and 
\( \hat{\tilde{a}}(\xi) = \widehat{a_{\tilde{m}}^B}(\xi) = 2^{-\tilde{m}}(1 + e^{-i\xi})^{\tilde{m}} \) be two B-spline filters.

\[
\hat{a}(0)\hat{\tilde{a}}(0) = 1,
\]

\[
[\hat{a}\hat{\tilde{a}}]'(0) = \frac{i(m - \tilde{m})}{2},
\]

\[
[\hat{a}\hat{\tilde{a}}]''(0) = \frac{(m - \tilde{m})^2 + m + \tilde{m}}{4}.
\]
Theorem

\((\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})_\Theta\) has the following generalized perfect reconstruction property:

\[
\Theta * \nu = \frac{1}{2} S_{\tilde{a}}(\Theta * T_a \nu) + \frac{1}{2} \sum_{\ell=1}^{s} S_{\tilde{b}_\ell} T_{b_\ell} \nu, \quad \forall \nu \in l(\mathbb{Z}),
\]

\[\iff \text{for all } \xi \in \mathbb{R}, \]

\[
\hat{\Theta}(2\xi) \hat{\tilde{a}}(\xi) \hat{\tilde{a}}(\xi) + \hat{b}_1(\xi) \hat{\tilde{b}}_1(\xi) + \cdots + \hat{b}_s(\xi) \hat{\tilde{b}}_s(\xi) = \hat{\Theta}(\xi),
\]

\[
\hat{\Theta}(2\xi) \hat{\tilde{a}}(\xi + \pi) \hat{\tilde{a}}(\xi) + \hat{b}_1(\xi + \pi) \hat{\tilde{b}}_1(\xi) + \cdots + \hat{b}_s(\xi + \pi) \hat{\tilde{b}}_s(\xi) = 0.
\]
Lemma

Let \( \{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\} \) be an OEP-based filter bank. Suppose that all \( b_1, \ldots, b_s \) have \( \tilde{m} \) vanishing moments and all \( \tilde{b}_1, \ldots, \tilde{b}_s \) have \( m \) vanishing moments, where \( m, \tilde{m} \in \mathbb{N} \). Then

\[
\hat{\Theta}(\xi) - \hat{\Theta}(2\xi) \hat{a}(\xi) \hat{\tilde{a}}(\xi) = O(\|\xi\|^{m+\tilde{m}}), \quad \xi \to 0.
\]

If in addition \( \hat{\Theta}(0) \neq 0 \), then the primal low-pass filter \( a \) must have \( m \) sum rules and the dual low-pass filter \( \tilde{a} \) must have \( \tilde{m} \) sum rules.
OEP-based Tight Framelets

Proposition

Let \( \theta, a, b_1, \ldots, b_s \in l_0(\mathbb{Z}) \). Then

\[
\| \theta^* T_a v \|_{l_2(\mathbb{Z})}^2 + \| T_{b_1} v \|_{l_2(\mathbb{Z})}^2 + \cdots + \| T_{b_s} v \|_{l_2(\mathbb{Z})}^2 = 2\| \theta^* v \|_{l_2(\mathbb{Z})}^2, \quad \forall \ v \in l_2(\mathbb{Z}),
\]

\[\iff (\{ a; b_1, \ldots, b_s \}, \{ a; b_1, \ldots, b_s \})_\Theta \text{ has PR:} \]

\[
[ \hat{b}_1(\xi) \quad \cdots \quad \hat{b}_s(\xi) ] [ \hat{b}_1(\xi + \pi) \quad \cdots \quad \hat{b}_s(\xi + \pi) ]^* = M_{\Theta,a}.
\]

where \( \Theta := \theta^* \theta^* \) and

\[
M_{\Theta,a} := \begin{bmatrix}
\hat{\Theta}(\xi) - \hat{\Theta}(2\xi)\hat{a}(\xi)^2 & -\hat{\Theta}(2\xi)\hat{a}(\xi + \pi)\hat{a}(\xi) \\
-\hat{\Theta}(2\xi)\hat{a}(\xi)\hat{a}(\xi + \pi) & \hat{\Theta}(\xi + \pi) - \hat{\Theta}(2\xi)|\hat{a}(\xi + \pi)|^2
\end{bmatrix}.
\]
OEP-based Dual Framelets

- \( \{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\} \) having PR is called an OEP-based dual framelet filter bank.

- \( \{a; b_1, \ldots, b_s\} \) having PR is called an OEP-based tight framelet filter bank.
Fejér-Riesz Lemma

**Lemma**

Let $Θ$ be a $2\pi$-periodic trigonometric polynomial with real coefficients (or with complex coefficients) such that $Θ(ξ) \geq 0$ for all $ξ \in \mathbb{R}$. Then there exists a $2\pi$-periodic trigonometric polynomial $θ$ with real coefficients (or with complex coefficients) such that $|θ(ξ)|^2 = Θ(ξ)$ for all $ξ \in \mathbb{R}$. Moreover, if $Θ(0) \neq 0$, then we can further require $θ(0) = \sqrt{Θ(0)}$. 
Lemma

Let \( \{a; b_1, \ldots, b_s\}_\Theta \) be an OEP-based tight framelet filter bank. If \( \hat{\Theta}(0) > 0 \), then \( \hat{\Theta}(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \) and consequently, there exists \( \theta \in l_0(\mathbb{Z}) \) such that \( \Theta = \theta \ast \theta^* \) holds and \( \hat{\theta}(0) = \sqrt{\hat{\Theta}(0)} \).
Theorem

Let \( \{a; b\}, \{\tilde{a}; \tilde{b}\} \) be an OEP-based dual framelet filter bank with \( \hat{\Theta}(0) \neq 0 \). Then

\[
\hat{\Theta}(2\xi)\hat{\Theta}(\pi) = \hat{\Theta}(\xi)\hat{\Theta}(\xi + \pi)
\]

\[
\begin{bmatrix}
\hat{a}(\xi) & \hat{b}(\xi) \\
\hat{a}(\xi + \pi) & \hat{b}(\xi + \pi)
\end{bmatrix}
\begin{bmatrix}
\hat{a}(\xi) & \hat{b}(\xi) \\
\hat{a}(\xi + \pi) & \hat{b}(\xi + \pi)
\end{bmatrix}^* = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\]
Theorem

where all filters are finitely supported and are given by

\[ \hat{a}(\xi) := \hat{a}(\xi)\hat{\Theta}(2\xi)/\hat{\Theta}(\xi) \quad \text{and} \quad \hat{b}(\xi) := \hat{b}(\xi)/\hat{\Theta}(\xi). \]

That is, \((\{a; b\}, \\{\hat{a}; \hat{b}\})\) is a biorthogonal wavelet filter bank.

Moreover,

\[ \hat{b}(\xi) = c e^{i(2n-1)\xi} \hat{\bar{a}}(\xi + \pi), \quad \hat{b}(\xi) = c^{-1} e^{i(2n-1)\xi} \hat{\bar{a}}(\xi + \pi) \quad \text{for some} \]

for some
An OEP-based $J$-level discrete framelet decomposition is exactly the same as the one before.

For given low-pass framelet coefficients $\hat{v}_0$ and high-pass framelet coefficients $\hat{w}_{j;\ell}, \ell = 1, \ldots, s$ and $j = 0, \ldots, J - 1$, an OEP-based $J$-level discrete framelet reconstruction is

$$
\check{v}_0 := \Theta \ast \hat{v}_0,
$$

$$
\check{v}_j := \frac{\sqrt{2}}{2} S_{\check{a}} \check{v}_{j-1} + \frac{\sqrt{2}}{2} \sum_{\ell=1}^{s} S_{\check{b}_\ell} \hat{w}_{j-1;\ell}, \quad j = 1, \ldots, J,
$$

recover $\hat{v}_J$ from $\check{v}_J$ via the relation $\hat{v}_J = \Theta \ast \check{v}_J$. 
Diagram of DFrT using OEP

Figure: Diagram of a two-level discrete framelet transform using an OEP-based dual framelet filter bank \((\{a; b_1, \ldots, b_s\}, (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s)_{\Theta})\).

Bin Han (University of Alberta)  Algorithms of Wavelets and Framelets  UofA 51 / 64
Avoiding Deconvolution using OEP

Figure: Diagram of an \( n \)-branch multi-level discrete averaging framelet transform using dual framelet filter banks

\[
\left\{ \left( \{ a_m; b_{m,1}, \ldots, b_{m,s_m} \}, \{ \tilde{a}_m; \tilde{b}_{m,1}, \ldots, \tilde{b}_{m,s_m} \} \right) \right\}_{\Theta_m}, \ m = 1, \ldots, n.
\]

Therefore, deconvolution is completely avoided.
Signals on Intervals

- Signals: \( v^b = \{ v^b(k) \}_{k=0}^{N-1} : [0, N - 1] \cap \mathbb{Z} \to \mathbb{C} \).
- Extend \( v^b \) from interval \([0, N - 1]\) to \( \mathbb{Z} \).
- \( T_{u(\cdot - 2m)} v = [T_u v](\cdot - m), \quad m \in \mathbb{Z} \).
- Extend \( v^b \) from \([0, N - 1] \cap \mathbb{Z}\) to a sequence \( v \) on \( \mathbb{Z} \) by any method that the reader prefers.
- Zero-padding: \( v(k) = 0 \) for \( k \in \mathbb{Z} \setminus [0, N - 1] \).
- Must recover \( v^b(0), \ldots, v^b(N - 1) \) exactly.
- Keep all \( [S_{\tilde{u}_\ell} T_{u_\ell} v](n), n = 0, \ldots, N - 1 \).
- \( \text{fsupp}(\tilde{u}) = [n_-, n_+] \) with \( n_- \leq 0 \) and \( n_+ \geq 0 \).
\[ \frac{1}{2} [S_{\tilde{u}} \mathcal{T}_u \nu](n) = \sum_{k \in \mathbb{Z}} [\mathcal{T}_u \nu](k) \tilde{u}(n - 2k) \], n = 0, \ldots, N - 1. \]

Record all the framelet coefficients:

\[ [\mathcal{T}_u \nu](k), \quad k = \left[ \frac{-n_+}{2} \right], \ldots, \left[ \frac{N-1-n_-}{2} \right]. \]

we always have \( \left[ \frac{-n_+}{2} \right] \leq 0 \) and \( \left[ \frac{N-1-n_-}{2} \right] \geq \frac{N}{2} - 1. \)

framelet coefficients \( \{[\mathcal{T}_u \nu](k)\}_{k=0}^{N-1} \) must be recorded.

extra: \( [\mathcal{T}_u \nu](k), \; k = \left[ \frac{-n_+}{2} \right], \ldots, -1 \) and \( k = \frac{N}{2}, \ldots, \left[ \frac{N-1-n_-}{2} \right] \).

Ideal situation can happen \( \iff u \) vanishes outside \([0, 1]\).
Proposition

Let \( u \in l_1(\mathbb{Z}) \) be a filter and \( v^b = \{ v^b(k) \}_{k=0}^{N-1} \). Extend \( v^b \) into an \( N \)-periodic sequence \( v \) on \( \mathbb{Z} \) as follows:

\[
v(Nn + k) := v^b(k), \quad k = 0, \ldots, N - 1, \quad n \in \mathbb{Z}.
\]

Then the following properties hold:

(i) \( u \ast v \) is an \( N \)-periodic sequence on \( \mathbb{Z} \);

(ii) \( S_\mu v \) is a \( 2N \)-periodic sequence on \( \mathbb{Z} \);

(iii) If \( N \) is even, then \( T_\mu v \) is an \( \frac{N}{2} \)-periodic;

(iv) If \( N \) is odd, then \( T_\mu v \) is an \( N \)-periodic.
DFrT using Periodic Extension

- A one-level discrete periodic framelet decomposition:

\[ w^b_\ell = \left\{ w^b_\ell(k) := \frac{\sqrt{2}}{2} \left[ T_{u_\ell} \nu \right](k) \right\}_{k=0}^{N^2-1}, \quad \ell = 0, \ldots, s, \]

where \( \nu \) is the \( N \)-periodic extension of \( \nu^b \).

- Grouping all framelet coefficients,

\[ W^\text{per}(\nu^b) = (w^b_0, w^b_1, \ldots, w^b_s)^T. \]

- A one-level discrete periodic framelet reconstruction:

\[ \hat{\nu}^b = W^\text{per}(\hat{w}^b_0, \ldots, \hat{w}^b_s) := \left\{ \hat{\nu}^b(k) := \frac{\sqrt{2}}{2} \sum_{\ell=0}^{s} [S_{\tilde{u}_\ell} \hat{w}_\ell](k) \right\}_{k=0}^{N^2-1}, \]

\[ W^\text{per} W^\text{per} = I_N. \]
A tight framelet filter bank by Ron-Shen, where

\[ u_0 = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right\}[-1,1], \]
\[ u_1 = \left\{ \frac{1}{4}, -\frac{1}{2}, \frac{1}{4} \right\}[-1,1], \]
\[ u_2 = \left\{ -\frac{\sqrt{2}}{4}, 0, -\frac{\sqrt{2}}{4} \right\}[-1,1]. \]

A test input data:

\[ v = \{1, 0, -1, -1, -4, 60, 58, 56\}[0,7] \]
We extend \( v^b \) to an 8-periodic sequence \( v \) on \( \mathbb{Z} \), given by

\[ v = \{ \ldots, 1, 0, -1, -1, -4, 60, 58, 56, 1, 0, -1, -1, -4, 60, 58, 56, 1, 0, -1, -1, \ldots \}. \]

Then all sequences \( \mathcal{T}_{u_0} v, \mathcal{T}_{u_1} v, \mathcal{T}_{u_2} v \) are 4-periodic and

\[ w_0 = \frac{\sqrt{2}}{2} \mathcal{T}_{u_0} v = \frac{\sqrt{2}}{2} \{ \ldots, \frac{29}{2}, -\frac{3}{4}, \frac{51}{4}, 58, \frac{29}{2}, -\frac{3}{4}, \frac{51}{4}, 58, \frac{29}{2}, -\frac{3}{4}, \frac{51}{4}, 58, \ldots \}, \]

\[ w_1 = \frac{\sqrt{2}}{2} \mathcal{T}_{u_1} v = \frac{\sqrt{2}}{2} \{ \ldots, \frac{27}{2}, \frac{1}{4}, \frac{67}{4}, 0, \frac{27}{2}, \frac{1}{4}, \frac{67}{4}, 0, \frac{27}{2}, \frac{1}{4}, \frac{67}{4}, 0, \ldots \}, \]

\[ w_2 = \frac{\sqrt{2}}{2} \mathcal{T}_{u_2} v = \{ \ldots, -14, \frac{1}{4}, -\frac{59}{4}, -29, -14, \frac{1}{4}, -\frac{59}{4}, -29, -14, \frac{1}{4}, -\frac{59}{4}, -29, \ldots \}. \]

It is also easy to check that \( \frac{\sqrt{2}}{2} (S_{u_0} w_0 + S_{u_1} w_1 + S_{u_2} w_2) = v. \)
Proposition

Let \( u \in l_1(\mathbb{Z}) \) such that \( u(2c - k) = \epsilon u(k) \). Extend \( v^b = \{ v^b(k) \}_{k=0}^{N-1} \), with both endpoints non-repeated (EN), into \((2N - 2)\)-periodic \( v \) by \( v(k) = v^b(2N - 2 - k) \).

(i) Then \( u^* \ast v \) is \((2N - 2)\)-periodic with

\[
[u^* \ast v](-2c - k) = [u^* \ast v](2N - 2 - 2c - k) = \epsilon [u^* \ast v](k),
\]

and \([-\lfloor c \rfloor, N - 1 - \lceil c \rceil]\) is its control interval.

(ii) If \( c \in \mathbb{Z} \), then \( T_u v \) is \((N - 1)\)-periodic with

\[
[T_u v](-c - k) = [T_u v](N - 1 - c - k) = \epsilon [T_u v](k),
\]

and \([-\left\lfloor \frac{c}{2} \right\rfloor, \left\lceil \frac{N-1-c}{2} \right\rceil]\) is a control interval of \( T_u v \).
Proposition

Let $u \in l_1(\mathbb{Z})$ with $\epsilon \in \{-1, 1\}$ and $c \in \frac{1}{2} \mathbb{Z}$. Extend $v^b$, with both endpoints repeated (ER), into $2N$-periodic $v$ by $v(k) = v^b(2N - 1 - k)$.

(i) Then $u^* * v$ is $2N$-periodic with

$$[u^* * v](-1 - 2c - k) = [u^* * v](2N - 1 - 2c - k) = \epsilon[u^* * v](k),$$

and $[-\lfloor \frac{1}{2} + c \rfloor, N - \lceil \frac{1}{2} + c \rceil]$ is its control interval.

(ii) If $c - \frac{1}{2} \in \mathbb{Z}$, then $T_u v$ is an $N$-periodic sequence:

$$[T_u v](-\frac{1}{2} - c - k) = [T_u v](N - \frac{1}{2} - c - k) = \epsilon[T_u v](k),$$

and $[-\lfloor -\frac{1}{4} - \frac{c}{2} \rfloor, \lfloor \frac{N}{2} - \frac{1}{4} - \frac{c}{2} \rfloor]$ is its control interval.
Table 1: Endpoint Nonrepeated (EN)

<table>
<thead>
<tr>
<th>filter $u$</th>
<th>$u^* \ast v$ with $v$ extended by EN</th>
<th>$T_u v$ with $v$ extended by EN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c = 0$</td>
<td>(2$N - 2$)-periodic,</td>
<td>$(N - 1)$-periodic,</td>
</tr>
<tr>
<td>$\epsilon = 1$</td>
<td>symmetric about 0 and $N - 1$,</td>
<td>symmetric about 0 and $\frac{N-1}{2}$,</td>
</tr>
<tr>
<td></td>
<td>a control interval $[0, N - 1]$.</td>
<td>a control interval $[0, \frac{N}{2} - 1]$.</td>
</tr>
<tr>
<td>$c = 0$</td>
<td>(2$N - 2$)-periodic,</td>
<td>$(N - 1)$-periodic,</td>
</tr>
<tr>
<td>$\epsilon = -1$</td>
<td>antisymmetric about 0 and $N - 1$,</td>
<td>antisymmetric about 0 and $\frac{N-1}{2}$,</td>
</tr>
<tr>
<td></td>
<td>a control interval $[0, N - 1]$,</td>
<td>a control interval $[0, \frac{N}{2} - 1]$,</td>
</tr>
<tr>
<td></td>
<td>$<a href="0">u^* \ast v</a> = [u^* \ast v](N - 1) = 0$.</td>
<td>$<a href="0">T_u v</a> = 0$.</td>
</tr>
<tr>
<td>$c = 1$</td>
<td>(2$N - 2$)-periodic,</td>
<td>$(N - 1)$-periodic,</td>
</tr>
<tr>
<td>$\epsilon = 1$</td>
<td>symmetric about $-1$ and $N - 2$,</td>
<td>symmetric about $-\frac{1}{2}$ and $\frac{N}{2} - 1$,</td>
</tr>
<tr>
<td></td>
<td>a control interval $[-1, N - 2]$.</td>
<td>a control interval $[0, \frac{N}{2} - 1]$.</td>
</tr>
<tr>
<td>$c = 1$</td>
<td>(2$N - 2$)-periodic,</td>
<td>$(N - 1)$-periodic,</td>
</tr>
<tr>
<td>$\epsilon = -1$</td>
<td>antisymmetric about $-1$ and $N - 2$,</td>
<td>antisymmetric about $-\frac{1}{2}$ and $\frac{N}{2} - 1$,</td>
</tr>
<tr>
<td></td>
<td>a control interval $[-1, N - 2]$,</td>
<td>a control interval $[0, \frac{N}{2} - 1]$,</td>
</tr>
<tr>
<td></td>
<td>$<a href="-1">u^* \ast v</a> = [u^* \ast v](N - 2) = 0$.</td>
<td>$[T_u v](\frac{N}{2} - 1) = 0$.</td>
</tr>
</tbody>
</table>

The decomposition filter $u$ has the symmetry $S \hat{u}(\xi) = \epsilon e^{-i2c\xi}$, where $\epsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$. $v$ is a symmetric extension with both endpoints non-repeated (EN) of an input signal $v^b = \{v^b(k)\}_{k=0}^{N-1}$. 
The decomposition filter $u$ has the symmetry $S\hat{u}(\xi) = \epsilon e^{-i2c\xi}$, where $\epsilon \in \{-1, 1\}$ and $c \in \{-\frac{1}{2}, \frac{1}{2}\}$. $v$ is a symmetric extension with both endpoints repeated (ER) of an input signal $v^b = \{v^b(k)\}_{k=0}^{N-1}$.

<table>
<thead>
<tr>
<th>filter $u$</th>
<th>$u^* \ast v$ with $v$ extended by ER</th>
<th>$T_u v$ with $v$ extended by ER</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c = \frac{1}{2}$, $\epsilon = 1$</td>
<td>2N-periodic, symmetric about $-1$ and $N - 1$, a control interval $[-1, N - 1]$.</td>
<td>$N$-periodic, symmetric about $-\frac{1}{2}$ and $\frac{N-1}{2}$, a control interval $[0, \frac{N}{2} - 1]$.</td>
</tr>
<tr>
<td>$c = \frac{1}{2}$, $\epsilon = -1$</td>
<td>2N-periodic, antisymmetric about $-1$ and $N - 1$, a control interval $[-1, N - 1]$, $<a href="-1">u^* \ast v</a> = [u^* \ast v](N - 1) = 0$.</td>
<td>$N$-periodic, antisymmetric about $-\frac{1}{2}$ and $\frac{N-1}{2}$, a control interval $[0, \frac{N}{2} - 1]$</td>
</tr>
<tr>
<td>$c = -\frac{1}{2}$, $\epsilon = 1$</td>
<td>2N-periodic, symmetric about 0 and $N$, a control interval $[0, N]$.</td>
<td>$N$-periodic, symmetric about 0 and $\frac{N}{2}$, a control interval $[0, \frac{N}{2}]$.</td>
</tr>
<tr>
<td>$c = -\frac{1}{2}$, $\epsilon = -1$</td>
<td>2N-periodic, antisymmetric about 0 and $N$, a control interval $[0, N]$, $<a href="0">u^* \ast v</a> = <a href="N">u^* \ast v</a> = 0$.</td>
<td>$N$-periodic, antisymmetric about 0 and $\frac{N}{2}$, a control interval $[0, \frac{N}{2}]$, $<a href="0">T_u v</a> = <a href="%5Cfrac%7BN%7D%7B2%7D">T_u v</a> = 0$.</td>
</tr>
</tbody>
</table>
Example: Biorthogonal Wavelet Filter Bank

Since $\mathbf{S}\hat{u}_0 = 1$ and $\mathbf{S}\hat{u}_1 = e^{-i2\xi}$, extend $v^b$ by both endpoints non-repeated (EN):

$v = \{ \ldots, 58, 60, -4, -1, -1, 0, 1, 0, -1, -1, -4, 60, 58, 56, 58, 60, -4, -1, -1 \}$

Then $\mathcal{T}_{u_0} v$ is 7-periodic and is symmetric about 0, $7/2$:

$$w_0 = \frac{\sqrt{2}}{2} \mathcal{T}_{u_0} v = \frac{\sqrt{2}}{2} \{ \ldots, \frac{37}{8}, -\frac{5}{8}, 1, -\frac{5}{8}, \frac{37}{8}, \frac{263}{4}, \frac{263}{4}, \frac{37}{8}, \ldots \},$$

and $\mathcal{T}_{u_1} v$ is 7-periodic and is symmetric about $-\frac{1}{2}, 3$:

$$w_1 = \frac{\sqrt{2}}{2} \mathcal{T}_{u_1} v = \frac{\sqrt{2}}{2} \{ \ldots, -\frac{3}{4}, 0, 0, -\frac{3}{4}, -\frac{33}{2}, 1, -\frac{33}{2}, -\frac{3}{4}, \ldots \}.$$
Example: Tight Framelet Filter Bank

Since $S\hat{u}_0 = S\hat{u}_1 = S\hat{u}_2 = 1$, extend $v^b$ with both endpoints non-repeated (EN). Then all $T_{u_0}v$, $T_{u_1}v$, $T_{u_2}v$ are 7-periodic and symmetric about 0 and $7/2$:

$$w_0 = \frac{\sqrt{2}}{2} T_{u_0}v = \frac{\sqrt{2}}{2} \{ \ldots, 58, \frac{51}{4}, -\frac{3}{4}, \frac{1}{2}, -\frac{3}{4}, \frac{51}{4}, 58, 58, \frac{51}{4}, -\frac{3}{4}, \ldots \},$$

$$w_1 = \frac{\sqrt{2}}{2} T_{u_1}v = \frac{\sqrt{2}}{2} \{ \ldots, 0, \frac{67}{4}, \frac{1}{4}, -\frac{1}{2}, \frac{1}{4}, \frac{67}{4}, 0, 0, \frac{67}{4}, \frac{1}{4}, \ldots \}.$$ 

$$w_2 = \frac{\sqrt{2}}{2} T_{u_1}v = \{ \ldots, -29, -\frac{59}{4}, \frac{1}{4}, 0, \frac{1}{4}, -\frac{59}{4}, -29, -29, -\frac{59}{4}, \frac{1}{4}, 0, \ldots \}.$$